# On the generating sequences of regular languages on $k$ symbols

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The main result is a characterization of the generating sequences of the length of words in a regular language on $k$ symbols. We say that a sequence $s$ of integers is regular if there is a finite graph $G$ with two vertices $i, t$ such that $s_n$ is the number of paths of length $n$ from $i$ to $t$ in $G$. Thus the generating sequence of a regular language is regular. We prove that a sequence $s$ is the generating sequence of a regular language on $k$ symbols if and only if both sequences $s = (s_n)_{n \geq 0}$ and $t = (k^n - s_n)_{n \geq 0}$ are regular.

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## 1. INTRODUCTION

The notion of a generating sequence for a formal language $L$ is a simple one: it is the sequence $(s_n)_{n \geq 0}$ where $s_n$ is the number of words of length $n$ in $L$. Even if the non-commutative nature of words is lost, this sequence carries important information concerning a formal language since it measures in a sense the size of the language. It is moreover of interest in coding. In fact, a length-preserving encoding defines a one-to-one correspondence between words. The two sets of words in such a correspondence will have the same length distribution.

The characterization of the generating sequences of regular languages has long been known. Indeed, a sequence $(s_n)_{n \geq 0}$ is the generating sequence of a regular language on some alphabet if and only if it is regular, i.e., there exists a finite graph $G$ with two vertices $i, t$ such that $s_n$ is the number of paths of length $n$ from $i$ to $t$ in $G$.

The idea of fixing the cardinality of the alphabet in this problem has surprisingly never been considered. In other terms, for a given integer $k$, when is an integer sequence the generating sequence of a regular language on $k$ symbols?

Suppose for example that we consider the regular language on three symbols $L = (a+b)^* c^+$. Its number of words of length $n$ is $2^{n-1}$. It has the same generating sequences as the regular language on two symbols $L' = (a+b)^* ab^*$. We address here the problem of characterizing the regular languages $L$ for which such a coding on a smaller alphabet is possible and we describe explicitly how to realize it. Our main result is a characterization of the generating sequences of regular languages on $k$ symbols.

Our characterization is the following. We prove that a sequence $s$ is the generating sequence of a regular language on $k$ symbols if and only if both sequences $s = (s_n)_{n \geq 0}$ and the complementary sequence $t = (k^n - s_n)_{n \geq 0}$ are regular (Theorem 3.2). Observe that the second condition implies the obviously necessary condition that $s_n \leq k^n$ for all $n$.

The proof is based on the use of forward and backward elementary equivalences, which we define as follows. A representation over a semiring $K$ of a sequence $s = (s_n)_{n \geq 0}$ is a triple $(\mathbf{i}, M, \mathbf{t})$, where $\mathbf{i}$ is a row vector over $K$, $\mathbf{t}$ is a column vector over $K$, and $M$ a matrix over $K$, with $s_n = \mathbf{i}M^n\mathbf{t}$ for any non-negative integer $n$. The representation is said to specify $s$. We say that a matrix $U$ such that $NU = UM, \mathbf{j}U = \mathbf{i}$, $\mathbf{x} = U\mathbf{t}$ defines a forward elementary equivalence from $(\mathbf{i}, M, \mathbf{t})$ to $(\mathbf{j}, N, \mathbf{x})$. It defines a backward elementary equivalence in the opposite direction. It is easy to verify that both representations specify the same sequence. This notion of forward elementary equivalence extends to representations of sequences the classical notion of multiset construction used in automata theory, and the notion of graph extension introduced in [Bassino et al. 2000]. This notion is also linked to the notion of intertwining between representations introduced in [Flouret 1999].

The classical computation of a reduced representation of an integer sequence is actually obtained by the composition of a forward elementary equivalence followed by a backward one (or the converse) with transfer matrices with integer coefficients (see [Berstel and Reutenauer 1988] on this notion).

An important step in the proof of the main result is a forward elementary equivalence obtained by extending to representations over $\mathbb{Z}$ a theorem from Lind [Lind and Marcus 1995] which states that for any Perron number, there is a primitive integer matrix whose spectral radius is this Perron number. By taking into account the row and column vectors, we prove that a representation over $\mathbb{N}$ can be obtained by only one forward elementary equivalence from any reduced representation over $\mathbb{Z}$ of the sequence (Theorem 6.1).

Our main result is a particular case of the following more general one. Let $k$ be a positive integer and let $s^{(1)}, s^{(2)}, \ldots, s^{(l)}$ be $l$ regular sequences whose $n$-terms add to $k^n$ for all $n \geq 0$. Then there is a deterministic automaton $\mathcal{A} = (Q, A, \delta, i, Q)$ on a $k$-letter alphabet $A$ with an initial state $i$, a transition function $\delta$ and a set of terminal states equal to set $Q$ of all states such that the following holds: There is a partition of the set of states $Q$ in $l$ sets $T_j$ such that for each $1 \leq j \leq l$, the automaton $(Q, A, \delta, i, T_j)$ recognizes a regular language on $k$ symbols whose generating sequence is exactly $s^{(j)}$. We prove this more general formulation (Theorem 7.5).

The paper is organized as follows. Section 2 contains the definitions of representations and the main result is stated in Section 3. In Section 4 we define the notion of a forward or backward elementary equivalence. Section 5 establishes some lemmas based on Perron theory which are used in Section 6 to show that, for any reduced representation of a non-negative Perron sequence, there is a forward elementary equivalence from this representation to an $\mathbb{N}$-representation. Section 7 presents the proof of the characterization of generating sequences of regular languages over $k$ symbols. The proof is constructive in the sense that the regular language over $k$ symbols can be built in an effective way, although with a high complexity. The construction process is composed of two forward elementary equivalences followed by one backward elementary equivalence. We give an example of this computation.

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## 2. RATIONAL AND REGULAR SEQUENCES

Let $K$ be a semiring. In most cases, we have in mind $K = \mathbb{Z}$ or $\mathbb{N}$. In the most general case, $K$ is not even supposed to be commutative. However, we shall often make the hypothesis that $K$ is a principal ideal domain (this is the same as a commutative principal ideal ring without zero divisors and holds in particular when $K = \mathbb{Z}$).

We consider sequences of elements of $K$ denoted by $s = (s_n)_{n\geq 0}$. We shall not distinguish between such a sequence and the formal series in one variable $s(z) = \sum_{n\geq 0} s_n z^n$. We usually denote a vector with coefficients in $K$ and indexed by elements of a set $Q$, also called a $Q$-vector, with boldface symbols. A $Q \times Q$ matrix on $K$ is a family $M_{pq}$ of elements of $K$ indexed by $Q \times Q$.

A sequence $s$ is said to be *$K$-rational* if there exist a set $Q$ of cardinality $d$ and a triple $(\mathbf{i}, M, \mathbf{t})$, where $\mathbf{i}$ is a row $Q$-vector, $\mathbf{t}$ is a column $Q$-vector, and $M$ is a $Q \times Q$ matrix, all with coefficients in $K$, such that, for any non-negative integer $n$, $s_n = \mathbf{i}M^n\mathbf{t}$.

Such a triple is called a *representation* over $K$, or a $K$-representation of the sequence $s$, and $d$ is its *dimension.* We say that the representation $(\mathbf{i}, M, \mathbf{t})$ *specifies* the sequence $s$.

A word about our terminology. A sequence of elements of $K$ can be considered as a $K$-subset of $\Sigma^*$, where $\Sigma$ has only one symbol. Our definition of a $K$-rational sequence corresponds to what is called a recognizable $K$-subset in Eilenberg's book [Eilenberg 1974]. A rational $K$-subset is defined using rational expressions with multiplicities, and a classical result proves the equivalence of the notions of recognizable or rational $K$-subsets when $\Sigma$ is finite (this is the Kleene-Schützenberger theorem, see [Eilenberg 1974, p. 175]). We shall occasionally use rational expressions to denote rational sequences. For example, $(kz)^*$ is the same as $\frac{1}{1-kz}$.

A representation over $K$ is *reduced* if it has a minimal dimension among all representations over $K$ that specify the same sequence. If $K$ is a principal ideal domain, this minimal dimension is the same over $K$ and over the quotient field of $K$ [Berstel and Reutenauer 1988, p. 77]. This minimal dimension is called the *rank* of the rational sequence.

If $K$ is a principal ideal domain, a representation over $K$ is said to be *left reduced* (respectively *right reduced*) if and only the module generated by the vectors $\mathbf{i}M^n$ (respectively $M^n\mathbf{t}$), for all $n \geq 0$, is the full space $K^{1\times d}$ (respectively $K^{d\times 1}$). The representation is then reduced if and only if it is both left and right reduced (see [Berstel and Reutenauer 1988, p. 26]). We define the *left minimal* representation over $K$ of a sequence $s$ as the unique reduced representation $(\mathbf{i}, M, \mathbf{t})$ of $s$ over $K$, where $\mathbf{i} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$ and $M$ is a companion matrix, i.e., of the form

$$M = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{r-1} \end{bmatrix}.$$

We say that a sequence of integers is *non-negative* if all its terms are non-negative. An $\mathbb{N}$-rational sequence is also called *regular*. In the case of a regular sequence, there is an equivalent form of a representation. Let us consider a triple $(I, G, T)$, where $G$ is a directed multigraph and $I, T$ two sets of vertices. Such a triple *specifies* the sequence $s$ where $s_n$ is the number of paths of length $n$ going from $I$ to $T$. The sequence $s$ is regular since it is also specified by the representation $(\mathbf{i}, M, \mathbf{t})$, where $M$ is the adjacency matrix of $G$ and $\mathbf{i}$, $\mathbf{t}$ are the characteristic vectors of the sets $I$ and $T$ respectively. It can be shown conversely that any regular sequence can be specified by such a triple.

A matrix or vector with coefficients equal to zero or one is called a 0-1 *matrix* or a 0-1 *vector*. Let $k$ be a positive integer. A *$k$-ary matrix* is a matrix with non-negative integral coefficients such that the sum of each row is $k$. In a similar way, a graph $G$ is called $k$-ary if its adjacency matrix is $k$-ary. This means that each vertex of $G$ has out-degree $k$.

An $\mathbb{N}$-representation $(\mathbf{i}, M, \mathbf{t})$ with $M$ a $Q \times Q$ matrix, is said to be *trim* if for each index $p \in Q$ there is a non-negative integer $n$ such that $(\mathbf{i}M^n)_p > 0$ and there is a non-negative integer $m$ such that $(M^m\mathbf{t})_p > 0$.

A sequence $s = (s_n)_{n\geq 0}$ is said to be the *merge* of the sequences $s^{(0)}, \ldots, s^{(p-1)}$, where $p$ is a positive integer, if $s_n^{(i)} = s_{i+np}$ for $0 \leq i \leq p-1$. Equivalently, $s(z) = \sum_{i=0}^{p-1} z^i s^{(i)}(z^p)$. If $(\mathbf{i}, M, \mathbf{t})$ is an $\mathbb{N}$-representation of $s$, then $(\mathbf{i}M^i, M^p, \mathbf{t})$ is an $\mathbb{N}$-representation of $s^{(i)}$ for each integer $0 \leq i \leq p-1$.

A $\mathbb{Z}$-rational sequence is said to have a *dominating pole* if it can be written as a rational fraction $s(z) = p(z)/q(z)$, with $p, q$ relatively prime, where $q$ has a simple root $r$ such that $r' > r$ for any other root $r'$.

The following theorem is known as Soittola's theorem. We state it without proof (see [Berstel and Reutenauer 1988, p. 90] or [Salomaa and Soittola 1978, p. 74]).

THEOREM 2.1. *A $\mathbb{Z}$-rational sequence with non-negative terms is regular if and only it is the merge of $\mathbb{Z}$-rational sequences with a dominating pole.*

As a consequence of Soittola's theorem, given a triple $(\mathbf{i}, M, \mathbf{t})$, it is decidable whether the specified $\mathbb{Z}$-rational sequence is regular. If $s$ is a regular sequence, there is a computable positive integer $p$ (the period) such that $s_{j+np} \sim c_j n^{l_j} \alpha_j^n$ as $n \to \infty$ $(j = 0, \ldots, p-1)$, where $c_j > 0$, $l_j \in \mathbb{N}$ and $\alpha_j$ is a non-negative real (see for instance [Salomaa and Soittola 1978, p. 62]). Furthermore, $\alpha_j$ and $l_j$ are computable.

## 3. GENERATING SEQUENCE OF A REGULAR LANGUAGE ON $K$ SYMBOLS

In this section, we state the main result of this paper, which is a characterization of the generating sequences of regular languages on $k$ symbols.

Let $A$ be a $k$-letter alphabet and $L$ be a language over $A$, that is, a subset of $A^*$, where $A^*$ is the set of all finite words whose letters are in $A$. The *generating sequence* of $L$ is defined as the sequence $s = (s_n)_{n\geq 0}$, where $s_n$ is the number of words of $L$ of length $n$.

The generating sequence of a formal language $L$ gives useful information on $L$. For example, assuming that the letters are chosen at random uniformly and independently, the probability that a word of length $n$ is in $L$ is equal to $\frac{s_n}{2^n}$. The sequence $s_n$ is also used to define the notion of *entropy* of $L$ as the superior limit of the sequence $\frac{1}{n}\log s_n$ (see [Lind and Marcus 1995] or [Kuich 1970]).

It is known that the generating sequence of a regular language is a regular sequence. It is also clear that the generating sequence of a regular language over a $k$-letter alphabet satisfies the following two conditions of being the generating sequence of

—a language over a $k$-letter alphabet,

—a regular language.

The first condition is equivalent to the fact that the generating sequence $s$ satisfies $s_n \leq k^n$, for any $n \geq 0$. The second condition is equivalent to the fact that the sequence is regular. A natural question is the sufficiency of the two conditions to ensure that $s$ is the generating sequence of a regular language over a $k$-letter alphabet. This question is similar to one solved in [Bassino et al. 2000] (see also [Bassino et al. 2001] and [Bassino et al. 1999]), where it is shown that a sequence is the generating sequence of a regular $k$-ary tree if and only if it is the generating sequence of $k$-ary tree and if it is regular.

The situation is quite different here since we give below an example of a regular sequence $s$ that is not the generating sequence of a regular language over a $k$-letter alphabet, although $s_n \leq k^n$ for any $n \geq 0$. The counterexample is based on an example of a $\mathbb{Z}$-rational sequence with non-negative terms that is not regular (see [Eilenberg 1974, p. 216-218] or [Berstel and Reutenauer 1988, p. 95]).

*Example* 3.1. Let $r$ be the sequence such that, for any $n \geq 0$, $r_n = b^{2n}\cos^2(n\theta)$, with $\cos\theta = \frac{a}{b}$, where the integers $a, b$ are such that $b \neq 2a$ and $0 < a < b$. We also assume that $b^2 < k$. The sequence $r$ is $\mathbb{Z}$-rational, has non-negative integer terms and is not regular [Eilenberg 1974, p. 216-218]. Note that, for any $n \geq 0$, $r_n \leq k^n$. We now define the sequence $s$ by $s_n = k^n - r_n$. By Soittola's theorem, the sequence $s$ is regular since it is a merge of rational sequences having a dominating pole, and it satisfies $s_n \leq k^n$ for any $n \geq 0$. If $s$ were the generating sequence of a regular language $L$ over a $k$-letter alphabet $A$, its complementary sequence $r$ would be the generating sequence of the complement of $L$. Thus $r$ would be regular, a contradiction.

Example 3.1 leads us to state the following result which completely characterizes the sequences that are generating sequences of languages over a $k$-letter alphabet. It is proved in Section 7.

THEOREM 3.2. *A sequence $s$ is the generating sequence of a regular language over a $k$-letter alphabet if and only if both sequences $s = (s_n)_{n\geq 0}$ and $t = (k^n - s_n)_{n\geq 0}$ are regular.*

Observe first that the second condition implies that $s_n \leq k^n$ for all $n$ since by definition a regular sequence has non-negative terms. If $s$ is a given $\mathbb{Z}$-rational sequence and $k$ a positive integer, the two conditions are decidable as seen above. Moreover if $s$ is regular, one can compute the least integer $k_0$ such that $s_n \leq k_0^n$, for any integer $n \geq 0$. For $k > k_0$, the second condition is automatically satisfied again by Soittola's theorem. It follows that, given some regular sequence, one can characterize the minimal alphabet such that $s$ is the generating sequence of a regular language on this alphabet.

## 4. EQUIVALENCE OF REPRESENTATIONS

In this section, we define a transformation on a representation of a sequence over a semiring that extends the notion of multiset extension introduced in [Bassino et al. 2000] to representations.

Let $(\mathbf{i}, M, \mathbf{t})$ and $(\mathbf{j}, N, \mathbf{x})$ be two representations, and $U$ be a matrix such that

$$\begin{aligned} NU &= UM, \\ \mathbf{j}U &= \mathbf{i}, \\ \mathbf{x} &= U\mathbf{t}. \end{aligned}$$

The transformation from $(\mathbf{i}, M, \mathbf{t})$ to $(\mathbf{j}, N, \mathbf{x})$ is called a *forward elementary equivalence.* The matrix $U$ is called the *transfer matrix* of the elementary equivalence, denoted $(\mathbf{i}, M, \mathbf{t}) \xrightarrow{U} (\mathbf{j}, N, \mathbf{x})$, or $(\mathbf{i}, M, \mathbf{t}) \xrightarrow[K]{U} (\mathbf{j}, N, \mathbf{x})$ to specify that $U$ has its coefficients in $K$. In this case, we also talk of a $K$-forward elementary equivalence. Note that $M$ or $N$ may have coefficients outside $K$.

Notice that, if we identify an element of $S$ to the row $Q$-vector of $U$ of the corresponding index, the equality $NU = UM$ is equivalent to the fact that, for any element $\mathbf{u}$ of $S$,

$$\mathbf{u}M = \sum_{\mathbf{v} \in S} N_{\mathbf{u},\mathbf{v}} \mathbf{v}.$$

The inverse transformation is called a *backward elementary equivalence*, denoted $(\mathbf{i}, M, \mathbf{t}) \xleftarrow{U} (\mathbf{j}, N, \mathbf{x})$. A forward or backward elementary equivalence is called an *elementary equivalence.* The symmetric and transitive closure of the relation of forward elementary equivalence with transfer matrices with coefficients in $K$, is called the *equivalence* over $K$, denoted by $\sim_K$.

Our definition of an elementary equivalence is connected with classical notions on matrices. Indeed, the definition of a forward elementary equivalence uses a relation between the matrices $M$, $N$ which generalizes the conjugacy relation. The general solution of the matrix equation $NX = XM$ is given in [Gantmacher 1977, p. 219]. A nonzero solution exists if and only if $M$ and $N$ have a common characteristic eigenvalue. It is also known [Lind and Marcus 1995, p. 285] that, when $M, N$ are non-negative real matrices with the same dominant eigenvalue, the equation $NX = XM$ has a non-negative and nonzero solution.

A simple example of forward (or backward) elementary equivalence is similarity. Two $K$-representations $(\mathbf{i}, M, \mathbf{t})$ and $(\mathbf{i}', M', \mathbf{t}')$ are said to be *similar* over $K$ if and only if there is a matrix $P$, invertible in $K$, such that $(\mathbf{i}, M, \mathbf{t}) \xrightarrow{P} (\mathbf{i}', M', \mathbf{t}')$.

Another example of a backward elementary equivalence is the out-splitting that comes from symbolic dynamics [Lind and Marcus 1995, p. 55]. Let $(\mathbf{j}, N, \mathbf{x})$ be an $\mathbb{N}$-representation. The matrix $N$ is the adjacency matrix of a graph $G$ on a set $S$ of vertices. Let us consider the graph $H$ on a set $Q = (S - \{i\}) \cup \{i', i''\}$ of vertices obtained from $G$ by splitting the vertex $i$ of $G$ into two vertices $i'$ and $i''$ according to a partition in two parts $P_1, P_2$ of edges going out of $i$. The edges coming in $i$ are duplicated in $H$ into edges coming in $i'$ and $i''$. Let $M$ be the adjacency matrix of $H$. Let $D$ be the $S \times Q$ matrix defined by $D_{pp} = 1$ if $p \neq i$, $D_{ii'} = D_{ii''} = 1$, and $D_{pq} = 0$ otherwise. Let $E$ be the $Q \times S$ matrix defined by $E_{pq} = M_{pq}$ if $p \neq i', i''$ and $E_{i'q}$ (respectively $E_{i''q}$) is the number of edges in $P_1$ (respectively $P_2$) going from $i$ to $q$. It can be easily checked (see [Lind and Marcus 1995, p. 55]) that

$$ED = M \text{ and } DE = N.$$

Then

$$DM = ND \text{ and } EN = ME.$$

The matrix $M$ is said to be obtained by an *out-splitting* of $N$. For any non-negative integral vector $\mathbf{x}$, there is an a non-negative integral vector $\mathbf{t}$ such that $\mathbf{x} = D\mathbf{t}$. By setting $\mathbf{i} = \mathbf{j}D$, we get $(\mathbf{i}, M, \mathbf{t}) \xrightarrow{D} (\mathbf{j}, N, \mathbf{x})$. This can be stated as follows.

PROPOSITION 4.1. *For any $\mathbb{N}$-representation $(\mathbf{j}, N, \mathbf{x})$ and any matrix $M$ that is obtained by out-splitting of $N$, there are non-negative integral vectors $\mathbf{i}, \mathbf{t}$ and a transfer matrix $D$ such that $(\mathbf{i}, M, \mathbf{t}) \xrightarrow[\mathbb{N}]{D} (\mathbf{j}, N, \mathbf{x})$.*

Similar results can be obtained for input state splitting. The notion of forward or backward elementary equivalence is nevertheless much weaker than the symbolic dynamics notion of conjugacy or even the notion of shift equivalence (see [Lind and Marcus 1995], [Kitchens 1997] for these notions).

The following two propositions are direct consequences of the definitions.

PROPOSITION 4.2. *Equivalent representations specify the same sequence.*

PROOF. If $(\mathbf{i}, M, \mathbf{t}) \xrightarrow{U} (\mathbf{j}, N, \mathbf{x})$, then $\mathbf{i}M^n\mathbf{t} = \mathbf{j}UM^n\mathbf{t} = \mathbf{j}N^nU\mathbf{t} = \mathbf{j}N^n\mathbf{x}$, for any non-negative integer $n$. □

PROPOSITION 4.3. *The composition of two forward (respectively backward) elementary equivalences is a forward (respectively backward) elementary equivalence. If $(\mathbf{i}, M, \mathbf{t}) \xrightarrow{U} (\mathbf{j}, N, \mathbf{x})$ and $(\mathbf{j}, N, \mathbf{x}) \xrightarrow{V} (\mathbf{j}', N', \mathbf{x}')$, then $(\mathbf{i}, M, \mathbf{t}) \xrightarrow{VU} (\mathbf{j}', N', \mathbf{x}')$.*

PROOF. The proof is straightforward. □

Checking whether two representations over a field $K$ are elementary equivalent is decidable, as shown in the following proposition.

PROPOSITION 4.4. *Let $K$ be a field. Given two $K$-representations, $(\mathbf{i}, M, \mathbf{t})$ and $(\mathbf{j}, N, \mathbf{x})$, it is decidable whether there is a $K$-forward elementary equivalence from $(\mathbf{i}, M, \mathbf{t})$ to $(\mathbf{j}, N, \mathbf{x})$.*

PROOF. If $(\mathbf{i}, M, \mathbf{t})$ has dimension $d$ and $(\mathbf{j}, N, \mathbf{x})$ dimension $d'$, the existence of a matrix $U$ such that $NU = UM$, $\mathbf{j}U = \mathbf{i}$ and $\mathbf{x} = U\mathbf{t}$, is obtained by solving a Cramer system of $dd' + d + d'$ equations with $dd'$ unknowns. This can be performed in cubic time. □

The converse of Proposition 4.2 is due to Schützenberger. His result states that if $K$ is a principal ideal domain, any $K$-rational sequence has a reduced representation that can be computed in two steps (see for instance [Berstel and Reutenauer 1988], [Salomaa and Soittola 1978] or [Sakarovitch 2003]). These two steps are respectively a forward elementary equivalence and a backward elementary equivalence (or conversely). This leads to the following statement in which $K$ is supposed to be a principal ideal domain.

PROPOSITION 4.5. *Let $(\mathbf{i}, M, \mathbf{t})$ be a representation over $K$ of a sequence, and $(\mathbf{j}, N, \mathbf{x})$ its left minimal representation over $K$. There is a forward elementary equivalence followed by a backward elementary equivalence from $(\mathbf{i}, M, \mathbf{t})$ to $(\mathbf{j}, N, \mathbf{x})$. As a consequence, two $K$-representations specify the same sequence if and only if they are equivalent over $K$.*

We briefly recall the construction of Schützenberger. Notice that $K$ is not necessarily a field.

PROOF. We already know that two $K$-representations that are equivalent over $K$ specify the same sequence.

Conversely, let $(\mathbf{i}, M, \mathbf{t})$ be a $K$-representation of dimension $d$. Let $F$ be the quotient field of $K$. We first show that $(\mathbf{i}, M, \mathbf{t})$ is equivalent over $K$ to a $K$-representation which is reduced over $K$ and over $F$. For any non-negative integer $n$, $\mathbf{i}M^n \in K^{1\times d}$. Thus the $K$-module $E$ generated by the vectors $\mathbf{i}M^n$ for $n \geq 0$, is a submodule of the free $K$-module $K^{1\times d}$. It is thus a free $K$-module. Let $d'$ be its dimension as $K$-module and let $e_1, \ldots, e_{d'}$ be one of its basis. Each $e_i$ is a linear combination over $K$ of the vectors $\mathbf{i}M^n$ for $n \geq 0$. Let $U$ be the $d' \times d$ matrix over $K$ whose rows are the vectors $e_i$, $1 \leq i \leq d'$. The $K$-module $E$ is stable by multiplication on the right by the matrix $M$. Let $N$ be $d' \times d$ matrix over $K$ that represents the action of $M$ in the basis $e_1, \ldots, e_{d'}$, that is, if $e_iM = a_1e_1 + \ldots a_{d'}e_{d'}$ for some elements $a_1, \ldots, a_{d'} \in K$, one defines the row of index $i$ of $N$ to be $[a_1, \ldots, a_{d'}]$. It is a consequence of the definition that $UM = NU$. Since $\mathbf{i}$ belongs to the $K$-module $E$, the vector $\mathbf{i}$ is a $K$-linear combination of the $e_i$. Thus there exists a vector $\mathbf{j}$, with coefficients in $K$, such that $\mathbf{i} = \mathbf{j}U$. We also set $\mathbf{x} = U\mathbf{t}$. Note that the $K$-module generated by the vectors $\mathbf{j}N^n$ for $n \geq 0$ has the same dimension $d'$ as the $K$-module $E$.

Symmetrically, let $r \leq d'$ be the dimension of the $K$-module generated by the vectors $N^n\mathbf{x}$ for $n \geq 0$. By considering the *transpose* $(\tilde{\mathbf{x}}, \tilde{N}, \tilde{\mathbf{t}})$ of the triple $(\mathbf{j}, N, \mathbf{x})$, where $\tilde{N}$ denotes the transpose of the matrix $N$, there is a $K$-representation of dimension $r$, $(\mathbf{k}, P, \mathbf{y})$, and a transfer matrix $V$ over $K$ such that $(\mathbf{k}, P, \mathbf{y}) \xrightarrow[K]{V} (\mathbf{j}, N, \mathbf{x})$. Since $(\mathbf{i}, M, \mathbf{t}) \xrightarrow[K]{U} (\mathbf{j}, N, \mathbf{x})$, we obtain that the representations $(\mathbf{i}, M, \mathbf{t})$ and $(\mathbf{k}, P, \mathbf{y})$ are equivalent over $K$.

Let us denote by $V(\mathbf{j}, N)$ the vector space over $F$ generated by the vectors $\mathbf{j}N^n$ for $n \geq 0$. Since $V(\mathbf{k}, P) = V(\mathbf{j}, N)$, and since $V(\mathbf{j}, N) = K^{1\times d'}$, the dimension of $V(\mathbf{j}, N) = V(\mathbf{k}, P)$ is the rank $r$ of $V$. Thus $(\mathbf{k}, P, \mathbf{y})$ is reduced over $F$ and thus also over $K$. It has been obtained from $(\mathbf{i}, M, \mathbf{t})$ with one forward elementary equivalence followed by one backward elementary equivalence.

A similar proof shows that there is a backward elementary equivalence followed by a forward elementary equivalence from $(\mathbf{i}, M, \mathbf{t})$ to a representation reduced over $F$.

We now show that if $(\mathbf{i}, M, \mathbf{t})$ is a $K$-representation of dimension $r$ of $s$, there is a forward elementary equivalence from $(\mathbf{i}, M, \mathbf{t})$ to the minimal left representation of $s$ which completes the proof since the composition of two forward elementary equivalences is a forward elementary equivalence.

Thus $V(\mathbf{i}, M)$ is a vector space of dimension $r$ over $F$. Then $(\mathbf{i}, \mathbf{i}M, \ldots, \mathbf{i}M^{r-1})$ is a basis of this space over $F$ and there are $a_0, a_1, \ldots a_{r-1} \in F$ such that $\mathbf{i}M^r = a_0\mathbf{i} + a_1\mathbf{i}M + \cdots + a_{r-1}\mathbf{i}M^{r-1}$. Let $U$ be the $r \times r$ matrix over $K$, invertible in $F$, defined by

$$U = \begin{bmatrix} \mathbf{i} \\ \mathbf{i}M \\ \vdots \\ \mathbf{i}M^{r-1} \end{bmatrix}.$$

Let $N$ be the matrix of dimension $r$ with coefficients in $F$ which represents the right multiplication by $M$ in the basis $(\mathbf{i}, \ldots, \mathbf{i}M^{r-1})$. We get $(\mathbf{i}, M, \mathbf{t}) \xrightarrow[K]{U} (\mathbf{j}, N, \mathbf{x})$ with

$$\mathbf{j} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ a_0 & a_1 & a_2 & \ldots & a_{r-1} \end{bmatrix}, \mathbf{x} = \mathbf{t}U.$$

Since $(\mathbf{j}, N, \mathbf{x})$ specifies $s$, we have

$$\mathbf{x} = \begin{bmatrix} s_0 \\ \vdots \\ s_{r-1} \end{bmatrix}.$$

Since $U$ is invertible in $F$, the characteristic polynomials of $M$ and $N$ are equal and the characteristic polynomial of $M$ has its coefficients in $K$. Since this polynomial is $X^r - a_{r-1}X^{r-1} - \cdots - a_1X - a_0$, the matrix $N$ has its coefficients in $K$. $\square$

It is also known (Schützenberger 1961, Fliess 1974) [Berstel and Reutenauer 1988, p. 27] that, if $K$ is a field, all reduced representations are similar over $K$. Note that the result is not true if $K$ is not a field. Consider for instance the two $\mathbb{Z}$-representations of dimension one: $(\mathbf{i} = [2], M = [1], \mathbf{t} = [3])$ and $(\mathbf{j} = [3], N = [1], \mathbf{x} = [2])$. They are similar over $\mathbb{Q}$ but not over $\mathbb{Z}$. It is known that checking whether two $K$-representations specify the same sequence is decidable in polynomial time (see for instance [Berstel and Reutenauer 1988]).

## 5. PERRON GEOMETRY

In this section, we consider $\mathbb{Z}$-rational sequences and regular sequences. We prove a series of lemmas used in the next section. The proofs rely on the Perron-Frobenius theory of non-negative matrices (see [Lind and Marcus 1995] for an introduction or [MacCluer 2000] for a recent survey).

If $\mathbf{v} = (v_q)_{q \in Q}$ is a vector with coefficients in $\mathbb{R}$, we say that $\mathbf{v}$ is non-negative, denoted $\mathbf{v} \geq 0$, (respectively positive, denoted $\mathbf{v} > 0$) if $v_q \geq 0$ (respectively $v_q > 0$) for all $q \in Q$. The same conventions are used for matrices.

An integer matrix has a *dominating eigenvalue*, i.e., has an eigenvalue $\lambda > 0$ such that $\lambda > |\mu|$ for all other eigenvalues $\mu$ of $M$. An integer matrix $M$ is said to be *spectrally Perron* if it has a dominating eigenvalue which is simple[1].

A sequence of integers $s$ is said to be *spectrally Perron* if it has a reduced representation over $\mathbb{Z}$ with a spectrally Perron matrix. A representation over $\mathbb{Z}$ with a spectrally Perron matrix is called a *spectrally Perron representation.* The left minimal representation, and more generally all reduced representations, of a spectrally Perron sequence are spectrally Perron representations. The spectral radius of the matrix of a reduced spectrally Perron representation is called the *Perron value* of the sequence specified.

Let $(\mathbf{i}, M, \mathbf{t})$ be a spectrally Perron representation. The matrix $M$ is a spectrally Perron $Q \times Q$ matrix whose spectral radius is $\lambda$, where $Q$ is the finite set of states of the representation. We denote by $d$ the dimension of $M$. The matrix $M$ has a nonzero left eigenvector $\mathbf{w}$ associated to the eigenvalue $\lambda$. All other eigenvectors associated to $\lambda$ are colinear to it.

Let $W$ be the span of $\mathbf{w}$ over $\mathbb{R}$. According to the Jordan canonical form of $M$, there is a complementary $M$-invariant subspace $V$ corresponding to eigenvalues $|\mu| < \lambda$. The space $\mathbb{R}^{1\times d}$ is a direct sum of $W$ and $V$. We denote by $\pi_1 : \mathbb{R}^{1\times d} \to W$ the projection to $W$ along $V$ and by $\pi_2 : \mathbb{R}^{1\times d} \to V$ the projection to $V$ along $W$. We also denote by $\alpha_1 : \mathbb{R}^{1\times d} \to \mathbb{R}$ the function associating to each vector $\mathbf{u}$ the real number $\alpha_1(\mathbf{u})$ such that $\pi_1(\mathbf{u}) = \alpha_1(\mathbf{u})\mathbf{w}$. The real number $\alpha_1(\mathbf{u})$ is called the *dominant coordinate* of $\mathbf{u}$.

Thus each vector $\mathbf{u}$ of $\mathbb{R}^{1\times d}$ can be written

$$\mathbf{u} = \alpha_1(\mathbf{u})\mathbf{w} + \pi_2(\mathbf{u}).$$

Note that

$$\mathbf{u}M = \lambda\alpha_1(\mathbf{u})\mathbf{w} + \pi_2(\mathbf{u})M.$$

Hence, since $V$ is invariant by $M$, $\alpha_1(\mathbf{u}M) = \lambda\alpha_1(\mathbf{u})$ and $\pi_2(\mathbf{u})M = \pi_2(\mathbf{u}M)$.

When $\mathbf{i}$ has a nonzero dominant coordinate, it will be convenient to choose a *left Perron eigenvector* $\mathbf{w}$ such that $\alpha_1(\mathbf{i}) > 0$. This is done by changing $\mathbf{w}$ to $-\mathbf{w}$ if $\alpha_1(\mathbf{i}) < 0$. Note that $\mathbf{w}$ depends only on $M$ and $\mathbf{i}$. When the representation is left-reduced, the vector $\mathbf{i}$ has a positive dominant coordinate.

For any real number $r$, we denote by $B(\mathbf{v}, r)$ the ball of radius $r$ centered on the point $\mathbf{v}$, which is the set of vectors $\mathbf{u}$ such that $\|\mathbf{v} - \mathbf{u}\| \le r$ where $\| \ \|$ is any equivalent norm of $\mathbb{R}^{1\times d}$. It will be convenient (in order to prove Lemma 5.4 below for instance) to use a norm that satisfies, for any vector $\mathbf{u}$,

$$\|\mathbf{u}\| = \|\pi_1(\mathbf{u})\| + \|\pi_2(\mathbf{u})\|.$$

Let $\mathbf{w}$ be a left Perron eigenvector of $(\mathbf{i}, M, \mathbf{t})$. We denote by $K_r(\mathbf{w})$ the set

$$K_r(\mathbf{w}) = \{\rho\mathbf{v} \mid \mathbf{v} \in B(\mathbf{w}, r), \rho \ge 0\}.$$

We also denote by $K_r^+(\mathbf{w})$ the nonzero vectors of $K_r(\mathbf{w})$.

The following lemma is from [Lind and Marcus 1995, p. 373].

[1]The definition taken from [Lind and Marcus 1995, p. 371] (see also [Lind and Marcus 1995, p. 369]) uses $\lambda \ge 1$ instead of $\lambda > 0$.

LEMMA 5.1. *Let $(\mathbf{i}, M, \mathbf{t})$ be a spectrally Perron representation. Let $\varepsilon$ be a positive real number and let $\mathbf{u}$ be an integer vector with a positive dominant coordinate. Then there is a positive integer $m$ such that $\mathbf{u}M^n$ belongs to $K_\varepsilon(\mathbf{w})$ for $n \geq m$.*

PROOF. We follow the lines of [Lind and Marcus 1995, p. 373] for the proof. We have

$$\mathbf{u} = \alpha_1(\mathbf{u})\mathbf{w} + \pi_2(\mathbf{u}).$$

Thus

$$\mathbf{u}M^n = \lambda^n \alpha_1(\mathbf{u})\mathbf{w} + \pi_2(\mathbf{u})M^n.$$

It follows from the Jordan canonical form that the growth rate of $M$ on $V$ is strictly less than $\lambda$, i. e. , for $\mathbf{v} \in V$, $\|\mathbf{v}M^n\|/\lambda^n \to 0$ as $n \to \infty$. Then, for a large enough $n$,

$$\|\pi_2(\mathbf{u})M^n\| < \lambda^n \alpha_1(\mathbf{u})\varepsilon.$$

Hence, for a large enough $n$, $\mathbf{u}M^n/\lambda^n\alpha_1(\mathbf{u})$ belongs to $K_\varepsilon(\mathbf{w})$ and thus $\mathbf{u}M^n$ also. □

Let $s$ be a $\mathbb{Z}$-ratio nal sequence of non-negative integers. The *complexity* of the sequence $s$ is defined as the inverse of its convergence radius, i.e., $\limsup_{n\to\infty} s_n^{1/n}$.

LEMMA 5.2. *Let $(\mathbf{i}, M, \mathbf{t})$ be a spectrally Perron representation with a spectral radius $\lambda$ such that the sequence specified is non-negative and has complexity $\lambda$. Then $\alpha_1(\mathbf{i}) > 0$ and $\mathbf{w} \cdot \mathbf{t} > 0$.*

PROOF. Since

$$\mathbf{i} = \alpha_1(\mathbf{i})\mathbf{w} + \pi_2(\mathbf{i}),$$

we have

$$\mathbf{i}M^n\mathbf{t} = \lambda^n\alpha_1(\mathbf{i})\mathbf{w} \cdot \mathbf{t} + \pi_2(\mathbf{i})M^n\mathbf{t},$$

with a growth rate of $M$ on $V$ strictly less than $\lambda$. If $s$ has complexity $\lambda$, $\alpha_1(\mathbf{i}) \neq 0$ and $\mathbf{w}\cdot\mathbf{t} \neq 0$. Moreover, since the sequence specified is non-negative, $\alpha_1(\mathbf{i}).(\mathbf{w}\cdot\mathbf{t}) \geq 0$. Under the hypothesis on the choice of $\mathbf{w}$, we get $\alpha_1(\mathbf{i}) > 0$. Thus $\mathbf{w} \cdot \mathbf{t} > 0$. □

LEMMA 5.3. *Let $(\mathbf{i}, M, \mathbf{t})$ be a spectrally Perron representation with a spectral radius $\lambda$ such that the sequence specified is non-negative and has complexity $\lambda$. Then there exists a positive real number $\eta$ such that for any vector $\mathbf{u} \in K_\eta^+(\mathbf{w})$, we have $\mathbf{u} \cdot \mathbf{t} > 0$.*

PROOF. This follows directly from Lemma 5.2. □

LEMMA 5.4. *Let $(\mathbf{i}, M, \mathbf{t})$ be a spectrally Perron representation. For any positive real number $\eta$, there exists a positive real number $\varepsilon$ such that, for any positive integer $n$, $\mathbf{u} \in K_\varepsilon(\mathbf{w})$, then $\mathbf{u}M^n \in K_\eta(\mathbf{w})$.*

PROOF. Let $\mathbf{u} \in K_\varepsilon(\mathbf{w})$. Thus $\mathbf{u} = \rho(\mathbf{w} + \mathbf{z})$, where $\mathbf{z} \in B(\mathbf{0}, \varepsilon)$ and $\rho$ is a positive real number. One has $\mathbf{z} = \alpha_1(\mathbf{z})\mathbf{w} + \mathbf{v}$, where $\mathbf{v} \in V$.

Then for any non-negative integer $n$

$$\frac{\mathbf{u}M^n}{\lambda^n} = \rho(\mathbf{w} + \alpha_1(\mathbf{z})\mathbf{w} + \frac{\mathbf{v}M^n}{\lambda^n}).$$

Since $\|\mathbf{z}\| = \|\alpha_1(\mathbf{z})\mathbf{w}\| + \|\mathbf{v}\|$, $\|\alpha_1(\mathbf{z})\mathbf{w}\| \le \varepsilon$ and $\|\mathbf{v}\| \le \varepsilon$.

It follows from the Jordan canonical form that the growth rate of $M$ on $V$ is strictly less than $\lambda$. Thus there is an integer $m$ such that for any $n > m$, $\|M^n\|/\lambda^n \le 1$ on the space $V$. Let $N$ be the maximum of $\|M^n\|/\lambda^n$ for all $0 \le n \le m$.

We now choose $\varepsilon = \min(\frac{\eta}{2}, \frac{\eta}{2N})$. Then $\|\alpha_1(\mathbf{z})\mathbf{w} + \frac{\mathbf{v}M^n}{\lambda^n}\| < \eta$ for any non-negative integer $n$. It follows that, for any non-negative integer $n$, $\frac{\mathbf{u}M^n}{\lambda^n} \in K_\eta(\mathbf{w})$ and thus $\mathbf{u}M^n$ also. $\square$

*Remark* 5.5. We note for future use that, if moreover $\eta < 1$ and $\mathbf{u} \neq \mathbf{0}$, then, for any non-negative integer $n$, $\mathbf{u}M^n \neq \mathbf{0}$.

We now state and prove a geometrical lemma which is used in the construction of Section 6. The lemma is essentially due to Lind (see [Lind and Marcus 1995, p. 374]), who proved that there is a positive real number $\varepsilon$ such that *all* integer vectors in $K_\varepsilon(\mathbf{w})$ are non-negative integral combinations of a *finite number* of integer vectors. With a slight modification, we show below that there is a positive real number $\varepsilon$ such that *all* integer vectors in $K_\varepsilon(\mathbf{w})$ are non-negative integral combinations of a *finite number* of integer vectors in $K_{2\varepsilon}(\mathbf{w})$.

LEMMA 5.6. *For a small enough positive real $\varepsilon$, there is a finite set $P$ of integer points in $K_{2\varepsilon}(\mathbf{w})$ such that each integer point of $K_\varepsilon(\mathbf{w})$ is a non-negative integral combination of points of $P$.*

PROOF. For technical reasons that will appear below, we choose $\varepsilon < 1/2$.

We choose a left eigenvector $\mathbf{w}$ with norm 1. If $\mathbf{q}$ is a point of $B(\mathbf{w}, \varepsilon)$, $B(\mathbf{q}, \varepsilon) \subset B(\mathbf{w}, 2\varepsilon) \subset K_{2\varepsilon}(\mathbf{w})$. As a consequence, for any positive real number $R$, any ball $B(R\mathbf{q}, R\varepsilon)$ is contained in $K_{2\varepsilon}(\mathbf{w})$. Let $D$ be the minimal value such that any ball of size $D$ contains at least one integer point. This value depends on the norm $\|\|$ chosen. We fix a large enough $R$ such that $r = R\varepsilon > 2D$. Note that $R > r$.

We define the finite set of integer points $P = K_{2\varepsilon}(\mathbf{w}) \cap B(\mathbf{0}, 3R) \cap \mathbb{N}^d$. We show that all integer points of $K_\varepsilon(\mathbf{w})$ are non-negative integral combinations of points of $P$.

Let us assume that this property is false, and denote by $\mathbf{x}$ an integer point of minimal norm which is in $K_\varepsilon(\mathbf{w})$ and which is not a non-negative integral combination of points of $P$. Then $\mathbf{x}$ does not belong to $P$, and its norm is greater than $2R$.

Let $\mathbf{p}'$ be a point of $B(\mathbf{w}, \varepsilon)$ which belongs to the semi-line defined by the point $\mathbf{x}$ and the null origin (see Figure 1). Such a point exists since $\mathbf{x}$ belongs to $K_\varepsilon(\mathbf{w})$. Note that $\mathbf{x} = l'\mathbf{p}'$, where $l'$ is a positive real number. Since $\mathbf{w}$ has norm 1, we have $1 - \varepsilon \le \|\mathbf{p}'\| \le 1 + \varepsilon$.

Let $\mathbf{p} = R\mathbf{p}'$, and $l = l'/R$. We have

$$(1 - \varepsilon)R \le \|\mathbf{p}\| \le (1 + \varepsilon)R < \|\mathbf{x}\|/2. \tag{1}$$

Let $\mathbf{m}_1 = \frac{R\mathbf{w}+\mathbf{p}}{2}$ and $\mathbf{m}_2 = 2\mathbf{p} - \mathbf{m}_1$. Thus the ball $B(\mathbf{m}_1, r/2)$ is included in $B(\mathbf{p}, r)$ and in $B(R\mathbf{w}, r)$. Thus $B(\mathbf{m}_1, r/2) \subset K_\varepsilon(\mathbf{w})$. It follows that

$$B(\mathbf{m}_2, r/2) \subset B(\mathbf{p}, r) \subset B(R\mathbf{w}, 2r) \subset K_{2\varepsilon}(\mathbf{w}). \tag{2}$$

A point $\mathbf{u}$ is in $B(\mathbf{m}_2, r/2)$ if and only if $2\mathbf{p} - \mathbf{u}$ is in $B(\mathbf{m}_1, r/2)$.

Since $r/2 \geq D$, there is an integer point $\mathbf{u}$ in $B(\mathbf{m}_2, r/2)$. Thus $2\mathbf{p} - \mathbf{u}$ belongs to $B(\mathbf{m}_1, r/2) \subset K_\varepsilon(\mathbf{w})$. We get $2\mathbf{p} - \mathbf{u} \in K_\varepsilon(\mathbf{w})$ and $\mathbf{u} \in K_{2\varepsilon}(\mathbf{w})$.

We have a succession of inequalities. First, $\|\mathbf{x} - \mathbf{u}\| \leq \|\mathbf{x} - \mathbf{p}\| + \|\mathbf{p} - \mathbf{u}\|$. Since $\mathbf{x}$ and $\mathbf{p}$ are colinear, we have $\|\mathbf{x} - \mathbf{p}\| = \|\mathbf{x}\| - \|\mathbf{p}\|$. Also $\|\mathbf{p} - \mathbf{u}\| \leq r$ since $\mathbf{u} \in B(\mathbf{m}_2, r/2)$ and $B(\mathbf{m}_2, r/2) \subset B(\mathbf{p}, r)$ by Equation (2). Thus

$$\|\mathbf{x} - \mathbf{u}\| \leq \|\mathbf{x}\| - \|\mathbf{p}\| + r.$$

This implies by Inequality (1)

$$\|\mathbf{x} - \mathbf{u}\| \leq \|\mathbf{x}\| + (\varepsilon - (1 - \varepsilon))R.$$

Since $\varepsilon < 1/2$, we obtain $\|\mathbf{x} - \mathbf{u}\| < \|\mathbf{x}\|$.

Since $\mathbf{x} = l\mathbf{p}$ with $l > 2$, $\mathbf{x} - \mathbf{u} = (l-2)\mathbf{p} + (2\mathbf{p} - \mathbf{u})$. Since $(l-2)\mathbf{p}$ and $2\mathbf{p} - \mathbf{u}$ belong to $K_\varepsilon(\mathbf{w})$, the point $\mathbf{x} - \mathbf{u}$ is also in $K_\varepsilon(\mathbf{w})$.

Thus $\mathbf{x} - \mathbf{u}$ is an integer point of $K_\varepsilon(\mathbf{w})$ which is strictly closer to the origin than $\mathbf{x}$. By hypothesis, $\mathbf{x} - \mathbf{u}$ is then a non-negative integral combination of points of $P$. From $\|\mathbf{u}\| \leq \|\mathbf{p}\| + \|\mathbf{p} - \mathbf{u}\|$, we get $\|\mathbf{u}\| \leq (1 + \varepsilon)R + \varepsilon R \leq 2R$. This shows that $\mathbf{u} \in P$. Then $\mathbf{x} = (\mathbf{x} - \mathbf{u}) + \mathbf{u}$ is a non-negative integral combination of points of $P$. This contradicts the hypothesis, concluding the proof of this lemma. $\square$

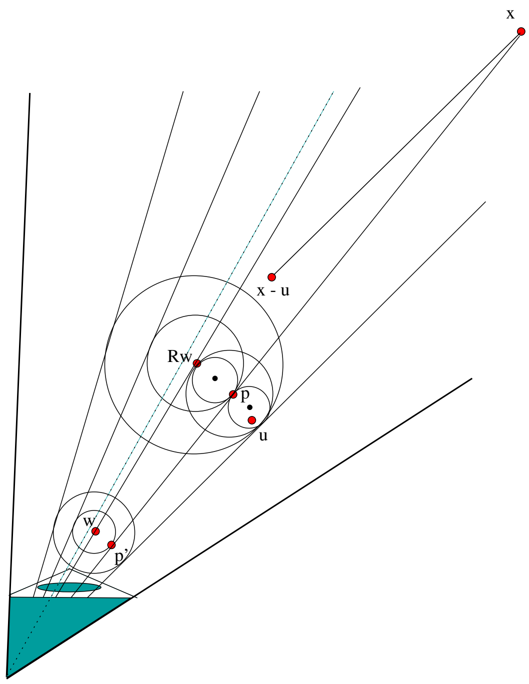


Fig. 1. The geometrical lemma (Lemma 5.6).

## 6. FROM A $\mathbb{Z}$-REPRESENTATION TO AN $\mathbb{N}$-REPRESENTATION

In this section, we prove a result which gives a sufficient condition for a sequence to be regular (Theorem 6.1).

It is known that a non-negative $\mathbb{Z}$-rational sequence that has a dominating pole is regular (Soittola 1976, Katayama et al. 1978, see [Berstel and Reutenauer 1988, p. 83] or also [Salomaa and Soittola 1978]). From this result and the results of Section 4 follows that any $\mathbb{Z}$-representation of a non-negative sequence that has a dominating pole is equivalent over $\mathbb{Z}$ to a regular representation. In the particular case of a spectrally Perron non-negative sequence, we show that an $\mathbb{N}$-representation can be obtained by only one forward elementary equivalence from any reduced $\mathbb{Z}$-representation of the sequence. This result is an adaptation to representations of a result from Lind ([Lind 1983], [Lind 1984], see also [Lind and Marcus 1995, Theorem 11.1.4 p. 369]) which says that for any Perron number, there is a primitive integral matrix whose spectral radius is this Perron number.

THEOREM 6.1. *Let $(\mathbf{i}, M, \mathbf{t})$ be a $\mathbb{Z}$-representation of a sequence $s$ of non-negative integers. If the two following conditions are satisfied,*

(i) *$M$ is spectrally Perron,*

(ii) *the complexity of $s$ is the spectral radius of $M$,*

*then there exists a forward elementary equivalence from $(\mathbf{i}, M, \mathbf{t})$ to an $\mathbb{N}$-representation.*

PROOF. Let $(\mathbf{i}, M, \mathbf{t})$ be a spectrally Perron $\mathbb{Z}$-representation of a non-negative sequence $s$. The matrix $M$ is thus spectrally Perron with a spectral radius $\lambda$. Let $\mathbf{w}$ be a left Perron eigenvector such that $\mathbf{i}$ has a non-negative dominant coordinate. By Lemma 5.2, $\mathbf{i}$ has a positive dominant coordinate.

By Lemma 5.3, there is a positive real number $\eta$ such that for any vector $\mathbf{u} \in K_\eta^+(\mathbf{w})$, we have $\mathbf{u}\cdot\mathbf{t} > 0$. We moreover choose $\eta$ small enough such that any vector in $K_\eta^+(\mathbf{w})$ has a positive dominant coordinate.

By Lemma 5.4 there exists a positive real number $\varepsilon$ such that, for any positive integer $n$, if $\mathbf{u} \in K_{2\varepsilon}(\mathbf{w})$ then $\mathbf{u}M^n \in K_\eta(\mathbf{w})$. Let us fix such a positive real number $\varepsilon$ with moreover $\varepsilon < 1/2$ and $2\varepsilon < \eta$. Thus $K_\varepsilon(\mathbf{w}) \subset K_{2\varepsilon}(\mathbf{w}) \subset K_\eta(\mathbf{w})$.

By Lemma 5.6, there is finite set $P$ of integer points in $K_{2\varepsilon}(\mathbf{w})$ such that each integer point of $K_\varepsilon(\mathbf{w})$ is a non-negative integral combination of points of $P$.

By Lemma 5.1 and since $P$ is a finite set of points of $K_{2\varepsilon}(\mathbf{w})$, there is an integer $n_0$ such that for any vector $\mathbf{v} \in P \cup \{\mathbf{i}\}$, the vector $\mathbf{v}M^{n_0} \in K_\varepsilon(\mathbf{w})$.

We define a forward elementary equivalence from $(\mathbf{i}, M, \mathbf{t})$ to a representation $(\mathbf{j}, N, \mathbf{x})$ as follows. The rows of the transfer matrix $U$ are the nonzero row vectors $\mathbf{v}M^j$, with $\mathbf{v} \in P \cup \{\mathbf{i}\}$ and $0 \le j \le n_0 - 1$. We define the matrix $N$ as a matrix of the multiplication by $M$ on the right on the set $S$ formed by these row vectors. If $\mathbf{u}$ is in $S$, either $\mathbf{u}M$ is in $S$ or $\mathbf{u}M$ belongs to $K_\varepsilon(\mathbf{w})$. In the latter case, it is a consequence of the geometrical lemma that $\mathbf{u}M$ is a non-negative integral combination of points of $P$. If $\mathbf{u}$ is in $S$ and $\mathbf{u}M = \sum_{\mathbf{v}\in S} \alpha_{\mathbf{u},\mathbf{v}}\mathbf{v}$, with $\alpha_{\mathbf{u},\mathbf{v}} \in \mathbb{N}$, we define the coefficient of index $\mathbf{u}, \mathbf{v}$ of $N$ as $\alpha_{\mathbf{u},\mathbf{v}}$. Thus the matrix $N$ has non-negative integral coefficients. Note that since $S$ is not necessarily a basis, $N$ is not necessarily unique. By definition, $UM = NU$. We only keep in $S$ the vertices accessible from $\mathbf{i}$ in the graph defined by the matrix $N$. Moreover, we order the rows of $U$ in such a way that the first row of $U$ is the vector $\mathbf{i}$.

We define the row vector $\mathbf{j}$ of length $|S|$ by $\mathbf{j} = \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}$ and the column vector $\mathbf{x}$ by $\mathbf{x} = U\mathbf{t}$. If $\mathbf{u} = \mathbf{i}M^j$ for $0 \leq j \leq n_0 - 1$, then $x_{\mathbf{u}} = \mathbf{u} \cdot \mathbf{t} = \mathbf{i}M^j\mathbf{t} = s_j \geq 0$.

By Remark 5.5, if $\mathbf{u}$ is a nonzero vector in $K_{2\varepsilon}(\mathbf{w})$, then $\mathbf{u}M^j \in K_\eta^+(\mathbf{w})$ for any $j \geq 0$. Then $\mathbf{u}M^j\mathbf{t} > 0$ for any $j \geq 0$. Thus $\mathbf{x}$ has non-negative integral coefficients. Note that the transfer matrix $U$ has its coefficients in $\mathbb{Z}$. Thus we have proved that $(\mathbf{i}, M, \mathbf{t}) \xrightarrow{U} (\mathbf{j}, N, \mathbf{x})$ is a backward elementary equivalence over $\mathbb{Z}$ and $(\mathbf{j}, N, \mathbf{x})$ is an $\mathbb{N}$-representation of $s$. This concludes the proof of the theorem. $\square$

We add two remarks on further consequences of the above proof for future reference. Both follow from the last paragraph of the proof.

*Remark* 6.2. If the sequence $s$ satisfies the additional hypothesis $s_n > 0$ for any non-negative integer $n$, then the vector $\mathbf{x}$ is a positive integral vector.

*Remark* 6.3. If $\mathbf{t}'$ is another non-negative vector such that $\mathbf{u} \cdot \mathbf{t}' > 0$ for any vector $\mathbf{u} \in K_\eta^+(\mathbf{w})$, and such that $(\mathbf{i}, M, \mathbf{t}')$ specifies a sequence of non-negative integers, $\mathbf{x}' = U\mathbf{t}'$ is a non-negative vector.

Corollary 6.4. *From any reduced spectrally Perron representation of a non-negative sequence, there exits a forward elementary equivalence to a regular representation.*

Proof. If $(\mathbf{i}, M, \mathbf{t})$ is a reduced $\mathbb{Z}$-representation of a sequence of non-negative integers $s$, the complexity of $s$ is equal to the spectral radius of $M$. $\square$

*Example* 6.5. Let us consider the regular sequence $s$ defined by, for $n \geq 2$,

$$\begin{aligned} s_0 &= 1, \\ s_1 &= 2, \\ s_n &= 4s_{n-1} - 3s_{n-2}. \end{aligned}$$

A $\mathbb{Z}$-representation of this sequence is

$$\mathbf{i} = \begin{bmatrix}1 & 0\end{bmatrix}, M = \begin{bmatrix}0 & 1 \\ -3 & 4\end{bmatrix}, \mathbf{t} = \begin{bmatrix}1 \\ 2\end{bmatrix}.$$

The matrix $M$ is spectrally Perron with a spectral radius 3 since its characteristic polynomial is $(X-3)(X-1)$. The computation of the first powers $\mathbf{i}M^n$ gives

$$\begin{aligned} \mathbf{i}M^0 &= \begin{bmatrix}1 & 0\end{bmatrix}, \\ \mathbf{i}M^1 &= \begin{bmatrix}0 & 1\end{bmatrix} = \mathbf{u} + \mathbf{i}, \end{aligned}$$

where $\mathbf{u} = \begin{bmatrix}-1 & 1\end{bmatrix}$. We have $\mathbf{u}M = 3\mathbf{u}$ and thus the set of non-negative integral combinations of the vectors $\mathbf{i}$, $\mathbf{u}$ is stable by $M$. As in the proof of Theorem 6.1, we choose as transfer matrix

$$U = \begin{bmatrix}1 & 0 \\ -1 & 1\end{bmatrix}.$$

Thus $s$ has the following $\mathbb{N}$-representation which is elementary equivalent to $(\mathbf{i}, M, \mathbf{t})$:

$$\mathbf{j} = \begin{bmatrix}1 & 0\end{bmatrix}, N = \begin{bmatrix}1 & 1 \\ 0 & 3\end{bmatrix}, \mathbf{x} = \begin{bmatrix}1 \\ 1\end{bmatrix}.$$

The sequence $s$ is thus specified by the triple $(\{1\}, H, \{1, 2\})$, where $H$ is the graph of Figure 2.

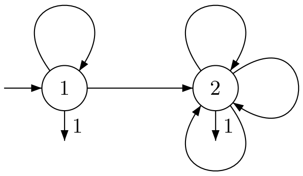


Fig. 2. An $\mathbb{N}$-representation of the sequence $s$ defined by $s_0 = 1$, $s_1 = 2$, and $s_n = 4s_{n-1} - 3s_{n-2}$ for $n \geq 2$. The vertex 1 (marked with an incoming arrow) is the initial vertex and $1, 2$ (marked with an outgoing arrow) are the terminal vertices.

## 7. THE MAIN RESULT

We now prove the main result in a slightly more general form (Theorem 7.5 below). The characterization of the generating sequences of regular languages on $k$ symbols given in Theorem 3.2 of Section 3 is a consequence of Theorem 7.5. We first state several lemmas. Lemma 7.4 constitutes one of the main parts of the proof of Theorem 7.5.

We recall the notion of approximate eigenvector. Let $k$ be a positive integer. A right *$k$-approximate* eigenvector of a non-negative matrix $M$ is an integer column vector $\mathbf{v} \geq 0$ such that $M\mathbf{v} \leq k\mathbf{v}$. When $M$ is the adjacency matrix of a graph $G$, we also say that $\mathbf{v}$ is a $k$-approximate eigenvector of $G$.

LEMMA 7.1. *Let $(\mathbf{j}, N, \mathbf{x})$ be an $\mathbb{N}$-representation such that $\mathbf{x}$ is a positive right $k$-approximate eigenvector (respectively a positive right $k$-eigenvector) of $N$. Then there is an $\mathbb{N}$-representation $(\mathbf{j}, N, \mathbf{x})$ and a backward elementary equivalence $(\mathbf{i}, M, \mathbf{t}) \xleftarrow{U} (\mathbf{j}, N, \mathbf{x})$, such that $\mathbf{t}$ is a positive right $k$-approximate eigenvector (respectively a positive $k$-eigenvector) of $M$ which has all its coefficients equal to 1.*

*Moreover if $\mathbf{x} = \sum_{i=1}^{l} \mathbf{x}_i$, where each $\mathbf{x}_i$ is a non-negative integral vector, then there are non-negative integral vectors $\mathbf{t}_i$ such that $\mathbf{x}_i = U\mathbf{t}_i$.*

PROOF. We give the proof in the case of approximate eigenvectors. The other alternative is similar. Let us denote by $Q$ the set of indices of $\mathbf{j}$. Let $Q'$ be the set of pairs $(q, j)$ with $q \in Q$ and $1 \leq j \leq x_q$. For each $p \in Q$, let us consider the set of triples $\{(q, j, l) \mid q \in Q, 1 \leq j \leq x_q, 1 \leq l \leq N_{pq}\}$. Its cardinality is $\sum_{q \in Q} N_{pq} x_q$. Since for each $p \in Q$, we have

$$\sum_{q \in Q} N_{pq} x_q \leq k x_p,$$

it is possible to partition this set in $x_p$ sets $X_{(p,1)}, X_{(p,2)}, \ldots, X_{(p,x_p)}$ of at most $k$ elements. We now define the square $Q' \times Q'$ matrix $M$ by defining $M_{(p,i)(q,j)}$, for $p, q \in Q$, $1 \leq i \leq x_p$ and $1 \leq j \leq x_q$, as the number of triples in $X_{(p,i)}$ whose first two components are $(q, j)$. Let $U$ be the $Q \times Q'$ matrix defined by $U_{q(q,j)} = 1$, for any $1 \leq j \leq x_q$, the other coefficients being zero. By construction, we get

$$NU = UM.$$

Indeed, the coefficient of index $p, (q, j)$, where $1 \leq j \leq x_q$, of $NU$ is

$$\sum_{r \in Q} N_{pr} U_{r(q,j)} = N_{pq}.$$

And the coefficient of index $p,(q,j)$ of $UM$ is

$$\sum_{(r,i)\in Q'} U_{p(r,i)} M_{(r,i)(q,j)} = \sum_{1\le i\le x_p} M_{(p,i)(q,j)} = N_{pq}.$$

We define the row $Q'$-vector $\mathbf{i}$ by $\mathbf{i} = \mathbf{j}U$. Let $\mathbf{t}$ be the column $Q'$-vector with all its coefficients equal to 1. It is straightforward that $\mathbf{x} = U\mathbf{t}$. Thus

$$(\mathbf{i}, M, \mathbf{t}) \xrightarrow{U} (\mathbf{j}, N, \mathbf{x}).$$

Since the sum of each row of $M$ is less than or equal to $k$, $\mathbf{t}$ is a right $k$-approximate eigenvector of $M$.

Let us now assume that $\mathbf{x} = \sum_{i=1}^{l} \mathbf{x}^{(\mathbf{i})}$, where $\mathbf{x}^{(i)}$ is a non-negative integral vector. Let us define the column $Q'$-vector $\mathbf{t}^{(i)}$ by $t^{(i)}_{(q,j)} = 1$ if and only if $1 \le j \le x^{(i)}_q$ and $t^{(i)}_{(q,j)} = 0$ otherwise for $1 \le i \le l$. Then

$$(U\mathbf{t}^{(i)})_p = \sum_{(q,j)\in Q'} U_{p(q,j)} t^{(i)}_{(q,j)} = \sum_{1\le j\le x_p} t^{(i)}_{(p,j)} = x^{(i)}_p.$$

We get $\mathbf{x}^{(i)} = U\mathbf{t}^{(i)}$ for $1 \le i \le l$. $\square$

We mention that a stronger form of this lemma can be proved by the use of the ACH algorithm of [Adler et al. 1983] which is based on state splitting.

Lemma 7.2. *Let $(\mathbf{i}, M, \mathbf{t}) \xleftarrow{U} (\mathbf{j}, N, \mathbf{x})$ be a forward elementary equivalence between $\mathbb{Z}$-representations. If $\mathbf{t}$ is a right $k$-eigenvector of $M$, then $\mathbf{x}$ is a right $k$-eigenvector of $N$.*

Proof. The proof is straightforward. If $M\mathbf{t} = k\mathbf{t}$ and $(\mathbf{i}, M, \mathbf{t}) \xrightarrow{U} (\mathbf{j}, N, \mathbf{x})$, then $N\mathbf{x} = NU\mathbf{t} = UM\mathbf{t} = Uk\mathbf{t} = k\mathbf{x}$. $\square$

Lemma 7.3. *Any left reduced $\mathbb{Z}$-representation $(\mathbf{j}, N, \mathbf{x})$ of $m(kz)^*$, where $m$ and $k$ are positive integers, is such that $\mathbf{x}$ is a right $k$-eigenvector of $N$.*

Proof. We consider a left reduced representation $(\mathbf{j}, N, \mathbf{x})$ of $m(kz)^*$. By Proposition 4.5, there is a backward elementary equivalence from $(\mathbf{j}, N, \mathbf{x})$ to $([m], [k], [1])$, which is the right minimal representation of $m(kz)^*$. Thus

$$([m], [k], [1]) \xrightarrow{V} (\mathbf{j}, N, \mathbf{x}),$$

where $V$ is the transfer matrix of this elementary equivalence. Since

$$V[k] = NV, \;\; [m] = \mathbf{j}V, \;\; \mathbf{x} = V[1],$$

we get $V = \mathbf{x}$ and $\mathbf{x}$ is a right $k$-eigenvector of $N$. $\square$

The following lemma constitutes the main part of the proof of Theorem 7.5. We use here a variant of the terminology of finite automata. Let $\mathcal{A} = (Q, A, \delta, I, T)$ be a finite automaton with set of states $Q$, alphabet $A$, transition function $\delta$, set of initial states $I$, and set of terminal states $T$. Let $L$ be the language recognized by $\mathcal{A}$. We can define a labelled graph $G$ with $Q$ as set of vertices and the pairs $(p, a, q)$ where $q \in \delta(p, a)$ as edges. Conversely any such graph corresponds uniquely to an automaton $\mathcal{A}$. When $\mathcal{A}$ is deterministic we say that $G$ is deterministically labelled. We also say that $(I, G, T)$ is a deterministic automaton that recognizes $L$.

LEMMA 7.4. *Let $l$ be a positive integer and $s_1, \dots s_l$ be $l$ regular sequences specified by $\mathbb{N}$-representations $(\mathbf{i}, M, \mathbf{t}_i)$ respectively, such that $s_1(z) + \cdots + s_l(z) = m(kz)^*$, where $m$ and $k$ are positive integers. Let us assume that $M$ has a dominating eigenvalue $k$, that all $s_i$ have complexity $k$, and that $(\mathbf{i}, M, \sum_{i=1}^{l} \mathbf{t}_i)$ is trim. Then there is a finite deterministically labelled graph $G$ on a $k$-letter alphabet, with $m$ initial states and a partition of the set of states of $G$ in $l$ sets $T_i$, with $1 \leq i \leq l$, such that the automaton $(I, G, T_i)$ recognizes a regular language on $k$ symbols whose generating sequence is exactly $s_i$.*

PROOF. We denote by $\mathbf{t}$ the column vector $\sum_{i=1}^{l} \mathbf{t}_i$ and thus $(\mathbf{i}, M, \mathbf{t})$ specifies $s(z) = \sum_{i=1}^{l} s_i(z) = m(kz)^*$. We denote by $J_r(k)$ the Jordan block of size $r$:

$$J_r(k) = \begin{bmatrix} k & 1 & 0 & \dots & 0 & 0 \\ 0 & k & 1 & \dots & 0 & 0 \\ 0 & 0 & k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & k & 1 \\ 0 & 0 & 0 & \dots & 0 & k \end{bmatrix}.$$

Since $(\mathbf{i}, M, \mathbf{t})$ is a trim $\mathbb{N}$-representation which specifies $m(kz)^*$, the Jordan canonical form of $M$ has no block $J_r(k)$ where $r > 1$. Indeed, let us assume that the Jordan form of $M$ contains such a block. Then there is a positive real number $c$ such that for any large enough integer $n$, $s_n \geq cn^{r-1}k^n$. Thus the sequence $s(z)$ cannot be equal to $m(kz)^*$.

We compute from $(\mathbf{i}, M, \mathbf{t})$ a left reduced $\mathbb{Z}$-representation $(\mathbf{j}, N, \mathbf{x})$ of $m(kz)^*$. We know from Proposition 4.5 that there exists a transfer matrix $U$ such that

$$(\mathbf{i}, M, \mathbf{t}) \xrightarrow[\mathbb{Z}]{U} (\mathbf{j}, N, \mathbf{x}).$$

Since $(\mathbf{j}, N, \mathbf{x})$ is left reduced, the dimension of the $\mathbb{Z}$-module generated by the vectors $\mathbf{j}N^n$, for $n \geq 0$, is the size $d$ of the square matrix $N$. This dimension is also equal to the dimension of the vector space $E$ generated by the vectors $\mathbf{j}N^n$, for $n \geq 0$, over the field $\mathbb{R}$. Let $E'$ be the eigenspace of $N$ associated to the eigenvalue $k$ in $E$ and let $E''$ be a complementary $N$-invariant subspace. Thus $d$ is the sum of the dimensions of $E'$ and of $E''$. We claim that $E'$ has dimension one. Indeed, the vector $\mathbf{j}$ can be written

$$\mathbf{j} = \mathbf{u} + \mathbf{v},$$

where $\mathbf{u} \in E'$ and $\mathbf{v} \in E''$. Since for any integer $n \geq 0$,

$$\mathbf{j}N^n = k^n\mathbf{u} + \mathbf{v}N^n,$$

the vector space over $\mathbb{R}$ generated by the vectors $\mathbf{j}N^n$, for $n \geq 0$, is included in $\langle\mathbf{u}\rangle + E''$, where $\langle\mathbf{u}\rangle$ denotes the vector space over $\mathbb{R}$ generated by $\mathbf{u}$. Thus the dimension of $E'$ is one.

The Jordan canonical form of $N$ has thus a dominating eigenvalue, has no block $J_r(k)$, where $r > 1$ , and has a one dimensional eigenspace associated to the spectral radius $k$. The matrix $N$ is thus a spectrally Perron matrix. Moreover, by Lemma 7.3, $\mathbf{x}$ is an integer right $k$-eigenvector of $N$. For each integer $1 \leq i \leq l$, we define $\mathbf{x}_i = U\mathbf{t}_i$.

By Theorem 6.1, there exists a forward elementary equivalence to a regular representation from $(\mathbf{j}, N, \mathbf{x})$ to an $\mathbb{N}$-representation $(\mathbf{k}, L, \mathbf{y})$. Let $V$ be its transfer matrix. Since the sequence specified is $m(kz)^*$, it has positive terms. Thus the vector $\mathbf{y}$ is a positive vector (see Remark 6.2 at the end of the proof of Theorem 6.1). By Lemma 7.2, the vector $\mathbf{y}$ is a right $k$-eigenvector of $L$. It is thus a positive integral eigenvector of $L$.

Since $(\mathbf{j}, N, \mathbf{x}_i)$ is a left reduced representation which specifies $s_i$, and since $s_i$ has complexity $k$, one chooses by Lemma 5.3 a positive real number $\eta$ such, that for each integer $1 \leq i \leq l$, for any vector $\mathbf{u} \in K_\eta^+(\mathbf{w})$, we have $\mathbf{u} \cdot \mathbf{x} > 0$ and $\mathbf{u} \cdot \mathbf{x}_i > 0$. It follows from Remark 6.3 at the end of the proof of Theorem 6.1 that the $l$ vectors $\mathbf{y}_i = V\mathbf{x}_i$ are non-negative integral vectors.

The final step is given by Lemma 7.1. There is a regular backward elementary equivalence from $(\mathbf{k}, L, \mathbf{y})$ to an $N$-representation $(\mathbf{i}', M', \mathbf{t}')$ such that $\mathbf{t}'$ is a positive right $k$-approximate eigenvector of $M'$ which has all its coefficients equal to 1. Let us denote by $W$ the transfer matrix of this backward elementary equivalence. Since $\mathbf{y} = \sum_{i=1}^{l} \mathbf{y}_i$ where the vectors $\mathbf{y}_i$ are non-negative integral vectors, there are two non-negative integral vectors $\mathbf{t}'_i$ such that $\mathbf{y}_i = W\mathbf{t}'_i$.

The two previous forward elementary equivalences and the backward elementary equivalence can be summarized in

$$(\mathbf{i}, M, \mathbf{t}) \xrightarrow[\mathbb{Z}]{U} (\mathbf{j}, N, \mathbf{x}) \xrightarrow[\mathbb{Z}]{V} (\mathbf{k}, L, \mathbf{y}) \xleftarrow[\mathbb{N}]{W} (\mathbf{i}', M', \mathbf{t}').$$

We also have for each integer $1 \leq i \leq l$,

$$(\mathbf{i}, M, \mathbf{t}_i) \xrightarrow[\mathbb{Z}]{U} (\mathbf{j}, N, \mathbf{x}_i) \xrightarrow[\mathbb{Z}]{V} (\mathbf{k}, L, \mathbf{y}_i) \xleftarrow[\mathbb{N}]{W} (\mathbf{i}', M', \mathbf{t}'_i).$$

Thus, for each integer $1 \leq i \leq l$, we get an $\mathbb{N}$-representation $(\mathbf{i}', M', \mathbf{t}'_i)$ of the sequence $s_i$. The coefficients of all $\mathbf{t}'_i$ are 0 or 1 and the sum of the vectors $\mathbf{t}'_i$ is the vector $\mathbf{t}'$ whose coefficients are all equal to 1. Let us denote by $T_i$ the set of indices of $\mathbf{t}'_i$ corresponding to a coefficient 1. Since $\mathbf{t}'$ is a right $k$-eigenvector of $M'$, the sum of each row of $M'$ is equal to $k$. The matrix $M'$ is thus the transition matrix of a $k$-ary directed multigraph $G$. Let $Q$ be the set of states of $G$. Since $\mathbf{i}' \cdot \mathbf{t}' = m$, the sum of the coefficients of the vector $\mathbf{i}'$ is $m$. We define a new graph $G'$ by adding to $G$ a new set $I$ of $m$ states $(p, j)$, for $p \in Q$ and $1 \leq j \leq i_p$, and $n$ edges from $(p, j)$ to $q$ if there are $n$ edges from $p$ to $q$ in $G$. This last transformation is again a backward elementary equivalence. Since the graph $G'$ is still $k$-ary, one can label it with $k$ symbols in a deterministic way. Then the automaton $(I, G', T_i)$ recognizes a regular language on $k$ symbols whose generating sequence is exactly $s_i$. $\square$

We now state and prove the main result. Theorem 3.2 is a formulation of Theorem 7.5 in the case of two sequences.

THEOREM 7.5. *Let $m$ and $k$ be two positive integers. Let $s^{(1)}, s^{(2)}, \ldots, s^{(l)}$ be $l$ regular sequences such that $s^{(1)} + s^{(2)} + \ldots + s^{(l)}(z) = m(kz)^*$. Then there is a finite deterministically labelled graph $G$ on a $k$-letter alphabet, with $m$ initial states and a partition of the set of states of $G$ in $l$ sets $T_i$, with $1 \leq i \leq l$, such that the automaton $(I, G, T_i)$ recognizes a regular language on $k$ symbols whose generating sequence is exactly $s^{(i)}$.*

The proof contains two main parts. The first part corresponds to sequences that all have complexity $k$ and relies mainly on Lemma 7.4. The second part treats the other case.

Proof. We first order the sequences in such a way that there is an integer $0 \leq l' \leq l-1$ such that $s^{(1)}, s^{(2)}, \ldots, s^{(l')}$ have a complexity strictly less than $k$ and that $s^{(l'+1)}, \ldots, s^{(l)}$ have complexity $k$. Note that at least one of the sequences has complexity $k$ since the sum of the sequences is $m(kz)^*$.

Let us consider first the case where $l' = 0$, that is, where all sequences $s^{(i)}$ have complexity $k$. Let $(\mathbf{i}^{(i)}, M^{(i)}, \mathbf{t}^{(i)})$ be a trim regular representation of $s^{(i)}$ for $1 \leq i \leq l$. Thus the regular representation $(\mathbf{i}, M, \mathbf{x}^{(i)})$ defined by

$$(\mathbf{i}, M, \mathbf{x}^{(i)}) = (\begin{bmatrix}\mathbf{i}^{(1)} & \ldots \mathbf{i}^{(l)}\end{bmatrix}, \begin{bmatrix} M^{(1)} & 0 & \cdots & 0 \\ 0 & M^{(2)} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & M^{(l)} \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ \mathbf{t}^{(i)} \\ \vdots \\ 0 \end{bmatrix}).$$

specifies the sequence $s^{(i)}$ for $1 \leq i \leq l$. We denote by $\mathbf{t}$ the column vector $\sum_{i=1}^{l} \mathbf{x}^{(i)}$ and thus $(\mathbf{i}, M, \mathbf{t})$ specifies $s(z) = \sum_{i=1}^{l} s^{(i)}(z) = m(kz)^*$. Since $(\mathbf{i}^{(i)}, M^{(i)}, \mathbf{t}^{(i)})$ are trim representations, $(\mathbf{i}, M, \mathbf{t})$ is also trim.

By the Perron-Frobenius theorem [MacCluer 2000], the eigenvalues of maximal modulus of $M$ are equal to $\lambda\rho$, where $\lambda$ is a positive real number and where $\rho$ is a root of unity. Thus there is an integer $p$ such that $M^p$ has a dominating eigenvalue. Each sequence $s^{(i)}$ is a merge of $p$ sequences $s^{(i,j)}$ specified by $(\mathbf{i}M^j, M^p, \mathbf{x}^{(i)})$, for $0 \leq j \leq p-1$. These representations may not be trim but $M^p$ cannot have a Jordan canonical form that contains a block $J_r(k^p)$ with $r > 1$. Indeed, let us assume that it is not true. Then there is at least one coefficient $(M^p)^n_{qr}$ of $(M^p)^n$ whose growth rate is at least $cn^{r-1}k^{pn}$, where $c$ is a positive real number. Since $(\mathbf{i}, M, \mathbf{t})$ is trim, there is a non-negative integer $n_1$ such that $(\mathbf{i}M^{n_1})_q > 0$, and there is a non-negative integer $n_2$ such that $(M^{n_2}\mathbf{t})_r > 0$. Thus $(s_{n_1+n_2+pn})_{n \geq 0}$ would have a growth rate which is, up to a positive constant, at least $n^{r-1}k^{pn}$ which is too much. Note that the sequence $\sum_{i=1}^{l} s^{(i,j)}(z)$ is equal to $mk^j(k^pz)^*$, for $0 \leq j \leq p-1$.

Let $j$ be an integer such that $0 \leq j \leq p-1$. Either all sequences $s^{(i,j)}$ have complexity $k^p$ or at least one of them has a complexity strictly less than $k^p$. In the former case, Lemma 7.4 constructs automata $(I^{(j)}, G^{(j)}, T_i^{(j)})$ that recognize $s^{(i,j)}$ on the alphabet $A^p$, where $A$ is a finite alphabet with $k$ symbols, and where $I^{(j)}$ has cardinal $mk^j$. For a given $j$, these representations define $l$ disjoint regular languages $L_j^{(i)}$ on a $k$-letter alphabet with generating sequences $s^{(i,j)}$. The latter case corresponds to an instance of the statement of the theorem when at least one sequence has a complexity strictly less than $k' = k^p$ with $m' = mk^j$. This case is proved below. Then the sets $\cup_{j=0}^{p-1} L_j^{(i)}$, with $1 \leq i \leq l$, are disjoint regular languages on a $k$-letter alphabet having $s^{(i)}$ as generating sequences.

We now consider the case where there is at least one sequence with a complexity strictly less than $k$. We denote by $t^{(l'+1)}$ the sequence $s^{(1)} + \ldots + s^{(l')} + s^{(l'+1)}$.

Thus $t^{(l'+1)}$ is a regular sequence which has complexity $k$. Thus, by applying the construction used in the case where all sequences have complexity $k$, we get regular representations with 0-1 coefficients $(\mathbf{j}, N, \mathbf{y}^{(i)})$ of $s^{(i)}$ for $l'+2 \le i \le l$, and $t^{(l'+1)}$ for $i = l'+1$, such that $\mathbf{j}$ has exactly $m$ coefficients equal to 1, and such that $N$ is a $k$-ary matrix. Note that the first case where all sequences have complexity $k$ is applied to at most $l-1$ sequences and thus that we can reason by induction.

We denote by $Q$ the set of indices of $N$, also called states, and by $d'$ the cardinality of $Q$. If $q$ is a state, we denote by $\overline{\mathbf{q}}$ the characteristic row vector of $q$ of size $d'$. A state $q$ is said to be a final for $(\mathbf{j}, N, \mathbf{y}^{(l'+1)})$ if $y_q^{(l'+1)} = 1$. Moreover the vector $\mathbf{z} = \sum_{i=l'+1}^{l} \mathbf{y}^{(i)}$ has all its coefficients equal to 1. A principal component of $N$ is an irreducible component of $N$ whose spectral radius is $k$. Since $N$ is a $k$-ary matrix, each principal component of $N$ is a sink, that is, $n$ has no nonzero coefficient $N_{pq}$ with $p$ inside the component and $q$ outside (i.e., a component from which there is "no escape"). A principal component that contains a final state for $(\mathbf{j}, N, \mathbf{y}^{(l'+1)})$ is called a final component. Since $t^{(l'+1)}$ has complexity $k$, there is at least a final component $C$ of $N$ that contains a final state $q$ for $(\mathbf{j}, N, \mathbf{y}^{(l'+1)})$. Moreover, there is a positive integer $s$ such that $(\mathbf{j}N^s)_q > 0$.

We denote by $p$ the period of $N$, which is the least common multiple of the periods of the irreducible components of $N$. Recall that the period of an irreducible matrix $N'$ is the gcd of positive integers $n$ such that the trace of $N'^n$ is positive. By applying the construction, with other values of $m$ and $k$, to the sequences $s^{(i,j)}$ defined, for $1 \le i \le l$ and $0 \le j \le p-1$, by $s_n^{(i,j)} = s_{j+pn}^{(i)}$, we can assume that $N$ has period 1 and thus that $C$ is a primitive matrix.

As a consequence of the Perron-Frobenius theorem, there is a positive real number $\rho$ and a positive integer $n_0$ such that for any integer $n \ge n_0$ and any two states $p, q$ of the final component $C$, $C_{pq}^n \ge \rho k^n$.

We get that for any integer $n \ge n_0 + s$, any state $q$ of the component $C$, and any index $l'+1 \le i \le l$, $\mathbf{j}N^n\mathbf{y}^{(i)} \ge \rho k^{n-s}$ and $\overline{\mathbf{q}}N^n\mathbf{y}^{(i)} \ge \rho k^{n-s}$. Without loss of generality, by increasing the value of $s$, we can assume that $\rho$ is the positive integer 1. Thus for any $n \ge n_0 + s$, any $q \in C$, and any $l'+1 \le i \le l$,

$$
\begin{aligned}
\overline{\mathbf{q}}N^n\mathbf{z} &= k^n \text{ (since } N \text{ is } k\text{-ary)} && (3)\\
\mathbf{j}N^n\mathbf{z} &= mk^n && (4)\\
\overline{\mathbf{q}}N^n\mathbf{y}^{(i)} &\ge k^{n-s} && (5)\\
\mathbf{j}N^n\mathbf{y}^{(i)} &\ge k^{n-s} && (6)
\end{aligned}
$$

Let us now consider the sequences having a complexity strictly less than $k$. Let $(\mathbf{i}^{(i)}, M^{(i)}, \mathbf{t}^{(i)})$ be a trim $\mathbb{N}$-representation of $s^{(i)}$ for $1 \le i \le l'$. We can moreover assume that $\mathbf{i}^{(i)}$ and $\mathbf{t}^{(i)}$ have 0-1 coefficients. Thus the $\mathbb{N}$-representation $(\mathbf{i}, M, \mathbf{x}^{(i)})$ defined by

$$
(\mathbf{i}, M, \mathbf{x}^{(i)}) = \left( \begin{bmatrix} \mathbf{i}^{(1)} & \dots \mathbf{i}^{(l')} \end{bmatrix}, \begin{bmatrix} M^{(1)} & 0 & \cdots & 0 \\ 0 & M^{(2)} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & M^{(l')} \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ \mathbf{t}^{(i)} \\ \vdots \\ 0 \end{bmatrix} \right).
$$

specifies the sequence $s^{(i)}$ for $1 \le i \le l'$. The vectors $\mathbf{i}$ and $\mathbf{x}^{(i)}$ have 0-1 coefficients. Let $\lambda$ be the spectral radius of $M$. For an infinite number of indices $n$, the terms of $s^{(i)}$ are, up to constant, at most $P^{(i)}(n)\lambda^n$, where $P^{(i)}(n)$ is a nonzero polynomial in $n$. Since each $s^{(i)}$, for $1 \le i \le l'$, has a complexity strictly less than $k$, $\lambda < k$.

Let $d$ be the size of $M$ and $P$ the set of indices of $M$. If $p$ is in $P$, we denote by $\overline{\mathbf{p}}$ the characteristic row vector of $p$ of size $d$. We denote by $\mathbf{t}$ the 0-1 column vector $\sum_{i=1}^{l'} \mathbf{x}^{(i)}$. We denote by $\mathbf{w}$ the column vector of size $d$ with all coefficients equal to 1.

Since $\lambda < k$, there is a positive integer $r$ such that for any $n \ge r$ and any $p \in P$, the following inequalities hold

$$\overline{\mathbf{p}}M^n\mathbf{w} \le k^{n-s}, \tag{7}$$

$$\mathbf{i}M^n\mathbf{w} \le k^{n-s}. \tag{8}$$

As a consequence, for any $n \ge r$ and any $p \in P$,

$$\overline{\mathbf{p}}M^n\mathbf{t} \le k^{n-s}, \tag{9}$$

$$\mathbf{i}M^n\mathbf{t} \le k^{n-s}. \tag{10}$$

We moreover choose $r \ge n_0 + s$.

We define a product of size $d+d'$ of the representations $(\mathbf{i}, M, \mathbf{x}^{(i)})$ and $(\mathbf{j}, N, \mathbf{y}^{(j)})$ as follows. For $1 \le i \le l'$, and $l'+1 \le j \le l$, let

$$\mathbf{k} = \begin{bmatrix}\mathbf{i} & \mathbf{j}\end{bmatrix}, \quad L = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}, \quad \mathbf{X}^{(\mathbf{i})} = \begin{bmatrix} \mathbf{x}^{(i)} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{Y}^{(j)} = \begin{bmatrix} \mathbf{0} \\ \mathbf{y}^{(j)} \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} \mathbf{0} \\ \mathbf{z} \end{bmatrix}.$$

Then the $\mathbb{N}$-representation $(\mathbf{k}, L, \mathbf{X}^{(i)})$ specifies $s^{(i)}$ for $1 \le i \le l'$. The $\mathbb{N}$-representation $(\mathbf{k}, L, \mathbf{Y}^{(l'+1)})$ specifies $t^{(l'+1)}$. The regular representation $(\mathbf{k}, L, \mathbf{Y}^{(j)})$ specifies $s^{(j)}$ for $l'+2 \le j \le l$. Finally $(\mathbf{k}, L, \mathbf{Z})$ specifies $m(kz)^*$.

We now define forward elementary equivalences from these $\mathbb{N}$-representations with a transfer matrix denoted by $U$ of size $d'' \times (d+d')$. Let $U$ be the matrix whose set of rows is formed by row vectors of size $(d+d')$, the vectors $\mathbf{k}L^n$, with $0 \le n \le (2r-1)$, the vectors $(\overline{\mathbf{p}}M^n, \overline{\mathbf{q}}N^n)$, with $r \le n \le (2r-1)$, $p \in P$, $q \in C$, and the vectors $(\mathbf{0}, \overline{\mathbf{q}}N^r)$ for $q \in Q$.

Let us consider a linear transformation of the rows of $U$ defined as follows.

—Each vector $\mathbf{k}L^n$ for $0 \le n < (2r-1)$ is transformed to $\mathbf{k}L^{n+1}$.
—Each vector $(\overline{\mathbf{p}}M^n, \overline{\mathbf{q}}N^n)$, for $0 \le n < (2r-1)$, $p \in P$, $q \in C$, is transformed in $(\overline{\mathbf{p}}M^{n+1}, \overline{\mathbf{q}}N^{n+1})$.
—Each vector $(\mathbf{0}, \overline{\mathbf{q}}N^r)$ for $q \in Q$ is transformed in a sum of $k$ vectors $(\mathbf{0}, \overline{\mathbf{q}}_i N^r)$, where $\overline{\mathbf{q}}N = \sum_{i=1}^{k} \overline{\mathbf{q}}_i$.
—Let $\mathbf{p}$ be either $\mathbf{i}$ or a characteristic vector $\overline{\mathbf{p}}$ of a state $p \in P$, and $\mathbf{q}$ be either $\mathbf{j}$ or a characteristic vector $\overline{\mathbf{q}}$ of a state $q \in C$. Since $\mathbf{p}M^r\mathbf{w} \le k^{r-s}$, the vector $\mathbf{p}M^r$ is the sum of $K \le k^{r-s}$ characteristic vectors $\overline{\mathbf{p}}_i$. If $q \in C$, $\overline{\mathbf{q}}N^r$ is the sum of $k^r$ characteristic vectors of states in $C$. If $\mathbf{q} = \mathbf{j}$, $\mathbf{q}N^r$ is the sum of $mk^r$ characteristic vectors of states in $Q$ such that at least $k^{r-s}$ among them belong to $C$. Then in both cases, $\mathbf{q}N^r$ is then the sum of $K' \ge k^{r-s}$ vectors $\mathbf{q}_i$ such that $\mathbf{q}_i \in C$ for $1 \le i \le k^{r-s}$. We transform $(\mathbf{p}M^{2r-1}, \mathbf{q}N^{2r-1})$ in the sum of the $K$ vectors $(\mathbf{p}_i M^r, \mathbf{q}_i N^r)$, for $1 \le i \le K$ and the $K'-K$ vectors $(\mathbf{0}, \mathbf{q}_i N^r)$,

for $K+1 \leq i \leq K'$. Note that all these vectors are rows of the matrix $U$ and that their sum is equal to $(\mathbf{p}M^{2r}, \mathbf{q}N^{2r})$.

We denote by $R$ the transition matrix of this linear transformation. The matrix $R$ has non-negative integral coefficients and $RU = UL$. We denote by $\mathbf{l}$ the row vector $\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}$ of size $d''$. We have the following forward elementary equivalences over $\mathbb{N}$ between $\mathbb{N}$-representations for $1 \leq i \leq l'$ and $l'+1 \leq j \leq l$.

$$
\begin{aligned}
(\mathbf{k}, L, \mathbf{X}^{(i)}) &\xrightarrow[\mathbb{N}]{U} (\mathbf{l}, R, U\mathbf{X}^{(i)}),\\
(\mathbf{k}, L, \mathbf{Y}^{(j)}) &\xrightarrow[\mathbb{N}]{U} (\mathbf{l}, R, U\mathbf{Y}^{(j)}),\\
(\mathbf{k}, L, \mathbf{Z}) &\xrightarrow[\mathbb{N}]{U} (\mathbf{l}, R, U\mathbf{Z}).
\end{aligned}
$$

Then, $U$, $U\mathbf{Z}$ and $RU\mathbf{Z}$ have the following forms.

$$
U = \begin{bmatrix}
\mathbf{k} \\
\mathbf{k}L \\
\vdots \\
\mathbf{k}L^{2r-1} \\
(\overline{\mathbf{p}}M^r, \overline{\mathbf{q}}N^r) \\
\vdots \\
(\overline{\mathbf{p}}M^{2r-1}, \overline{\mathbf{q}}N^{2r-1}) \\
\vdots \\
(\mathbf{0}, \overline{\mathbf{q}}N^r) \\
\vdots
\end{bmatrix}, \quad
U\mathbf{Z} = \begin{bmatrix}
m \\
mk \\
\vdots \\
mk^{2r-1} \\
k^r \\
\vdots \\
k^{2r-1} \\
\vdots \\
k^r \\
\vdots
\end{bmatrix}, \quad
RU\mathbf{Z} = \begin{bmatrix}
mk \\
mk^2 \\
\vdots \\
mk^{2r} \\
k^{r+1} \\
\vdots \\
k^{2r} \\
\vdots \\
k^{r+1} \\
\vdots
\end{bmatrix}.
$$

Thus $U\mathbf{Z}$ is a positive right $k$-eigenvector of $R$. Moreover, it follows from Equations (3) to (10), and since $\sum_{i=1}^{l'} s^{(i)} \leq t^{(l'+1)}$, that $U\mathbf{X}^{(i)}$ and $U\mathbf{Y}^{(j)}$, for $1 \leq i \leq l'$ and $l'+1 \leq j \leq l$, are non-negative vectors such that

$$
\sum_{i=1}^{l'} U\mathbf{X}^{(i)} \leq U\mathbf{Y}^{(l'+1)} \leq U\mathbf{Z},
$$

and

$$
U\mathbf{Y}^{(j)} \leq U\mathbf{Z}.
$$

We now do backward elementary equivalences with a transfer matrix denoted by $V$. By Lemma 7.1, there is a backward elementary equivalence from $(\mathbf{l}, R, U\mathbf{Z})$ to an $\mathbb{N}$-representation $(\mathbf{i}', M', \mathbf{t}')$ such that $\mathbf{t}'$ is a positive right $k$-eigenvector of $M'$ which has all its coefficients equal to 1. Thus $M'$ is a $k$-ary matrix. The vector $\mathbf{i}'$ has $m$ entries 1, the other ones being 0. Moreover, for $1 \leq i \leq l$, there are non-negative integral vectors $\mathbf{t}'^{(i)}$, such that $\mathbf{t}'^{(i)} = VU\mathbf{X}^{(i)}$, for $1 \leq i \leq l'$, $\mathbf{t}'^{(l'+1)} = V(U\mathbf{Y}^{(l'+1)} - \sum_{i=1}^{l'} U\mathbf{X}^{(i)})$ and $\mathbf{t}'^{(i)} = VU\mathbf{Y}^{(i)}$, for $l'+2 \leq i \leq l$. Then $(\mathbf{i}', M', \mathbf{t}'^{(i)})$, for $1 \leq i \leq l$, specifies $s^{(i)}$ and $\sum_{i=1}^{l} \mathbf{t}'^{(i)} = \mathbf{t}'$. $\square$

*Example* 7.6. Let us consider the sequences $s_1$ and $s_2$ specified by the $\mathbb{N}$-representations $(\mathbf{i}, M, \mathbf{t}_1)$ and $(\mathbf{i}, M, \mathbf{t}_2)$ respectively, where

$$\mathbf{i} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}, \quad \mathbf{t}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{t}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

These $\mathbb{N}$-representations of $s_1$ and $s_2$ are pictured in Figure 3. The sequence $s(z) = s_1(z) + s_2(z)$ is equal to $(3z)^*$, and the sequence $s_1$ and $s_2$ have both a complexity equal to 3. The spectral radius of $M$ is 3. We successively get

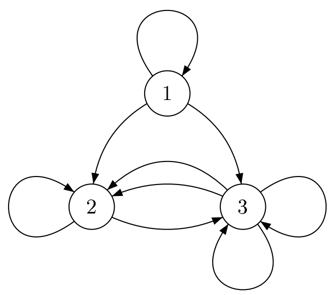


Fig. 3. The $\mathbb{N}$-representations $(\mathbf{i}, M, \mathbf{t}_1)$ and $(\mathbf{i}, M, \mathbf{t}_2)$.

$$\begin{aligned} \mathbf{i} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \\ \mathbf{i}M &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \\ \mathbf{i}M^2 &= \begin{bmatrix} 1 & 4 & 4 \end{bmatrix} = 4\mathbf{i}M - 3\mathbf{i}, \end{aligned}$$

Thus one can choose for $U$ the $2 \times 3$ matrix whose rows are $\mathbf{i}$ and $\mathbf{i}M$ with

$$\mathbf{j} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The matrix $N$ is spectrally Perron with spectral radius 3, and $\mathbf{x}$ is a right eigenvector of $N$ for the eigenvalue 3. The next computation is detailed in the example of Section 5. We can choose for $V$ the $2 \times 2$ matrix whose rows are $\mathbf{j}$ and $\mathbf{u}$, where $\mathbf{u} = \begin{bmatrix} -1 & 1 \end{bmatrix}$ (see Section 5), with

$$\mathbf{k} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The $\mathbb{N}$-representation $(\mathbf{k}, L, \mathbf{y}_1)$ of $s_1$ is pictured in Figure 2.

The final representation is indexed by the set $\{(1,1), (2,1), (2,2)\}$ and one can choose

$$\mathbf{i}' = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad M' = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad \mathbf{t}' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{t}'_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{t}'_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus the sequence $s_1$ is specified by the graph of Figure 4 where the final states are $(1,1)$ and $(2,2)$, and where the initial state is $(1,1)$. The sequence $s_2$ is specified by the same graph where the final state is $(2,1)$.

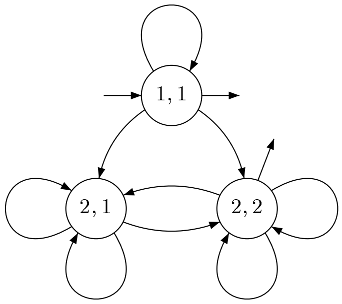


Fig. 4. An $\mathbb{N}$-representation of $s_1$ whose graph is 3-ary.

From the previous result, we get the following corollary.

COROLLARY 7.7. *Let $k$ be a positive integer and $s$ be a regular sequence of non-negative integers that has a complexity strictly less than $k$. Then $s$ is the sum of generating sequences of regular languages on $k$ symbols.*

PROOF. Since $s$ is regular and has a complexity strictly less than $k$, there is a positive integer $m$ such that its terms $s_n$ are bounded by $mk^n$. Moreover the complementary sequence of $s$ is regular by Soittola's theorem. The result is then a consequence of Theorem 7.5 for the case $l = 2$. □

Finally, we mention an open problem and a general question. Suppose that we are given a regular language $X$ and two regular sequences $s, t$ such that $s + t$ is the generating sequence of $X$. Is it true that there exists a partition $X = Y + Z$ such that $s$ is the generating sequence of $Y$ and $t$ is the generating sequence of $Z$? By Theorem 3.2, the answer is yes when $X$ is the set of all words on $k$ symbols. We wonder whether the result holds in general.

A more general question is the following. Soittola's theorem characterizes regular sequences among $\mathbb{Z}$-rational ones. Such a characterization is not known in several variables. In particular it is not known when the difference of two $\mathbb{N}$-rational sets is $\mathbb{N}$-rational. An answer to this question would certainly enlighten the field of automata with multiplicities.

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