ELSEVIER

Theoretical Computer Science 178 (1997) 171–203

Theoretical Computer Science

# Sturmian words, Lyndon words and trees

Jean Berstel [a], Aldo de Luca [b,*]

[a] *LITP, Université Paris 6, 4, Place Jussieu, F-75252, Paris, France*
[b] *Dipartimento di Matematica Università di Roma "La Sapienza", Piazzale Aldo Moro 2, 00185, Roma, Italy*

Received May 1995; revised February 1996
Communicated by D. Perrin

**Abstract**

We prove some new combinatorial properties of the set *PER* of all words $w$ having two periods $p$ and $q$ which are coprimes and such that $w = p+q-2$ [4, 3]. We show that $aPERb \cup \{a,b\} = St \cap Lynd$, where *St* is the set of the finite factors of all infinite Sturmian words and *Lynd* is the set of the Lyndon words on the alphabet $\{a,b\}$. It is also shown that $aPERb \cup \{a,b\} = CP$, where *CP* is the set of Christoffel primitive words. Such words can be defined in terms of the 'slope' of the words and of their prefixes [1]. From this result one can derive in a different way, by using a theorem of Borel and Laubie, that the elements of the set *aPERb* are Lyndon words. We prove the following correspondence between the ratio $p/q$ of the periods $p, q, p \leqslant q$ of $w \in PER \cap a\{a,b\}^*$ and the slope $\rho = (|w|_b + 1)/(|w|_a + 1)$ of the corresponding Christoffel primitive word $awb$: If $p/q$ has the development in continued fractions $[0, h_1, \ldots, h_{n-1}, h_n + 1]$, then $\rho$ has the development in continued fractions $[0, h_n, \ldots, h_2, h_1 + 1]$. This and other related results can be also derived by means of a theorem which relates the developments in continued fractions of the Stern–Brocot and the Raney numbers of a node in a complete binary tree. However, one needs some further results. More precisely we label the binary tree with standard pairs (standard tree), Christoffel pairs (Christoffel tree) and the elements of *PER* (Farey tree). The main theorem is the following: If the node $W$ is labeled by the standard pair $(u, v)$, by the Christoffel pair $(x, y)$ and by $w \in PER$, then $uv = wab$, $xy = awb$. The Stern–Brocot number $SB(W)$ is equal to the slope of the standard word $uv$ and of the Christoffel word $xy$ while the Raney number $Ra(W)$ is equal to the ratio of the minimal period of $wa$ and the minimal period of $wb$. Some further auxiliary results are also derived.

## 1. Introduction

Sturmian words can be defined in several different but equivalent ways. Some definitions are 'combinatorial' and others of 'geometrical' nature. With regard to the first type of definition a Sturmian word is *a binary infinite word which is not ultimately periodic and is of minimal subword complexity*. A 'geometrical' definition is the following: A

* Corresponding author.


*PII* S0304-3975(96)00101-6

Sturmian word can be defined by considering the sequence of the intersections with a squared-lattice of a semi-line having a slope which is an irrational number. A horizontal intersection is denoted by the letter $b$, a vertical intersection by $a$ and an intersection with a corner by $ab$ or $ba$. Sturmian words represented by a semi-line starting from the origin are usually called *standard* or *characteristic*. They are of great interest from the language point of view since one can prove that the set of all finite subwords of a Sturmian word depends only on the slope of the corresponding semi-line. We denote by *St* the set of all finite subwords of all Sturmian words.

Infinite standard Sturmian words can be also introduced as follows: Let $(c_0, c_1, c_2, \ldots)$ be any infinite sequence of integers such that $c_0 \geqslant 0$ and $c_i > 0$ for all $i > 0$. We define the infinite sequence of words $\{s_n\}_{n\geqslant 0}$, where $s_0 = b$, $s_1 = a$ and for all $n \geqslant 1$, as

$$s_{n+1} = s_n^{c_{n-1}} s_{n-1}.$$

One can prove that the sequence $\{s_n\}_{n\geqslant 0}$ converges to an infinite sequence $\mathbf{s}$ which is a standard Sturmian word; moreover any infinite standard Sturmian word can be obtained by the preceding procedure. The sequence $\{s_n\}_{n\geqslant 0}$ is called the *approximating sequence* of $\mathbf{s}$ and $(c_0, c_1, c_2, \ldots)$ the *directive sequence* of $\mathbf{s}$. The Fibonacci word $\mathbf{f}$ is the standard Sturmian word whose directive sequence is $(1, 1, \ldots, 1, \ldots)$.

We denote by **Stand** the set of all infinite standard Sturmian words. A word $w \in \{a, b\}^*$ is called a *finite standard Sturmian word*, or a *generalized finite Fibonacci word*, if there exists $\mathbf{s} \in$ **Stand** and an integer $n \geqslant 0$ such that $w = s_n$, where $\{s_n\}_{n\geqslant 0}$ is the approximating sequence of $\mathbf{s}$.

We shall denote by *Stand* the set of all finite standard Sturmian words. We say that a standard word $s \in Stand$ has the *directive sequence* $(c_0, c_1, \ldots, c_n)$, with $c_0 \geqslant 0$, $c_i > 0$, $1 \leqslant i \leqslant n$, if there exists a sequence of standard words $s_0, s_1, \ldots, s_n, s_{n+1}, s_{n+2}$ such that

$$s_0 = b, \quad s_1 = a, \quad s_{i+1} = s_i^{c_{i-1}} s_{i-1}, \quad 1 \leqslant i \leqslant n+1$$

and $s = s_{n+2}$. One can prove that any standard word has a *unique* directive sequence [3].

A more set-theoretic definition of *Stand*, based on the Rauzy's [12] method of construction of infinite standard Sturmian words, is the following. Let $\mathscr{A} = \{a, b\}$. We consider the smallest subset $\mathscr{R}$ of $\mathscr{A}^* \times \mathscr{A}^*$ which contains the pair $(a, b)$ and closed under the property:

$$(u, v) \in \mathscr{R} \quad \Rightarrow \quad (u, uv), (vu, v) \in \mathscr{R}.$$

Let us set

$$\mathscr{R}_0 = \{(a, b)\}$$

and define for $n \geqslant 0$

$$\mathscr{R}_{n+1} = \{(u, v) \mid \exists (x, y) \in \mathscr{R}_n : u = x, \ v = xy \ or \ u = yx, \ v = y\}.$$

Thus

$$\mathscr{R} = \bigcup_{n\geqslant 0} \mathscr{R}_n.$$

We call $\mathscr{R}$ also the set of *standard pairs*. Let us denote by $Trace(\mathscr{R})$ the set:

$$Trace(\mathscr{R}) = \{u \in \mathscr{A}^* \mid \exists v \in \mathscr{A}^* \text{ such that } (u,v) \in \mathscr{R} \text{ or } (v,u) \in \mathscr{R}\}.$$

One can easily prove that

$$Stand = Trace(\mathscr{R}).$$

The set *Stand* has several characterizations, that we shall use in the following, based on quite different concepts:

(a) *Palindrome words*. Let $PAL$ be the set of all palindromes on $\mathscr{A}$. The set $\Sigma$ is the subset of $\mathscr{A}^*$ defined by

$$\Sigma = \mathscr{A} \cup (PAL^2 \cap PAL\{ab, ba\}).$$

Thus a word $w$ belongs to $\Sigma$ if and only if $w$ is a single letter or satisfies the equation

$$w = AB = Cxy,$$

where $A, B, C \in PAL$ and $\{x,y\} = \{a,b\}$. A remarkable result obtained by Pedersen [10] is that there exists, and it is *unique*, a word $W$ such that

$$W = AB = Cxy,$$

if and only if $gcd(|A|+2, |B|-2) = 1$.

It was proved in [4] that

$$Stand = \Sigma.$$

(b) *Periodicities of the words*. Let $w \in \mathscr{A}^*$ and $\Pi(w)$ be the set of its periods (cf.[7], recall that a positive integer $p < n$ is a proper *period* of $w = a_1 \cdots a_n$, where $a_1, \ldots, a_n$ are letters, if $a_i = a_{i+p}$ for $i = 1, \ldots, n-p$; however, any integer $p \geqslant n$ is assumed also as a period of $w$.) We define the set $PER$ of all words $w$ having two periods $p, q \in \Pi(w)$ which are coprimes and such that $|w| = p+q-2$. Thus a word $w$ belongs to $PER$ if it is a power of a single letter or is a word of maximal length for which the theorem of Fine and Wilf does not apply. In the sequel we assume that $\varepsilon \in PER$. This is, formally, coherent with the above definition if one takes $p = q = 1$. In [4] de Luca and Mignosi proved that

$$Stand = \mathscr{A} \cup PER\{ab, ba\}.$$

(c) *Special elements*. A word $w \in St$ is a *right* (*left*) *special element* of $St$ if $wa, wb \in St$ ($aw, bw \in St$). The word $w$ is *bispecial* if it is right and left special. It is called *strictly bispecial* if

$$awa, awb, bwa, bwb \in St.$$

Let us denote by *SBS* the set of all strictly bispecial elements of *St*. It has been proved in [4] that

$$PER = SBS.$$

If $S_R$ ($S_L$) denotes the set of all right (left) special elements of *St* one has that [3]:

$$SBS = S_R \cap PAL = S_L \cap PAL.$$

(d) *Palindrome closures*. Let us introduce the map $(-) : \mathscr{A}^* \to PAL$ which associates to any word $w \in \mathscr{A}^*$ the palindrome word $w^{(-)}$ defined as the smallest palindrome word having the suffix $w$. We call $w^{(-)}$ the *palindrome left-closure of* $w$.

If $X$ is a subset of $\mathscr{A}^*$ we denote by $X^{(-)}$ the set

$$\{w^{(-)} \in \mathscr{A}^* \mid w \in X\}.$$

Let us define inductively the sequence $\{X_n\}_{n\geqslant 0}$ of finite subsets of $\mathscr{A}^*$ by

$$X_0 = \{\varepsilon\}$$
$$X_{n+1} = (\mathscr{A}X_n)^{(-)}, \quad n \geqslant 0.$$

Thus $s \in X_{n+1}$ if and only if there exist $x \in \mathscr{A}$ and $t \in X_n$ such that $s = (xt)^{(-)}$.

We shall set $\mathscr{L} = \bigcup_{n\geqslant 0} X_n$. One can prove [3] that $\mathscr{L} = PER$. We can summarize the above results in the following basic theorem.

**Theorem 1.1.**
(1) $Stand = \Sigma$,
(2) $\Sigma = \mathscr{A} \cup PER\{ab, ba\}$
(3) $PER = SBS$,
(4) $PER = \mathscr{L}$.

This theorem has several applications. In particular, one can determine the subword complexity of *Stand* and derive in a simple and purely combinatorial way, the subword complexity formula for *St* [4]. An application of Theorem 1.1 to the study of Sturmian words generated by iterated morphisms was recently given in [2].

Theorem 1.1 shows that the 'kernel' of the set of standard Sturmian words is the set *PER*. In this paper we shall present some new combinatorial results concerning the structure, the combinatorics, and the arithmetics of *PER*. These results are relevant for all finite Sturmian words because the set *St* is equal to the set of all subwords of *PER*.

In Section 3 we show that $aPERb \cup \mathscr{A} = St \cap Lynd$, where *Lynd* is the set of the Lyndon words on the alphabet $\mathscr{A}$.

In Section 4 we prove that $aPERb \cup \mathscr{A} = CP$, where *CP* is the set of Christoffel primitive words. Such words can be defined in terms of the 'slope' of the words and of their prefixes [1]. From this result one can derive in a different way, by using a theorem of Borel and Laubie [1], that the elements of the set *aPERb* are Lyndon words.

In Section 5 we prove the existence of the following correspondence between the ratio $p/q$ of the periods $p, q,\ p \leqslant q$ of $w \in PER \cap a\mathscr{A}^*$ and the slope $\rho = (|w|_b + 1)/(|w|_a + 1)$ of the corresponding Christoffel primitive word $awb$ :If $p/q$ has the development in continued fractions $[0, h_1, \ldots, h_{n-1}, h_n + 1]$, then $\rho$ has the development in continued fractions $[0, h_n, \ldots, h_2, h_1 + 1]$.

In Section 6 we consider binary complete trees described by binary words and unimodular matrices of order 2. Each node can be labeled by an irreducible fraction in several ways. From a general theorem we derive an interesting result which relates the developments in continued fractions of the Stern–Brocot and the Raney number of a node.

In Section 7 we label the binary tree with standard pairs (standard tree), Christoffel pairs (Christoffel tree) and with the elements of *PER* (Farey tree). We prove the following main result: If the node $W$ is labeled by the standard pair $(u, v)$, by the Christoffel pair $(x, y)$ and by $w \in PER$, then $uv = wab$, $xy = awb$. The Stern–Brocot number $SB(W)$ is equal to the slope of the standard word $uv$ and of the Christoffel word $xy$ while the Raney number $Ra(W)$ is equal to the ratio of the minimal period of $wa$ and the minimal period of $wb$. Some further auxiliary results are also derived. In particular we give a different proof of the result of Section 5.

## 2. Preliminaries

For all notations and definitions not given in the text the reader is referred to the book of Lothaire [7].

In the following $\mathscr{A}$ will denote a finite alphabet and $\mathscr{A}^*$ the free monoid on it, i.e. the set of all finite sequences of symbols from $\mathscr{A}$, including the *empty* sequence $\varepsilon$. The elements of $\mathscr{A}$ are usually called *letters* and those of $\mathscr{A}^*$ *words*.

For any word $w \in \mathscr{A}^*$, $|w|$ denotes its *length*, i.e. the number of letters occurring in $w$. The length of $\varepsilon$ is taken to be equal to 0. For any letter $a \in \mathscr{A}$, $|w|_a$ will denote the number of occurrences of the letter $a$ in $w$. A word $u$ is a *factor*, or *subword*, of $w$ if $w \in \mathscr{A}^* u \mathscr{A}^*$, i.e. there exist $x, y \in \mathscr{A}^*$ such that $w = xuy$. The factor $u$ is called *proper* if $u \neq w$. If $x = \varepsilon (y = \varepsilon)$, then $u$ is called a *prefix* (*suffix*) of $w$. By $F(w)$ we denote the set of all factors of $w$. If $u \in \mathscr{A}^*$ and $X$ is a subset of $\mathscr{A}^*$, then $u^{-1}X$ will denote the set

$$u^{-1}X = \{v \in \mathscr{A}^* \mid uv \in X\}.$$

For any $w \in \mathscr{A}^*$, $alph(w)$ denotes the smallest subset of $\mathscr{A}$ containing all the letters occurring in $w$.

In the next we shall refer mainly to the alphabet $\mathscr{A} = \{a, b\}$. We totally order the alphabet $\mathscr{A}$ by setting $a < b$. We can then totally order $\mathscr{A}^*$ by the *lexicographic order* $\leqslant$ defined as [7]: for all $u, v \in \mathscr{A}^*$

$$u \leqslant v \iff v \in u\mathscr{A}^* \text{ or } u = ha\xi,\ v = hb\eta,\ h, \xi, \eta \in \mathscr{A}^*.$$

If $u \leqslant v$ and $u \neq v$, then we write $u < v$. We recall that two words $f, g \in \mathscr{A}^*$ are *conjugate* if there exist words $u, v \in \mathscr{A}^*$ such that $f = uv$ and $g = vu$. If $u$ and $v$ are different from the empty word then $f$ and $g$ are said *properly* conjugate. The conjugation relation is an equivalence relation in $\mathscr{A}^*$.

A word $f$ is *primitive* if it is not properly self-conjugate. This is also equivalent to the statement that $f \neq w^p$ for all $w \in \mathscr{A}^*$ and $p > 1$.

The *reversal operation* or *mirror image* ($\tilde{\ }$) is the unary operation in $\mathscr{A}^*$ recursively defined as $\tilde{\varepsilon} = \varepsilon$ and $\widetilde{(ua)} = a\tilde{u}$, for all $u \in \mathscr{A}^*$ and $a \in \mathscr{A}$. The mirror image is involutory and such that for all $u, v \in \mathscr{A}^*$, $\widetilde{(uv)} = \tilde{v}\tilde{u}$, i.e. it is an involutory antiautomorphism of $\mathscr{A}^*$. For any $L$ subset of $\mathscr{A}^*$ we set $\tilde{L} = \{\tilde{w} |\ w \in L\}$. A word $w$ which coincides with its mirror image is called *palindrome*. The set of all palindromes over $\mathscr{A}$ is denoted by $PAL(\mathscr{A})$, or simply, by $PAL$.

A word $w \in \mathscr{A}^*$ is called a *Lyndon* word over $\mathscr{A}$ if $w$ is less, w.r.t. the lexicographic order, than every proper right factor. We denote by $Lynd$ the set of all the Lyndon words over the alphabet $\{a, b\}$. There exist several characterizations, or equivalent definitions, of the Lyndon words [7]. We recall here that $w \in Lynd$ if and only if it is primitive and minimal in its conjugation class.

We recall [7] that if $l, m \in Lynd$ and $l < m$, then $lm \in Lynd$. Conversely, any word $w \in Lynd \backslash \mathscr{A}$ can be always factorized, and generally in several ways, as $w = lm$ with $l, m \in Lynd$. When $m$ is the suffix of $w \in Lynd$ of maximal length such that $m \in Lynd$ then one has $l \in Lynd$ and $l < lm < m$. This factorization $(l, m)$ is also called the *standard factorization* of $w$.

A word $w \in \mathscr{A}^*$ is called *anti-Lyndon* word if $w$ is primitive and maximal in its conjugation class. Let us introduce in $\mathscr{A}^*$ the automorphism ($\hat{\ }$) defined as : $\hat{a} = b, \hat{b} = a$. Thus $\hat{\varepsilon} = \varepsilon$ and for any $w \in \mathscr{A}^*$, $w \neq \varepsilon$, $\hat{w}$ is obtained from $w$ by interchanging the letter $a$ with $b$. For a subset $L$ of $\mathscr{A}^*$ we set $\hat{L} = \{\hat{w} |\ w \in L\}$. One easily verifies that if $u \leqslant v$ and $|u| = |v|$ then $\hat{u} \geqslant \hat{v}$ so that $w$ is a Lyndon word if and only if $\hat{w}$ is anti-Lyndon. Thus the set of all anti-Lyndon words coincides with $\widehat{Lynd}$.

An infinite word (from left to right) $\mathbf{x}$ over $\mathscr{A}$ is any map $\mathbf{x} : N \to \mathscr{A}$. For any $i \geqslant 0$, we set $x_i = \mathbf{x}(i)$ and write

$$\mathbf{x} = x_0 x_1 \ldots x_n \ldots .$$

The set of all infinite words over $\mathscr{A}$ is denoted by $\mathscr{A}^\omega$. A word $u \in \mathscr{A}^*$ is a (finite) factor of $\mathbf{x} \in \mathscr{A}^\omega$ if $u = \varepsilon$ or there exist integers $i, j$ such that $i \leqslant j$ and $u = x_i \ldots x_j$. The set of all finite factors of $\mathbf{x}$ is denoted by $F(\mathbf{x})$.

The *subword complexity* of the infinite word $\mathbf{x}$ is the map $g_{\mathbf{x}} : N \to N$, defined as follows: for all $n \geqslant 0$,

$$g_{\mathbf{x}}(n) = Card(F(\mathbf{x}) \cap \mathscr{A}^n).$$

An infinite word $\mathbf{x} \in \mathscr{A}^\omega$ is *ultimately periodic* if there exist words $u, v \in \mathscr{A}^*$ such that $\mathbf{x} = uv..v.. = uv^\omega$. One can easily prove that an infinite word $\mathbf{x}$ is ultimately periodic if and only if there exists an integer $n$ such that $g_{\mathbf{x}}(n) \leqslant n$.

Infinite Sturmian words are infinite words over the alphabet $\{a,b\}$ which are not ultimately periodic and of minimal subword complexity. This is equivalent to the following:

**Definition 1.** An infinite word $\mathbf{x} \in \mathscr{A}^{\omega}$ is Sturmian if and only if the subword complexity $g_{\mathbf{x}}$ satisfies the following condition: For all $n > 0$

$$g_{\mathbf{x}}(n) = n + 1.$$

Let us now define the set *St* of *finite* Sturmian words:

**Definition 2.** A word $w \in St$ if and only if there exists an infinite Sturmian word $\mathbf{x}$ such that $w \in F(\mathbf{x})$.

The following interesting and useful combinatorial characterization of the language *St* holds [9, 5]:

**Theorem 2.1.** *The language St is the set of all the words $w \in \{a,b\}^*$ such that for any pair $(u,v)$ of factors of $w$ having the same length one has*

$$||u|_a - |v|_a| \leqslant 1.$$

## 3. Lyndon and Sturmian words

In this section we shall prove that the set of Sturmian words which are Lyndon words coincides with the set $aPERb \cup \{a,b\}$. To this end we need some results concerning the set *PER* whose proof is in [3].

**Lemma 3.1.** *Let $w \in PER$ be such that $Card(alph(w)) > 1$. Then $w$ can be uniquely represented as*

$$w = PxyQ = QyxP,$$

*with $x, y$ fixed letters in $\{a,b\}$, $x \neq y$ and $P, Q \in PAL$. Moreover, $gcd(p,q) = 1$, where $p = |P| + 2$ and $q = |Q| + 2$.*

**Theorem 3.1.** *Let $w \in PER$ and $x \in \{a,b\}$. Then $(xw)^{(-)} \in PER$. Moreover, if $w = PxyQ$, with $P, Q \in PAL$ and $\{x,y\} = \{a,b\}$, then one has*

$$(xw)^{(-)} = QyxPxyQ, \quad (yw)^{(-)} = PxyQyxP.$$

**Lemma 3.2.** *If $w \in \mathscr{A}^*$ and $awb \in Lynd \cap St$, then $w \in PAL$.*

**Proof.** If $w = \varepsilon$ or $w \in \{a,b\}$, then the result is trivial. Let us then suppose that $|w| > 1$; we can write $w = w_1 \ldots w_n$ with $n > 1$ and $w_i \in \mathscr{A}$, $i \in [1,n]$. By hypothesis the word $awb$ is a Sturmian word which is Lyndon.

Let us first prove that $w_1 = w_n$. If $w_1 = a$ then $w_n$ has to be equal to $a$ in view of the fact that $awb$ is a Sturmian word. Indeed, by Theorem 2.1 the word $awb$ cannot begin with $aa$ and terminate with $bb$. If $w_1 = b$ then $w_n = b$ in view of the fact that $awb$ is a Lyndon word. Indeed, otherwise the Lyndon word $awb$ will begin and terminate with the same factor $ab$ which is a contradiction.

Suppose, by induction, that we have already proved that $w_j = w_{n-j+1}$ for $j = 1,\ldots,k$, $k < [n/2]$. We want to prove that $w_{j+1} = w_{n-j}$. By using the inductive hypothesis we can also write $awb$ as

$$awb = auw_j \ldots w_{n-j}\tilde{u}b.$$

If $w_j = a$, then $w_{n-j} = a$ since $awb$ is Sturmian. Otherwise one would have

$$||aua|_a - |b\tilde{u}b|_a| = 2,$$

which is a contradiction in view of Theorem 2.1. If $w_j = b$, then $w_{n-j} = b$. Indeed, suppose by contradiction that $w_{n-j} = a$. Since $awb \in St$ let $\mathbf{x}$ be an infinite standard Sturmian word such that $awb \in F(\mathbf{x})$. One has then $aub, a\tilde{u}b \in F(\mathbf{x})$. This implies [4, Proposition 9]

$$aub, b\tilde{u}a, a\tilde{u}b, bua \in F(\mathbf{x}).$$

Hence $u$ and $\tilde{u}$ are factors of $F(\mathbf{x})$ which are (right) special in $\mathbf{x}$ [4, Definition 10]. Since for any length, $\mathbf{x}$ admits only one (right) special factor in $\mathbf{x}$ of that length, it follows that $u = \tilde{u}$. Hence the Lyndon word $awb$ will begin and terminate with the same factor $aub$ which is a contradiction. □

**Theorem 3.2.** $\mathscr{A} \cup aPERb = Lynd \cap St$.

**Proof.** Let us first prove the inclusion $\subseteq$. If $u \in \mathscr{A}$ the result is trivial. Let us then suppose that $u \in aPERb$ and write $u = awb$ with $w \in PER$. We shall prove the result by using induction on the length of the word $w$.

*Base of the induction.* If $w = \varepsilon$ then $u = ab \in Lynd$. If $w = a$ or $w = b$ one has, respectively, $u = aab$ or $u = abb$ and $aab, abb \in Lynd$.

*Induction step.* Let $w \in PER$ and suppose that $|w| > 1$. By Theorem 1.1, $PER = \mathscr{L}$, so that an integer $n \geqslant 1$ exists such that $w \in X_{n+1}$. Hence we can write

$$w = (xv)^{(-)},$$

with $x \in \mathscr{A}$ and $v \in X_n$. Let us first consider the case when $Card(alph(v)) = 1$. If $x = a$ and $v = a^{|v|}$, then $xv = a^{|v|+1}$ so that $(xv)^{(-)} = a^{|v|+1}$ and $awb = a^{|v|+2}b \in Lynd$. If $x = a$ and $v = b^{|v|}$, then $w = (xv)^{(-)} = b^{|v|}ab^{|v|}$ and $u = awb = ab^{|v|}ab^{|v|+1} \in Lynd$. If $x = b$ and $v = b^{|v|}$ one has $u = awb = ab^{|v|+2} \in Lynd$. Finally, if $x = b$ and $v = a^{|v|}$ one derives $u = awb = a^{|v|+1}ba^{|v|}b \in Lynd$. Let us then suppose that $Card(alph(v)) = 2$. By Lemma 3.1 one can uniquelly represent $v$ as

$$v = PxyQ = QyxP,$$

with $P, Q \in PAL$ and $x, y \in \{a, b\}$ and $x \neq y$. By Theorem 3.1 one has $(xv)^{(-)} = QyxPxyQ$ and

$$u = awb = aQyxPxyQb.$$

Let us first suppose that $y = b$. The word $u$ becomes

$$u = aQbaPabQb.$$

Since $Q, PabQ \in PER$ and $|Q|, |PabQ| < |w|$ one has by induction

$$aQb = l_1, aPabQb = l_2 \in Lynd.$$

Thus

$$u = l_1 l_2.$$

Since $aPabQb = aQbaPb$ one has $l_1 < l_2$ so that [7] $u = l_1 l_2 \in Lynd$.

Suppose now that $y = a$. One has

$$u = aQabPbaQb.$$

Let us set $l_1 = aQabPb$ and $l_2 = aQb$. By the inductive hypothesis $l_1, l_2 \in Lynd$. Since $l_1 < l_2$ one derives that also in this case $u = l_1 l_2 \in Lynd$.

We have then proved that $\mathscr{A} \cup aPERb \subseteq Lynd$. Let $u \in \mathscr{A} \cup aPERb$. If $u \in \mathscr{A}$ then one, trivially, has $u \in St$. Let us then suppose that $u \in aPERb$, i.e. $u = awb$ with $w \in PER$. By Theorem 1.1, $PER = SBS$, so that $awb \in St$.

Let us now prove the inverse inclusion $\supseteq$. Let $u \in Lynd \cap St$. If $|u| = 1$ the result is trivial. Suppose then that $|u| > 1$. Since $u \in Lynd$ then $u \in a\mathscr{A}^* b$. Indeed, $u$ cannot begin and terminate with the same letter and, moreover, necessarily the first letter has to be $a$ and the last $b$. Let us write $u$ as $u = awb$ with $w \in \mathscr{A}^*$. We want to prove that $w \in PER$. From Lemma 3.2 one has that $w \in PAL$. Since $u = awb \in St$ there exists an infinite standard Sturmian word $\mathbf{x}$ such that $u = awb \in F(\mathbf{x})$. This implies [4] that

$$awb, bwa \in F(\mathbf{x}).$$

Hence $w$ is a palindrome (right) special element of $St$. This implies [3] $w \in SBS = PER$. $\square$

**Corollary 3.1.** $\mathscr{A} \cup bPERa = \widehat{Lynd} \cap St = \widetilde{Lynd} \cap St$.

**Proof.** It is a straightforward consequence of the above theorem. One has only to observe that the sets $\mathscr{A}$, $St$ and $PER$ are invariant under the reversal operation ( $\tilde{}$ ) and the involutory automorphism ( $\hat{}$ ) which interchanges the letter $a$ with the letter $b$. $\square$

**Remark 1.** From the preceding corollary it follows that the set of Lyndon words which are Sturmian words is invariant under the involutory anti-automorphism of $\mathscr{A}^*$

which is the composition of the operations ($\tilde{\ }$) and ($\hat{\ }$). This is not the case for the set *Lynd*. For instance, the word $w = aababbab \in Lynd$, whereas $\hat{\tilde{w}} = abaababb \notin Lynd$.

**Corollary 3.2.** *If $w \in aPERb$ then there exist and are unique two words $u, v \in aPERb$ such that $w = uv$. The factorization $(u, v)$ of $w$ is the standard factorization of $w$ in Lyndon words.*

**Proof.** If $w \in aPERb$, then from Theorem 3.2, $w \in Lynd$, so that $w$ can be factorized in the standard way as $w = lm$ with $l, m \in Lynd \cap St$. Hence, by Theorem 3.2 one has

$$l = aPb, \quad m = aQb, \quad w = lm = aPbaQb,$$

with $P, Q, PbaQ \in PER$. Let $(l', m')$ be another factorization of $w$, i.e. $w = l'm'$ with $l' = aP'b, m' = aQ'b$ with $P', Q', P'baQ' \in PER$. Thus $PbaQ = P'baQ'$ and this is absurd in view of Lemma 3.1. □

**Corollary 3.3.** *The enumeration function $g$ of the set $Lynd \cap St$ is given by $g(1) = 2$ and $g(n) = \phi(n)$ for $n > 1$, where $\phi$ is Euler's function.*

**Proof.** Clearly $g(1) = 2$. For $n > 1$ the above theorem has shown that there exists a bijection $\zeta_n : PER \cap \mathscr{A}^n \to Lynd \cap St \cap \mathscr{A}^{n+2}$ defined by: for $w \in PER$,

$$\zeta_n(w) = awb.$$

Since for any $n \geqslant 0$, $Card(PER \cap \mathscr{A}^n) = \phi(n+2)$ [4, Lemma 5], where $\phi$ is the Euler function, it follows that for $n \geqslant 2$

$$g(n) = Card(PER \cap \mathscr{A}^{n-2}) = \phi(n). \quad \square$$

Let us now set

$$\Sigma_a = \Sigma \cap \mathscr{A}^* a, \quad \Sigma_b = \Sigma \cap \mathscr{A}^* b.$$

**Corollary 3.4.** $\Sigma_a = a^{-1}(Lynd \cap St)a$, $\Sigma_b = b^{-1}(\widehat{Lynd} \cap St)b$.

**Proof.** By Theorem 3.2 one has $Lynd \cap St = aPERb \cup \mathscr{A}$. Thus

$$a^{-1}(Lynd \cap St)a = PERba \cup \{a\}.$$

By Theorem 1.1, $\Sigma = \mathscr{A} \cup PER\{ab, ba\}$ so that $\Sigma_a = PERba \cup \{a\}$. In a similar way by using Corollary 3.1 one obtains $\Sigma_b = b^{-1}(\widehat{Lynd} \cap St)b$. □

## 4. Christoffel words

In this section we shall prove that the set $aPERb \cup \mathscr{A}$ coincides with the set $CP$ of Christoffel primitive words. These words introduced by Christoffel, and recently reconsidered in depth by Borel and Laubie [1], may be defined in terms of the notion of ‘slope’ of a binary word. The proof of this equality is remarkable by the fact that the elements of the set *PER* are definable in terms of ‘periods’ of the words.

Let $\mathscr{A} = \{a, b\}$. We consider the map

$$\rho : \mathscr{A}^* \to \mathbb{Q} \cup \{\infty\},$$

defined by

$$\rho(\varepsilon) = 1, \quad \rho(w) = |w|_b/|w|_a, \quad \text{for } w \neq \varepsilon.$$

We assume that $1/0 = \infty$. For any $w \in \mathscr{A}^*$ we call $\rho(w)$ the ‘slope’ of $w$. For instance the words *abbabaa*, *aaa* and *bb* have, respectively, slopes $\frac{3}{4}$, 0 and $\infty$. For any $w \in \mathscr{A}^*$ and $k \in [1, |w|]$ we define the set

$$\delta_k(w) = \{u \in \mathscr{A}^k \mid \rho(u) \leqslant \rho(w)\}.$$

and denote by $\mu_k(w)$ the quantity

$$\mu_k(w) = max\{\rho(u) | \; u \in \delta_k(w)\}.$$

Note that $\delta_k(w)$ is always not empty since for any $k \in [1, |w|]$ the slope of the word $a^k$ is 0. For any word $w \in \mathscr{A}^*$ we denote by $w_{[k]}$ the prefix of $w$ of length $k$.

**Definition 3.** A word $w \in \mathscr{A}^*$ is called a (positive) word of Christoffel if for any $k \in [1, |w|]$ one has

$$\rho(w_{[k]}) = \mu_k(w).$$

Thus a word $w \in \mathscr{A}^*$ is a positive Christoffel word if and only if every prefix $w'$ of $w$ has a slope which is maximal with respect to the slope of any other word $u$ such that $|u| = |w'|$ and $\rho(u) \leqslant \rho(w)$.

In a symmetric way one can define a *negative* Christoffel word as a word $w$ such that every prefix $w'$ of $w$ has a slope which is minimal with respect to the slope of any other word $u$ such that $|u| = |w'|$ and $\rho(u) \geqslant \rho(w)$.

We denote by *CP* (*NCP*) the set of all positive (negative) Christoffel words which are primitive words. One can easily prove that $NCP = \widetilde{CP}$. In the following we shall refer mainly to positive Christoffel words that we simply call *Christoffel words*. Following Borel and Laubie [1] one can represent any word $w = w_1 \ldots w_n$, $w_i \in \{a, b\}$, $i \in [1, n]$, by a suitable graphic representation: to $w$ one can associate a ‘path’ in the lattice $N \times N$. One starts from the initial point $O$ of coordinates $(0, 0)$ and to each occurrence of the letter $a$ in $w$ corresponds a horizontal step oriented on the right and to each occurence of a letter $b$ corresponds a vertical step upwards. In this way one reaches a terminal point $Q$ of coordinates $(|w|_a, |w|_b)$. In Fig. 1 we drop the orientations and denote by $\mathscr{G}_w$ this graphic representation associated to $w$.

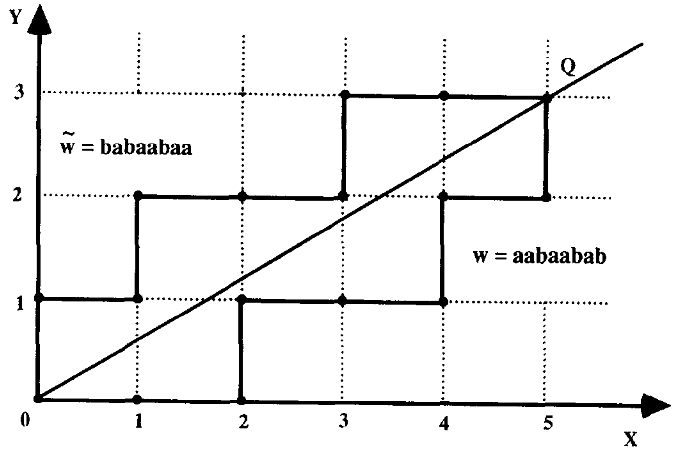


Fig. 1. The representation of Christoffel words.

Let $\rho = \rho(w)$ be the slope of the word $w$ and consider the semi-line $r$ of the equation

$$y = \rho x, \quad x \geqslant 0.$$

One has that the initial and terminal points $O$ and $Q$ of $\mathscr{G}_w$ lie on $r$. Moreover, one easily derives from Definition 3 that $w$ is a Christoffel word if and only if $\mathscr{G}_w$ lies in the plane's region $y \leqslant \rho x$, $x \geqslant 0$ and its vertices are the nearest possible to $r$. Indeed, for any prefix $w_{[k]}$, $k \in [1, n]$, one has $\rho(w_{[k]}) \leqslant \rho(w)$. This is equivalent to the statement that any vertex of $\mathscr{G}_w$ lies under or on the semi-line $r$. The fact that the vertices of $\mathscr{G}_w$ are the nearest possible to $r$ is equivalent to the statement that the slope of any prefix $w'$ of $w$ is maximal with respect to the slope of any other word $u \in \mathscr{A}^*$ such that $|u| = |w'|$ and $\rho(u) \leqslant \rho(w)$.

Christoffel words satisfy some noteworthy properties which follow from Definition 3 [1]:

**Proposition 4.1.** *Let $(p, q)$ be a pair of nonnegative integers such that $p + q > 0$. There exists, and it is unique, a Christoffel word $w$ having the slope $\rho(w) = p/q$ and length $|w| = p + q$.*

**Proposition 4.2.** *A Christoffel word $w$ is primitive if and only if any proper prefix $w'$ of $w$ has a slope $\rho(w') < \rho(w)$.*

**Proposition 4.3.** *Let $(p, q)$ be a pair of nonnegative integers such that $p + q > 0$. The corresponding word of Christoffel $w$ is primitive if and only if $d = gcd(p, q) = 1$. Moreover, if $d > 1$ then setting $p = dr$, $q = ds$ one has $w = u^d$ where $u \in CP$ and $|u| = r + s$.*

From the previous results one derives that there exists a one-to-one correspondence $\Phi$ of the set of all semi-lines starting from the origin and having a rational slope onto the set of all Christoffel primitive words. More precisely, if $r$ is a semi-line having the slope $p/q$ with $gcd(p,q) = 1$, then the corresponding Christoffel primitive word $w = \Phi(r)$ has the slope $\rho(w) = p/q$ and length $p+q$. Moreover, the graph $\mathscr{G}_w$ intersects the semi-line $r$ only in the points $O$ and $Q$ of coordinates $(0,0)$ and $(q,p)$, respectively.

However, one can associate to the semi-line $r$ a binary word by considering the sequence of crosses of $r$ with vertical and horizontal lines of the lattice $N \times N$. We refer to the open interval $(0,q)$ and an intersection of $r$ with a vertical line ($x = c,\ c > 0$) is denoted by the letter $a$ and an intersection with a horizontal line ($y = c,\ c > 0$) by the letter $b$. Let us denote by $\Psi(r)$ the word associated with the semi-line $r$. This is sometimes called "cutting sequence" [13] of $r$. The following proposition holds:

**Proposition 4.4.** *Let $r$ be a semi-line of equation $y = (p/q)x,\ x \geqslant 0$, having the rational slope $p/q$ with $p+q > 1$ and $gcd(p,q) = 1$. One has*

$$a\Psi(r)b = \Phi(r).$$

**Proof.** Let $r$ be a semi-line of equation $y = (p/q)x,\ x \geqslant 0$, having the rational slope $p/q$ with $gcd(p,q) = 1$. From this latter condition it follows that $r$ meets in the interval $(0,q)$ only the corners $(0,0)$ and $(q,p)$. Let us set now $\delta(x) = (p/q)x$ and consider for $n \in (0,q)$ the quantity:

$$\Delta_n = [\delta(n)] - [\delta(n-1)],$$

where $[x]$ denotes the integer part of $x$. For each $n = 1,\ldots,q-1$, $\Delta_n$ holds only 1 or 0. Moreover, one has that $\Delta_n = 0$ if and only if there is no intersection of $r$ with a horizontal line in the interval $(n-1,n)$. Let us then consider the word

$$u = u_1 \ldots u_{q-1},$$

where for $i \in (0,q)$,

$$u_i = a, \text{ if } \Delta_i = 0$$
$$u_i = ba, \text{ if } \Delta_i = 1.$$

One has then that $u$ is the cutting sequence $\Psi(r)$ of $r$. Moreover, one has that $|u|_b$ equals the number of crosses of $r$ in the interval $[1,q-1]$ with horizontal lines of the lattice $N \times N$. Since $r$ meets the corner $(q,p)$ it follows that $|u|_b = p-1$. In a similar way one derives that $|u|_a$ equals the number of crosses of $r$ in the interval $[1,q-1]$. Thus $|u|_a = q-1$. Hence,

$$|u| = |u|_a + |u|_b = p+q-2.$$

We can then write

$$u = a_1 \ldots a_{p+q-2}, \quad a_i \in \mathscr{A}, \quad i \in [1, p+q-2].$$

Let $w$ be the Christoffel primitive word corresponding to $r$, i.e. $w = \Phi(r)$. One has $|w| = p+q$. From the Definition 3 and Proposition 4.2 it follows that $w$ has to begin with $a$ and terminate with $b$. We can then write

$$w = avb,$$

with $|v| = p+q-2$. We want prove that $v = u$. Indeed, in the graph $\mathscr{G}_w$ we have a vertical line (which represents a $b$) joining two horizontal lines (and, therefore, two $a$) if and only if between two consecutive vertical crosses of $r$ (representing two letters $a$) we have a horizontal cross (which represents a $b$). Hence the sequence of letters in $v$ is the same that in $u$. This implies $u = v$. □

**Theorem 4.1.** $CP = aPERb \cup \mathscr{A}$.

**Proof.** Let us first prove the inclusion $\subseteq$. Let $w \in CP$. If $|w| = 1$, the inclusion is clear. Suppose then $|w| > 1$. We can write $w = aub$, $u \in \mathscr{A}^*$. If $u = \varepsilon$, then one has $w = ab \in CP$, $\rho(ab) = 1/1$ and $\varepsilon \in PER$. Let us then suppose that $u \neq \varepsilon$.

Since $w \in CP$ we can write $\rho(w) = p/q$, with $p+q > 1$, $gcd(p,q) = 1$ and $|w| = p+q$. We can consider the semi-line $r$ of equation $y = (p/q)x$, having the slope $p/q$. One has that $\Phi(r) = w$. Moreover, from Proposition 4.4 the cutting sequence $\Psi(r)$ of $r$, is such that $a\Psi(r)b = \Phi(r)$. Hence, $u = \Psi(r)$.

We can always rotate the semi-line $r$ a sufficiently small angle $\alpha$ in the anticlockwise sense in such a way that the slope of the new semi-line $r'$ becomes an irrational number and, moreover, $r'$ has the same sequence of crosses with horizontal and vertical lines in the interval $(0, q)$ as the semi-line $r$, i.e. $\Psi(r') = \Psi(r)$. In a similar way one can rotate the semi-line $r$ a sufficiently small angle $\alpha'$ in the clockwise sense in such a way that the slope of the new semi-line $r''$ becomes an irrational number and $\Psi(r'') = \Psi(r)$. Since the point $P$ of coordinates $(q, p)$ belongs to $r$, one has that in the interval $(0, q]$ the sequence of crosses of $r'$ is represented by the word $uba$, whereas the sequence of crosses of the semi-line $r''$ is represented by the word $ua$.

According to the usual definition of standard Sturmian infinite word, one derives that $ub$ and $ua$ are prefixes of two infinite standard Sturmian words $\mathbf{x}'$ and $\mathbf{x}''$, respectively. From a general result [8] it follows that

$$\widetilde{(ua)} = a\tilde{u} \quad \text{and} \quad \widetilde{(ub)} = b\tilde{u}$$

are right special elements of the words $\mathbf{x}''$ and $\mathbf{x}'$, respectively. Hence one derives that:

$$a\tilde{u}a,\ a\tilde{u}b \in St, \quad b\tilde{u}a,\ b\tilde{u}b \in St.$$

Thus $\tilde{u} \in SBS = PER$. Since $PER \subseteq PAL$ it follows $\tilde{u} = u \in PER$.

Let us now prove the inverse inclusion $\supseteq$. We know, from the general definition of the set $CP$ that $\mathscr{A} \subseteq CP$. Let us then consider the element $w = aub$ with $u \in PER$. Since $PER = SBS$, one has

$$aua, aub, bua, bub \in St.$$

Hence $au, bu \in S_R$. From a result proved in [3] $au$, as well as $bu$, is a right special element in an infinite standard Sturmian word. We can then write

$$au = \tilde{p}_1, \quad bu = \tilde{p}_2,$$

where $p_1$ and $p_2$ are two prefixes of two standard Sturmian words $\mathbf{x}_1$ and $\mathbf{x}_2$. Since $u$ is a palindrome one has

$$ua = p_1, \quad ub = p_2.$$

Let us consider now the semiline $r$ starting from the origin and whose slope is $\rho = (|u|_b + 1)/(|u|_a + 1)$. The word $\Psi(r)$ associated to $r$ in the interval $(0, |w|_a)$ is still $u$. This implies that $w = aub$ is a Christoffel word. Finally, $w = aub$ is certainly primitive since its conjugate $wba$ is primitive [4], so that $w \in CP$. □

**Corollary 4.1.** $NCP = bPERa \cup \mathscr{A}$.

**Proof.** Since $NCP = \widetilde{CP}$ and $PER \subseteq PAL$, one derives from the above theorem, $NCP = bPERa \cup \mathscr{A}$. □

Borel and Laubie proved in [1] that $CP \subseteq Lynd$ and $NCP \subseteq \widetilde{Lynd}$; moreover, the standard factorization of $w \in CP$ is the unique factorization of $w$ in Christoffel primitive words. A different proof of this result is easily obtained by using Theorem 4.1, Corollaries 3.1 and 3.2.

Let us now consider, in a way similar to that of standard pairs, the notion of *Christoffel pairs*. We define the set $\mathscr{C}$ as the smallest subset of $\mathscr{A}^* \times \mathscr{A}^*$ which contains the pair $(a, b)$ and closed under the property:

$$(u, v) \in \mathscr{C} \quad \Rightarrow \quad (u, uv), (uv, v) \in \mathscr{C}.$$

Let us set

$$\mathscr{C}_0 = \{(a, b)\}$$

and for $n \geqslant 0$,

$$\mathscr{C}_{n+1} = \{(u, v) | \exists (x, y) \in \mathscr{C}_n : u = x, \ v = xy \ or \ u = xy, \ v = y\}.$$

One has then

$$\mathscr{C} = \bigcup_{n \geqslant 0} \mathscr{C}_n.$$

The set $\mathscr{C}$ is called the set of *Christoffel pairs*.

The following lemma shows the existence for all $n \geqslant 0$ of a bijection between the sets $\mathscr{R}_n$ and $\mathscr{C}_n$.

**Lemma 4.1.** *For all $n \geqslant 0$ the elements of $\mathscr{R}_n$ are the standard pairs $(a, a^n b), (b^n a, b)$ and pairs of the kind $(Pba, Qab)$ with $P, Q \in PER$. Moreover, the elements of $\mathscr{C}_n$ are $(a, a^n b), (ab^n, b)$ and $(aPb, aQb)$ with $(Pba, Qab) \in \mathscr{R}_n$.*

**Proof.** By induction on $n$. For $n = 0, 1$ the result is trivial. For $n = 2$ the elements of $\mathscr{R}_2$ and $\mathscr{C}_2$ are respectively

$$(a, a^2 b), (b^2 a, b), (aba, ab), (ba, bab)$$

and

$$(a, a^2 b), (ab^2, b), (aab, ab), (ab, abb),$$

so that in this case the result is true. Let us then suppose the result true up to the integer $n$. We prove it for $n+1$. By hypothesis the elements of $\mathscr{C}_n$ are $(a, a^n b)$, $(ab^n, b)$ and $(aPb, aQb)$ with $(Pba, Qab) \in \mathscr{R}_n$. By this latter fact it follows that

$$(Pba, PbaQab), (QabPba, Qab) \in \mathscr{R}_{n+1}.$$

Since by Theorem 1.1 $P, Q, QabP \in PER$, one has $QabP = PbaQ$. The pair $(aPb, aQb) \in \mathscr{C}_n$ generates the two elements of $\mathscr{C}_{n+1}$

$$(aPb, aPbaQb), (aPbaQb, aQb),$$

so that the result for these pairs is true. In $\mathscr{C}_{n+1}$ there are also the other pairs

$$(a, a^{n+1} b), (ab^{n+1}, b), (a^{n+1} b, a^n b), (ab^n, ab^{n+1}).$$

To conclude the proof we have to show that the pairs

$$(a^n ba, a^n b), (b^n a, b^n ab) \in \mathscr{R}_{n+1}.$$

This is trivial since, by induction, the pairs $(a, a^n b)$ and $(b^n a, b)$ belong to $\mathscr{R}_n$. $\square$

Let us denote by $Trace(\mathscr{C})$ the set

$$Trace(\mathscr{C}) = \{u \in \mathscr{A}^* \mid \exists v \in \mathscr{A}^* : (u, v) \in \mathscr{C} \text{ or } (v, u) \in \mathscr{C}\}.$$

The following theorem was proved by Borel and Laubie in [1]. Here we report a different and simple proof.

**Theorem 4.2.** $CP = Trace(C)$.

**Proof.** Let $w \in Trace(\mathscr{C})$. There exists an integer $n \geqslant 0$ and $(u, v) \in \mathscr{C}_n$ such that $w = u$ or $w = v$. From the previous lemma one has

$$(u, v) = (a, a^n b) \quad \text{or} \quad (u, v) = (ab^n, b) \quad \text{or} \quad (u, v) = (aPb, aQb)$$

with $(Pba, Qab) \in \mathscr{R}_n$. In any case $w \in aPERb \cup \mathscr{A}$, so that by Theorem 4.1, $w \in CP$.

Let $w \in CP$. If $w \in \mathscr{A}$, then the result is trivial. Let us then suppose $w \in CP \setminus \mathscr{A}$. From Theorem 4.1 one has $w = aPb$ with $P \in PER$. Hence $Pab, Pba \in Stand$. This implies that there exists an integer $n \geqslant 0$ and a standard pair $(u, v) \in \mathscr{R}_n$ such that $u = Pba$ or $v = Pab$. By Lemma 4.1, $aPb = w$ will be one component of a pair of $\mathscr{C}_n$. Hence $w \in Trace(\mathscr{C})$. □

## 5. Slope and ratio of periods

We denote by $PER_{(a)}$ the set of all elements of $PER$ whose first letter is $a$, i.e. $PER_{(a)} = PER \cap a\mathscr{A}^*$. Similarly, $PER_{(b)}$ will be the set $PER_{(b)} = PER \cap b\mathscr{A}^*$. Hence $PER = \{\varepsilon\} \cup PER_{(a)} \cup PER_{(b)}$. One easily verifies that $s \in PER_{(a)}$ if and only if $\hat{s} \in PER_{(b)}$, so that the operation $(\hat{\ })$ determines a bijection of $PER_{(a)}$ in $PER_{(b)}$.

In the following we denote by $\mathscr{F}$ the set of all fractions $p/q$ such that $0 < p \leqslant q$ and $gcd(p, q) = 1$. From the results of the previous sections one easily derives [3] the following:

**Lemma 5.1.** *For any $s \in PER$ there exists a* unique *fraction $p/q \in \mathscr{F}$ such that $p, q \in \Pi(s)$, $p$ is the minimal period of $s$ and $|s| = p + q - 2$. The map $\eta : PER \to \mathscr{F}$ defined as:*

$$\eta(s) = p/q,$$

*is a surjection. Moreover, for $s \neq \varepsilon$:*

$$\eta(s) = (|s| - |Q|)/(|Q| + 2),$$

*where $Q$ is the maximal proper palindrome suffix of $s$.*

*The restrictions $\eta_a$ and $\eta_b$ of $\eta$, respectively, to $PER_{(a)} \cup \{\varepsilon\}$ and to $PER_{(b)} \cup \{\varepsilon\}$, are bijections.*

**Remark 2.** From the above lemma one has that the set $PER_{(a)} \cup \{\varepsilon\}$ faithfully represents all fractions $p/q$ with $0 < p \leqslant q$ and $gcd(p, q) = 1$. One can then use the set $PER_{(b)}$ either to represent the *negative* fractions $-p/q$ or the irreducible and improper fractions $q/p$. Let us observe that $PER$ can also represent the set of all Gauss integers $a + ib$, with $a, b > 0$ and $gcd(a, b) = 1$. More precisely one can consider the map $\gamma$ from $PER$ into the set of complex numbers such that $\gamma(s) = p + iq$ if $s \in PER_{(a)} \cup \{\varepsilon\}$ and $\gamma(s) = q + ip$, if $s \in PER_{(b)}$.

Let $\mathscr{G}$ the set of all irreducible fractions $p/q$ with $p$ and $q$ non negative integers such that $gcd(p, q) = 1$. We define the map

$$\zeta : \mathscr{G} \to PER,$$

as $\zeta(1/1) = \varepsilon$ and for $p/q \neq 1/1$,

$$\zeta(p/q) = \eta_a^{-1}(p/q) \quad \text{if } p < q,$$

$$\zeta(p/q) = \eta_b^{-1}(q/p) \quad \text{if } p > q.$$

From Lemma 5.1 the map $\zeta$ is a bijection. For all $w \in PER$ the irreducible fraction $p/q = \zeta^{-1}(w)$ will be called also the *Farey number* of $w$.

We know (cf. Theorems 3.2 and 4.1) that there exists a bijection

$$\lambda : PER \to CP\backslash\mathscr{A},$$

defined as: if $w \in PER$, then $\lambda(w) = awb \in CP$. From Proposition 4.3 one has that there exists also a bijection $\rho$ of $CP\backslash\mathscr{A}$ and $\mathscr{G}$ which associates to the word $u \in CP$ the slope $\rho(u) = |u|_b/|u|_a$. Hence, the map

$$\mu = \zeta \circ \lambda \circ \rho,$$

is a bijection $\mu : \mathscr{G} \to \mathscr{G}$. The map $\mu$ associates to any irreducible fraction $p/q$ seen as the ratio of the periods of a word $w \in PER$, the slope of the Christoffel primitive word $awb$. Hence, between the ratio of periods of the elements of $PER$ and the slopes of the corresponding Christoffel primitive words there exists a bijection. In this section we shall see that this correspondence can be expressed in terms of the continued fractions of the slopes and of the ratios of periods. We need to recall some propositions [3] and prove some lemmas.

We define the map

$$\psi : \mathscr{A}^* \to PER,$$

as

$$\psi(\varepsilon) = \varepsilon, \quad \psi(a) = a, \quad \psi(b) = b,$$

and for all $w \in \mathscr{A}^*$, $x \in \mathscr{A}$,

$$\psi(wx) = (x\psi(w))^{(-)}.$$

**Proposition 5.1.** *The map $\psi : \mathscr{A}^* \to PER$ is a bijection.*

We can represent any word $w \in \mathscr{A}^*$ uniquely by a finite sequence $(h_1, h_2, \ldots, h_n)$ of integers, where $h_1 \geqslant 0$, $h_i > 0$ for $1 < i \leqslant n$ and

$$w = a^{h_1} b^{h_2} a^{h_3} \ldots .$$

One has $|w| = \sum_{i=1}^{n} h_i$. We call such a representation of the words of $\mathscr{A}^*$ the *integral representation*.

**Proposition 5.2.** *Let $w \in \mathscr{A}^*$ and be $(h_1, h_2, \ldots, h_n)$ its integral representation. The standard words*

$$\psi(w)ab, \quad \psi(w)ba$$

*have, respectively, the directive sequences*

$$(h_1, \ldots, h_n, 1), \quad (h_1, \ldots, h_{n-1}, h_n + 1),$$

*if $n$ is even, and, respectively,*

$$(h_1,\ldots,h_{n-1},h_n+1), \quad (h_1,\ldots,h_n,1),$$

*if $n$ is odd.*

Let $a_0,a_1,\ldots,a_n$ be a finite sequence of integers such that $a_i > 0$, $1 \leqslant i < n$ and $a_0$, $a_n \geqslant 0$. We denote by $\langle a_0,a_1,\ldots,a_n\rangle$ the continued fraction $[a_0,a_1,\ldots,a_{n-1},a_n+1] = [a_0,a_1,\ldots,a_{n-1},a_n,1]$:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\begin{array}{c}\vdots\\ \cfrac{1}{a_{n-1} + \cfrac{1}{a_n+1}}\end{array}}}}$$

We note that we use the above notation for continued fractions since in this way the major part of the results can be expressed by more symmetric formulas.

**Proposition 5.3.** *Let $w$ be a word on the alphabet $\mathscr{A} = \{a,b\}$ and $\psi(w)$ be the corresponding element of PER. If $(h_1,\ldots,h_n)$ is the integral representation of $w$, then $\eta(\psi(w))$ has a development in continued fractions given by $\langle 0,h_n,\ldots,h_1\rangle$.*

**Lemma 5.2.** *If $s$ is a standard Sturmian word having the directive sequence $(h_1,\ldots,h_n)$, then the slope $\rho(s)$ of $s$ is given by*

$$\rho(s) = |s|_b/|s|_a = \langle 0,h_1,\ldots,h_n-1\rangle = [0,h_1,\ldots,h_n].$$

**Proof.** Let $s \in Stand$ have the directive sequence $(h_1,\ldots,h_n)$. If $n = 1$ then we have $s = a^{h_1}b$ so that $\rho(s) = 1/h_1 = [0,h_1]$.(Note that this result holds also when $h_1 = 0$. In this case $\rho(s) = \infty$.) Let us now give the proof for any $n$. We can write

$$s_0 = b, \quad s_1 = a, \quad s_2 = s_1^{h_1}s_0, \quad \ldots, \quad s_{n+1} = s_n^{h_n}s_{n-1} = s.$$

The lengths of the terms $s_i$, $i = 1,\ldots,n$, satisfy the following recurrence relations:

$$|s_{i+1}|_b = h_i|s_i|_b + |s_{i-1}|_b,$$

$$|s_{i+1}|_a = h_i|s_i|_a + |s_{i-1}|_a.$$

For $i = 1,\ldots,n$, we have

$$[0,h_1,\ldots,h_i] = p_i/q_i,$$

where for $i = 1,\dots,n$, $p_i/q_i$ is an irreducible fraction. From the general theory of continued fractions one has

$$p_i = h_i p_{i-1} + p_{i-2},$$

$$q_i = h_i q_{i-1} + q_{i-2}.$$

Moreover, $p_1 = 1$, $q_1 = h_1$ and $p_2/q_2 = h_2/(h_1h_2 + 1) = |s_3|_b/|s_3|_a$, so that $p_2 = |s_3|_b$ and $q_2 = |s_3|_a = 1 + h_1h_2$. By a comparison of the recurrence relations one derives that for $i = 1,\dots,n$ :

$$p_i = |s_{i+1}|_b \quad \text{and} \quad q_i = |s_{i+1}|_a,$$

so that

$$\rho(s) = |s_{n+1}|_b/|s_{n+1}|_a = p_n/q_n = [0, h_1, \dots, h_n]. \qquad \square$$

**Theorem 5.1.** *Let $w \in PER_{(a)}$ and be $\eta(w) = p/q$. If $p/q$ has the development in continued fractions $\langle 0, h_1, \dots, h_n\rangle$, then the slope $\rho$ of the Christoffel primitive word awb has a development in continued fractions given by $\langle 0, h_n, \dots, h_1\rangle$.*

**Proof.** Let $w \in PER_{(a)}$ and let $u \in \mathscr{A}^*$ be such that $w = \psi(u)$. If $(h_1,\dots,h_n)$ is the integral representation of $u$, then by Proposition 5.3, $\eta(w) = p/q$ has a development in continued fractions given by $\langle 0, h_n, \dots, h_1\rangle$. One has, moreover, that

$$\rho = \rho(awb) = \rho(wab) = \rho(wba).$$

It follows by Proposition 5.2 that either *wab* or *wba* is a standard Sturmian word having the directive sequence:

$$(h_1, \dots, h_n + 1).$$

Hence, by Lemma 5.2 the slope of such a word is $\langle 0, h_1, \dots, h_n\rangle$, i.e.

$$\rho = \rho(awb) = \langle 0, h_1, \dots, h_n\rangle. \qquad \square$$

A different proof of Theorem 5.1 will be given in Section 7.

**Example 1.** Let $u = a^2b^3a$. The word $w = \psi(u)$ is

$$w = (a^2b)^3a^3b(a^2b)^2a^2.$$

One has $\eta(w) = p/q = 10/13 = \langle 0, 1, 3, 2\rangle$ and $\rho(awb) = (|w|_b + 1)/(|w|_a + 1) = 7/16 = \langle 0, 2, 3, 1\rangle$.

**Corollary 5.1.** *Let $w \in PER_{(a)}$ and be $u \in \mathscr{A}^*$ such that $w = \psi(u)$. If $u$ has the integral representation $(h_1, \dots, h_n)$, then one has $\rho(awb) = \langle 0, h_1, \dots, h_n\rangle$.*

**Proof.** From Proposition 5.2, $\eta(w) = p/q = \langle 0, h_n, \dots, h_1\rangle$. Hence from Theorem 5.1 it follows $\rho(awb) = \langle 0, h_1, \dots, h_n\rangle$. $\square$

**Corollary 5.2.** *Let* $w \in PER_{(b)}$ *and* $\eta(w) = p/q$. *If* $p/q$ *has the development in continued fractions* $\langle 0, h_1, \ldots, h_n \rangle$ *then the slope* $\rho$ *of the Christoffel primitive word* $awb$ *is given by*

$$\rho(awb) = \langle h_n, \ldots, h_1 \rangle.$$

**Proof.** Let $\hat{w}$ be the word obtained from $w$ by interchanging the letter $a$ with the letter $b$. One has that

$$\eta(\hat{w}) = \eta(w) = p/q = \langle 0, h_1, \ldots, h_n \rangle$$

and

$$|\hat{w}|_a = |w|_b, \quad |\hat{w}|_b = |w|_a.$$

Hence

$$\rho(awb) = (|w|_b + 1)/(|w|_a + 1) = (|\hat{w}|_a + 1)/(|\hat{w}|_b + 1).$$

Since $\hat{w} \in PER_{(a)}$ from Theorem 5.1 one has $\rho(a\hat{w}b) = \langle 0, h_n, \ldots, h_1 \rangle$ and $\rho(awb) = (\rho(a\hat{w}b))^{-1} = \langle h_n, \ldots, h_1 \rangle$. $\square$

## 6. Matrices and trees

Let $\mathscr{D}_1$, or simply $\mathscr{D}$, be the set of all matrices:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c$ and $d$ are nonnegative integers and such that $det(M) = ad - bc = 1$. As is well known $\mathscr{D}$ is a monoid freely generated by the two matrices:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $\mathscr{A} = \{L, R\}$ be a two letter alphabet and $\mathscr{A}^*$ the free monoid on $\mathscr{A}$. We denote by $\iota$ the empty word of $\{L, R\}^*$. The map $\alpha : \mathscr{A} \to \mathscr{D}$ defined by

$$\alpha(L) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \alpha(R) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

can be extended to an isomorphism of $\{L, R\}^*$ onto $\mathscr{D}$. (The empty word $\iota$ is represented by the identity matrix.) We can then identify, when no confusion arises, each word $W \in \{L, R\}^*$ with the corresponding matrix $\alpha(W)$. We say also that $W$ is the *generating word* of the matrix $\alpha(W)$. We shall write any word $W \in \{L, R\}^*$ in the following form:

$$W = R^{a_0} L^{a_1} \ldots L^{a_{n-1}} R^{a_n},$$

for a suitable $n \geqslant 0$, $a_0, a_n \geqslant 0$ and $a_i > 0$ for $i \in [1, n-1]$. We denote, as usual, by $\tilde{W}$ and $\hat{W}$, respectively, the mirror image of $W$ and the word obtained from $W$ by interchanging the letter $R$ with the letter $L$. We denote also by $W'$ the word $\widetilde{(\hat{W})} = \widehat{(\tilde{W})}$. The operation ($'$) is an antiautomorphism of $\{L,R\}^*$. It holds the following [11].

**Proposition 6.1.** *Let $W \in \{L,R\}^*$ and be*

$$\alpha(W) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

*Then*

$$\alpha(\tilde{W}) = \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \quad \alpha(\hat{W}) = \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \quad \alpha(W') = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

The next proposition is stated in [11] without proof.

**Proposition 6.2.** *Let $p$ and $q$ be positive integers which are coprimes. Then there exists a unique word $W \in \{L,R\}^*$ such that*

$$\begin{pmatrix} p \\ q \end{pmatrix} = W \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

**Proof.** We make induction on the integer $n = p + q$. If $n = 2$ then $p = q = 1$ and $W = \iota$. Suppose then $n > 1$. We can write, and uniquely,

$$\begin{pmatrix} p \\ q \end{pmatrix} = L \begin{pmatrix} p \\ q - p \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} p \\ q \end{pmatrix} = R \begin{pmatrix} p - q \\ q \end{pmatrix},$$

if, respectively, $q > p$ or $q < p$. Let us consider only the first case. The second is dealt with in a similar way. Since $gcd(p, q-p) = 1$ and $p + (q-p) = q < n$, by induction there exists a unique word $V \in \{L,R\}^*$ such that

$$\begin{pmatrix} p \\ q - p \end{pmatrix} = V \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} p \\ q \end{pmatrix} = LV \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad \square$$

To each vector

$$u = \begin{pmatrix} p \\ q \end{pmatrix}, \quad p, q > 0$$

we associate the number $f(u) = p/q$. If

$$W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \{L,R\}^*,$$

then we define

$$f(W) = (a+b)/(c+d) = f\left(W\begin{pmatrix}1\\1\end{pmatrix}\right).$$

From the general theory [11, 6] one has

**Proposition 6.3.** *If $W = R^{a_0}L^{a_1}\dots L^{a_{n-1}}R^{a_n}$, then $f(W)$ has the development in continued fractions given by*

$$f(W) = [a_0, a_1, \dots, a_n + 1] = \langle a_0, a_1, \dots, a_n \rangle.$$

Let us now consider the complete binary tree. Each path from the root to a particular node can be represented by a word $W \in \{L,R\}^*$. More precisely, the sequence of letters of $W = R^{a_0}L^{a_1}\dots L^{a_{n-1}}R^{a_n}$, read from left to right, gives the sequence of right and left moves in order to reach the node starting from the root. Since for every node there exists a unique path going from the root to the node, one has that the nodes are faithfully represented by the words $W \in \{L,R\}^*$. In the following we shall identify the nodes of the tree with the binary words of $\{L,R\}^*$.

Let us now label each node of the tree with an irreducible fraction $p/q$, $p$ and $q$ positive integers, in the following way. The root has the label $1/1$. If a node has the label $p/q$, then the 'left son' has the label

$$p/(p+q) = f\left(L\begin{pmatrix}p\\q\end{pmatrix}\right)$$

and the 'right son' has the label

$$(p+q)/q = f\left(R\begin{pmatrix}p\\q\end{pmatrix}\right).$$

We call this labeled binary tree the *Raney tree*. The label of each node $W$ is called the *Raney number* of $W$ and denoted by $Ra(W)$. From the definition one has

$$Ra(W) = f(\tilde{W}) = p/q, \quad \text{with} \quad \begin{pmatrix}p\\q\end{pmatrix} = \tilde{W}\begin{pmatrix}1\\1\end{pmatrix}.$$

From Proposition 6.2 it follows that all irreducible fraction $p/q$, $p, q > 0$ can be faithfully represented by the Raney tree.

Let us now consider the *Stern–Brocot tree* [6] which is a complete binary tree labeled by irreducible fractions according to the following rule. The label $p/q$ in a node is given by $(p'+p'')/(q'+q'')$, where $p'/q'$ is the nearest ancestor above and to the left and $p''/q''$ is the nearest ancestor above and to the right.(In order to construct the tree one needs also to add to the binary tree two more nodes labeled by $1/0$ and $0/1$.) For each node $W$ we denote by $SB(W)$ the corresponding label; $SB(W)$ is called the *Stern–Brocot number* of $W$. One has that [6]

$$SB(W) = f(W) = p/q, \quad \text{with} \quad \begin{pmatrix}p\\q\end{pmatrix} = W\begin{pmatrix}1\\1\end{pmatrix}.$$

From Proposition 6.2 it follows that all the irreducible fractions $p/q$, $p, q > 0$, are faithfully represented in the Stern–Brocot tree. Hence, each node $W$ of a binary tree can be labeled by two irreducible fractions the Raney number $Ra(W)$ and the Stern–Brocot number $SB(W)$. We report below (see Fig. 2) a part of the binary tree. The tree is labeled by Raney's numbers and by the Stern–Brocot numbers in the brackets.

Let us consider the node

$$W = LRL^2R = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}.$$

One has

$$\tilde{W} = RL^2RL = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}.$$

Hence $Ra(W) = f(\tilde{W}) = 11/8$ and $SB(W) = f(W) = 7/12$.

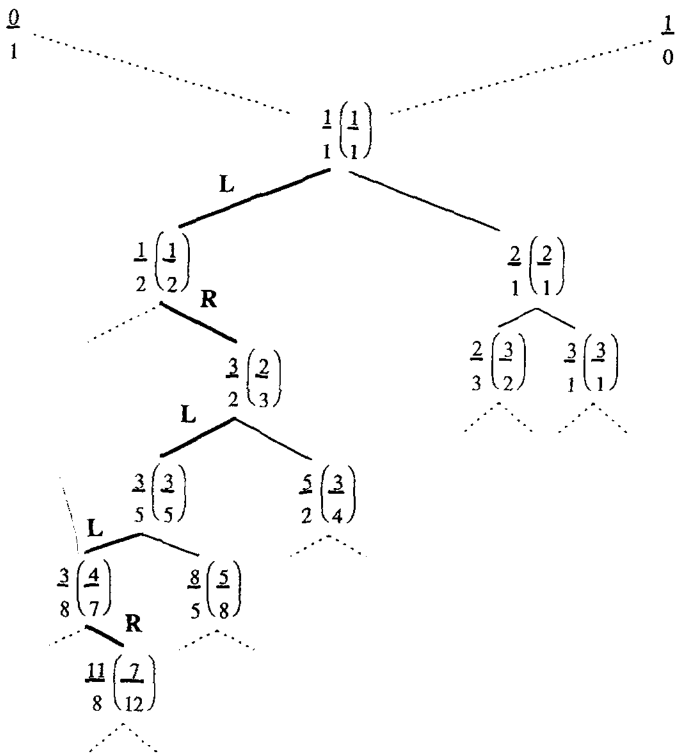

Fig. 2. The Raney and the Stern–Brocot trees.

The relation between Raney's number and Stern–Brocot's number is given by the following:

**Proposition 6.4.** *Let a node in the binary tree have the Raney number $p/q$ and the Stern–Brocot number $r/s$. The number $r/s$ has the development in continued fractions*

$$r/s = \langle a_0, a_1, \ldots, a_n \rangle$$

*with $n$ even, $a_0, a_n \geqslant 0$ and $a_i > 0$, $i = 1, \ldots, n-1$, if and only if $p/q$ has the development in continued fractions*

$$p/q = \langle a_n, a_{n-1}, \ldots, a_0 \rangle.$$

**Proof.** Let $W = R^{a_0}L^{a_1}\ldots L^{a_{n-1}}R^{a_n}$ be a node. One has by Proposition 6.3

$$SB(W) = f(W) = \langle a_0, a_1, \ldots, a_n \rangle.$$

Since $\tilde{W} = R^{a_n}L^{a_{n-1}}\ldots L^{a_1}R^{a_0}$, by using again Proposition 6.3 one has

$$Ra(W) = f(\tilde{W}) = \langle a_n, a_{n-1}, \ldots, a_0 \rangle. \qquad \square$$

Let us observe that in the case of the above example $S(W) = 7/12 = \langle 0, 1, 1, 2, 1 \rangle$ and $Ra(W) = 11/8 = \langle 1, 2, 1, 1 \rangle$.

## 7. Standard, Christoffel and Farey trees

We can label a complete binary tree by the standard pairs $(u, v) \in \mathscr{R}$ as follows. The root is labeled by the pair $(a, b) \in \mathscr{R}$ and if $(u, v) \in \mathscr{R}$ is a label of a node, then the label of the 'left son' is $(u, uv) \in \mathscr{R}$ and the label of the 'right son' is $(vu, v) \in \mathscr{R}$. We call this tree the *standard tree*.

Let us introduce in $\mathscr{R}$ two elementary maps, or operators, $L$ and $R$, defined by

$$(u, uv) = L(u, v), \qquad (vu, v) = R(u, v).$$

We can then consider a map $\sigma : \{L, R\}^* \to \mathscr{R}$ inductively defined as: for all $W \in \{R, L\}^*$

$$\sigma(\imath) = (a, b), \qquad \sigma(WR) = R\sigma(W), \qquad \sigma(WL) = L\sigma(W).$$

In this way one has for all $W \in \{L, R\}^*$

$$\sigma(W) = \tilde{W}(a, b).$$

From the above construction one has that $\sigma(W)$ is the standard pair which labels $W$ in the standard tree; $W$ is also called the *generating word* of the standard pair $\sigma(W)$.

The map $\sigma$ is obviously surjective. Moreover, it is also injective and then bijective. This can be proved in several ways. We shall prove it as an obvious consequence of the following:

**Lemma 7.1.** *For any standard pair $(u,v) \in \mathscr{R}$ there exists a unique word $W \in \{L,R\}^*$ such that*

$$(u,v) = W(a,b).$$

**Proof.** The proof is by induction on the integer $n = |u| + |v|$. For $n = 2$, i.e. $u = a$ and $u = b$, one has $W = \iota$. For $n > 2$ one can write, uniquely,

$$(u,v) = L(u,u^{-1}v) \quad \text{or} \quad (u,v) = R(v^{-1}u,v).$$

Let us consider only the first case. The second is dealt with in a similar way. Since $|u| + |u^{-1}v| = |v| < n$ then by induction there exists and is unique a word $V$ such that

$$(u,u^{-1}v) = V(a,b).$$

Hence,

$$(u,v) = LV(a,b),$$

so that $W = LV$. □

**Theorem 7.1.** *A word $W$ of $\{L,R\}^*$ generates the standard pair $(u,v)$ if and only if $W$ generates the matrix*:

$$\begin{pmatrix} |v|_b & |u|_b \\ |v|_a & |u|_a \end{pmatrix}.$$

**Proof.** The proof is by induction on the length $n$ of $W$. If $n = 0$, i.e. $W = \iota$, then the standard pair generated by $\iota$ is $(a,b)$ and the matrix generated by $\iota$ is the identity matrix, so that in this case the result is trivially true. Let us now suppose that the statement of theorem is true up to the length $n \geqslant 0$ for $W$; we want prove that it is true for a length of $W$ equal to $n+1$. Let $(u,v)$ be the standard pair generated by $W$. By using the inductive hypothesis, we have that

$$(u,v) = \tilde{W}(a,b), \quad \text{if and only if } \alpha(W) = \begin{pmatrix} |v|_b & |u|_b \\ |v|_a & |u|_a \end{pmatrix}.$$

Let us consider the words $WL$ and $WR$ generating, respectively, the standard pairs $(u,uv)$ and $(vu,v)$. One has

$$\alpha(WL) = \begin{pmatrix} |v|_b + |u|_b & |u|_b \\ |v|_a + |u|_a & |u|_a \end{pmatrix} = \begin{pmatrix} |uv|_b & |u|_b \\ |uv|_a & |u|_a \end{pmatrix},$$

and

$$\alpha(WR) = \begin{pmatrix} |v|_b & |v|_b + |u|_b \\ |v|_a & |v|_a + |u|_a \end{pmatrix} = \begin{pmatrix} |v|_b & |vu|_b \\ |v|_a & |vu|_a \end{pmatrix},$$

which proves the assertion. □

Let us introduce the map $ra : PER \to \mathscr{G}$ defined by

$$ra(\varepsilon) = 1/1, \quad ra(a^n) = 1/(n+1), \quad ra(b^n) = n+1, \quad n \geqslant 1$$

and, if $w \in PER$ and $Card(alph(w)) = 2$, then

$$ra(w) = p/q,$$

where $p = |P| + 2$, $q = |Q| + 2$ and $PbaQ = w$ is the unique factorization of $w$ of the previous kind with $P, Q \in PAL$. One can easily prove that $ra$ is a bijection. For any $w \in PER$ we call $ra(w)$ the *Raney number* of $w$.

As it is clear from the definition, the Raney number of $w \in PER$, as well as the Farey number of $w$, is a ratio of the two periods $p$ and $q$ such that $gcd(p, q) = 1$ and $|w| = p + q - 2$. For any $w \in PER$ the Farey number of $w$ is equal to $ra(w)$ or to its inverse.

A further interpretation of the Raney number of an element of $PER$ is given by the following:

**Lemma 7.2.** *For any $w \in PER$, $ra(w) = p/q$ where $p$ is the minimal period of $wa$ and $q$ the minimal period of $wb$. When $Card(alph(w)) > 1$ then $p$ is also the minimal period of $wab$ and $q$ the minimal period of $wba$.*

**Proof.** If $w = a^n$, $n > 0$, then $wa = a^{n+1}$ and $wb = a^n b$. Now $wa$ and $wb$ have respectively the minimal periods 1 and $n+1$. Since $ra(w) = 1/(n+1)$ the result is verified. In a similar way if $w = b^n$, $n > 0$, one has $ra(w) = n+1$ and the minimal periods of $wa$ and $wb$ are, respectively, $n+1$ and 1.

Let us then suppose that $Card(alph(w)) > 1$. We can write $w$ as

$$w = PbaQ = QabP.$$

Let $p = |P| + 2$ and $q = |Q| + 2$ the two periods of $w$ such that $gcd(p, q) = 1$ and $|w| = p + q - 2$. Let us consider the words:

$$wa = QabPa, \quad wb = PbaQb.$$

The word $wa$, as well as $wab$, has the period $p$, whereas the words $wb$ and $wba$ have the period $q$. Let us suppose $p < q$. One has that $p$ is the minimal period of $wa$ and of $wab$. In fact, otherwise, if $wa$ or $wab$ has a period $p' < p$, then also $w$ would have the period $p'$ which contradicts the minimality of $p$.

We want now to prove that $q$ is the minimal period of $wb$ and $wba$. Indeed, suppose that $wb$ or $wba$ has the period $q' < q$. The words $wb$ and $wba$ cannot have the period $p$ since, otherwise, by the theorem of Fine and Wilf [7] they will be powers of a single letter; hence $q' > p$. Now the word $w$ has the periods $p$, $q$ and $q'$. Moreover,

$$|w| = p + q - 2 \geqslant p + q' - 1,$$

so that by the theorem of Fine and Wilf $w$ has also the period $gcd(p, q')$. Since $p$ is the minimal period of $w$ it follows that $q' = kp$, $k > 1$. In view of the fact that $wb$

has the period $q'$ and $w = w_1 \dots w_n$, $w_i \in \{a,b\}$, $i = 1,\dots,n$, has the period $p$, one derives

$$b = w_{n-kp+1} = w_{n-p+1}.$$

This implies that $wb$ has the period $p$ which is a contradiction. □

**Corollary 7.1.** *If $W$ generates the standard pair $(u,v) \in \mathscr{R}$, then*

$$Ra(W) = |u|/|v|, \quad SB(W) = |uv|_b/|uv|_a.$$

*Moreover, if $uv = \pi ab$ with $\pi \in PER$ then*

$$Ra(W) = ra(\pi).$$

**Proof.** By definition one has

$$Ra(W) = f(\tilde{W}) \quad \text{and} \quad SB(W) = f(W).$$

By Theorem 7.1 we can write

$$W = \begin{pmatrix} |v|_b & |u|_b \\ |v|_a & |u|_a \end{pmatrix},$$

so that by Proposition 6.1

$$\tilde{W} = \begin{pmatrix} |u|_a & |u|_b \\ |v|_a & |v|_b \end{pmatrix}.$$

One has then

$$Ra(W) = f\left(\tilde{W}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = |u|/|v| \quad \text{and} \quad SB(W) = f\left(W\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

If the standard pair $(u,v) = (a, a^n b)$, $n \geqslant 0$, then one has $\pi = a^n$

$$Ra(W) = 1/(n+1) = ra(a^n).$$

In a similar way one proves the result if $(u,v) = (b^n a, b)$. Let us now suppose that $u = Pba$ and $v = Qab$, with $P, Q \in PER$. One has then

$$\pi = PbaQ \in PER,$$

so that

$$Ra(W) = |u|/|v| = p/q = ra(\pi).$$

In this case by the previous lemma $Ra(W)$ is equal to the ratio of the minimal period of $uv$ and the minimal period of $vu$. □

We label now the complete binary tree by the Christoffel pairs $(x,y) \in \mathscr{C}$ as follows. The root is labeled by the pair $(a,b) \in \mathscr{C}$ and if $(x,y) \in \mathscr{C}$ is a label of a node, then the label of the 'left son' is $(x, xy) \in \mathscr{C}$ and the label of the 'right son' is $(xy, y) \in \mathscr{C}$. We call this tree the *Christoffel tree.*

Let us introduce in $\mathscr{C}$ the two operators, $L$ and $R$, defined as: for any $(x, y) \in \mathscr{C}$

$$(x, xy) = L(x, y), \quad (xy, y) = R(x, y).$$

We can then consider a map $\tau : \{L, R\}^* \to \mathscr{C}$ inductively defined as: for all $W \in \{R, L\}^*$

$$\tau(\imath) = (a, b), \quad \tau(WR) = R\tau(W), \quad \tau(WL) = L\tau(W).$$

In this way one has for all $W \in \{L, R\}^*$

$$\tau(W) = \tilde{W}(a, b).$$

From the above construction one has that $\tau(W)$ is the Christoffel pair which labels $W$ in the Christoffel tree; $W$ is also called the *generating word* of the Christoffel pair $\tau(W)$.

The map $\tau$ is obviously surjective. Moreover, it is also injective and then bijective. This is a consequence of the following lemma whose prove is very similar to that of Lemma 7.1.

**Lemma 7.3.** *For any Christoffel pair* $(x, y) \in \mathscr{C}$ *there exists a unique word* $W \in \{L, R\}^*$ *such that*

$$(x, y) = W(a, b).$$

In a way similar to that of Theorem 7.1and Corollary 7.1 one can prove

**Theorem 7.2.** *A word* $W$ *of* $\{L, R\}^*$ *generates the Christoffel pair* $(x, y)$ *if and only if* $W$ *generates the matrix*

$$\begin{pmatrix} |y|_b & |x|_b \\ |y|_a & |x|_a \end{pmatrix}.$$

*Moreover,* $Ra(W) = |x|/|y|$ *and* $SB(W) = |xy|_b/|xy|_a$.

In the Fig. 3 we report the binary tree labeled by the standard pairs and by the Christoffel pairs.

Finally, we consider the *Farey tree* [3] which is obtained by labelling the complete binary tree with the elements of the set $PER$ in the following way: the root is labeled by the empty word $\varepsilon \in PER$. Moreover, if $u \in PER$ is a label of a node then the label of the 'left son' is $(au)^{(-)}$ and the label of the 'right son' is $(bu)^{(-)}$. Let $L, R$ be the two operators $L, R : PER \to PER$ defined for all $u \in PER$ as:

$$L \cdot u = (au)^{(-)}, \quad R \cdot u = (bu)^{(-)}.$$

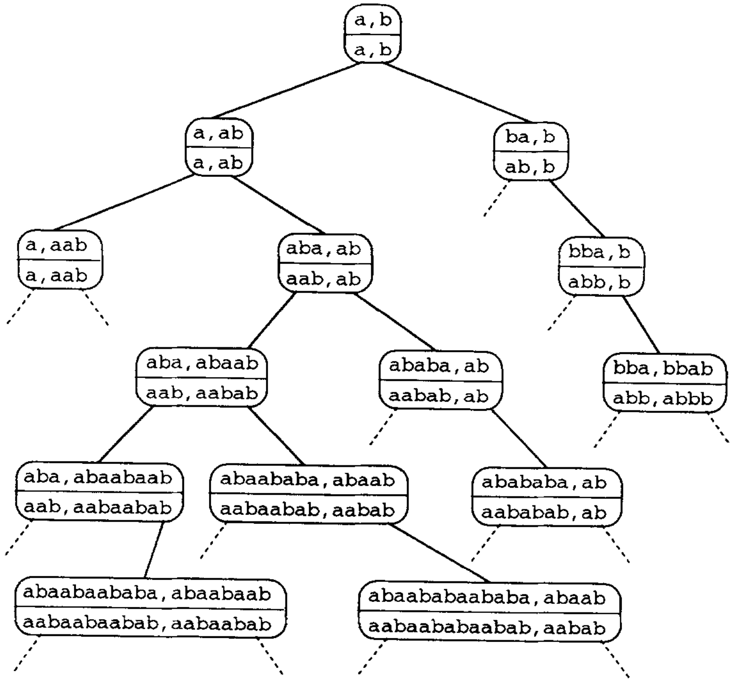


Fig. 3. The standard and the Christoffel trees.

We can then consider the map $\varphi : \{L,R\}^* \to PER$ inductively defined as: for all $W \in \{L,R\}^*$

$$\varphi(\iota) = \varepsilon, \quad \varphi(WL) = L \cdot \varphi(W), \quad \varphi(WR) = R \cdot \varphi(W).$$

In this way one has for all $W \in \{L,R\}^*$

$$\varphi(W) = \tilde{W} \cdot \varepsilon.$$

For each $W \in \{L,R\}^*$, $\varphi(W)$ gives the label of the node $W$ in the Farey tree. The map $\varphi$ is a bijection. In fact, if we introduce the isomorphism $j : \{L,R\}^* \to \{a,b\}^*$ defined by $j(L) = a$, $j(R) = b$. Then one has

$$\varphi(W) = \psi(j(W)),$$

where $\psi : \{a,b\}^* \to PER$ is the map defined in Section 5. Thus $\varphi = j \circ \psi$; since $\psi$ is a bijection, so will be $\varphi$.

To each node $W$ of the Farey tree one can associate the irreducible fraction $Fa(W)$ defined by

$$Fa(W) = \zeta^{-1}(\varphi(W)),$$

where $\zeta$ was defined in Section 5. We call $Fa(W)$ the *Farey number* of $W$. If a node $W$ is labeled by $p/q$, with $p < q$, then its sons will be labeled by $p/(p+q)$ and $q/(p+q)$. (Note that no necessarily $WL$ has the label $p/(p+q)$.)

Let $W = R^{a_0}L^{a_1}\dots L^{a_{n-1}}R^{a_n}$, $a_0, a_n \geqslant 0$ and $a_i > 0$, $i = 1,\dots,n-1$, we define $ord(W)$ the number of nonzero elements in the sequence $(a_0, a_1, \dots, a_n)$. There exists a relation between the Farey number $Fa(W)$ and the Raney number $Ra(W)$.

**Proposition 7.1.** *For all* $W \in \{L,R\}^*$

$$Fa(W) = \begin{cases} Ra(W) & \text{if } ord(W) \text{ is odd,} \\ 1/Ra(W) & \text{otherwise.} \end{cases}$$

**Proof.** Suppose first that $ord(W)$ is odd. Since $n$ is even then either $a_0 = a_n = 0$ or $a_0, a_n > 0$. In the first case $Ra(W) = \langle 0, a_{n-1}, \dots, a_1, 0\rangle = \langle 0, a_{n-1}, \dots, a_1\rangle$. Now $j(W) = a^{a_1}b^{a_2}\dots a^{a_{n-1}}$, so that the integral representation of $j(W)$ is $(a_1, \dots, a_{n-1})$. This implies from Proposition 5.3, $\eta(\varphi(W)) = \eta(\psi(j(W))) = \langle 0, a_{n-1}, \dots, a_1\rangle$. Since $\varphi(W) \in PER_{(a)}$ one has that

$$Fa(W) = \zeta^{-1}(\varphi(W)) = \eta(\varphi(W)) = Ra(W).$$

In the second case $Ra(W) = \langle a_n, a_{n-1}, \dots, a_1, a_0\rangle$ and the word $j(W) = b^{a_0}a^{a_1}\dots b^{a_n}$ has the integral representation $(a_0, a_1, \dots, a_n)$. Thus $\eta((j(W)) = \langle 0, a_n, \dots, a_0\rangle$. Since $\psi(j(W)) \in PER_{(b)}$ one has

$$Fa(W) = \zeta^{-1}(\varphi(W)) = (\eta(\varphi(W)))^{-1} = \langle a_n, \dots, a_1, a_0\rangle = Ra(W).$$

Let now $ord(W)$ be an even integer. One has two consider the two subcases: $a_0 = 0$, $a_n > 0$ and $a_0 > 0$, $a_n = 0$. In the first subcase $Ra(W) = \langle a_n, \dots, a_1\rangle$. Since the integral representation of $j(W)$ is $(a_1, \dots, a_n)$ and $\varphi(W) \in PER_{(a)}$, one has

$$Fa(W) = \zeta^{-1}(\varphi(W)) = \eta(\varphi(W)) = \langle 0, a_n, \dots, a_1\rangle = 1/Ra(W).$$

In the second subcase $Ra(W) = \langle 0, a_{n-1}, \dots, a_1, a_0\rangle$ and the integral representation of $j(W)$ is $(a_0, a_1, \dots, a_{n-1})$ one has $\eta(\psi(j(W)) = \langle 0, a_{n-1}, \dots, a_1, a_0\rangle$. Since $\varphi(W) \in PER_{(b)}$, it follows that

$$Fa(W) = \zeta^{-1}(\varphi(W)) = (\eta(\varphi(W))^{-1} = \langle a_{n-1}, \dots, a_1, a_0\rangle = 1/Ra(W). \qquad \square$$

**Proposition 7.2.** *Let* $W \in \{L,R\}^*$ *and* $\sigma(W)$ *and* $\varphi(W)$ *the standard pair* $(u,v)$ *and the element of PER which label W in the standard tree and in the Farey tree respectively. One has*

$$uv = \varphi(W)ab.$$

**Proof.** Suppose first that $a_0 = 0$, i.e. $\varphi(W) \in PER_{(a)}$. Let $(u,v)$ be the standard pair $\sigma(W)$. Let us write $uv = \pi ab$ with $\pi \in PER_{(a)}$. We know from Corollary 7.1 that $Ra(W) = ra(\pi) = p/q$ where $p = |u|$ and $q = |v|$ are the two periods of $\pi$ such that $gcd(p,q) = 1$ and $p + q - 2 = |\pi|$. Since $Fa(W) \leqslant 1$ from the previous proposition one has

$$Fa(W) = \begin{cases} p/q & \text{if } p \leqslant q, \\ q/p & \text{otherwise.} \end{cases}$$

Hence $\varphi(W) = \zeta(Fa(W)) = \pi$. The case $a_0 > 0$ is dealt with in a symmetric way. □

Let us now give a different proof of Theorem 5.1.

**Proof of Theorem 5.1** . Let $w \in PER_{(a)}$ and $W \in \{L,R\}^*$ be such that $w = \varphi(W)$. If $\sigma(W) = (u,v)$, then one has by the previous proposition $uv = wab$. By Corollary 7.1 one has that

$$SB(W) = |uv|_b/|uv|_a = |wab|_b/|wab|_a = \rho(awb).$$

Moreover, from Propositions 6.3 and 6.4 one derives $SB(W) = \langle a_0, a_1, \ldots, a_n \rangle$ and $Ra(W) = \langle a_n, \ldots a_1, a_0 \rangle$. Since $w \in PER_{(a)}$ then $a_0 = 0$.

If $ord(W)$ is odd then $a_n = 0$ and by Proposition 7.2

$$\eta(w) = Fa(W) = Ra(W).$$

Hence, $\rho(awb) = \langle 0, a_1, \ldots, a_{n-1} \rangle$ and $\eta(w) = \langle 0, a_{n-1} \ldots, a_1 \rangle$. If $ord(W)$ is even then $a_n > 0$ and

$$\eta(w) = Fa(W) = 1/Ra(W).$$

Hence, $\rho(awb) = \langle 0, a_1, \ldots, a_n \rangle$ and $\eta(w) = \langle 0, a_n \ldots, a_1 \rangle$. □

**Example 2.** In the case of the word $W = LRL^2R$ one has (cf. Fig. 3) $\sigma(W) = (u,v)$, $\tau(W) = (x,y)$ and $\varphi(W) = w$, with

$$u = abaabaababa, \quad v = abaabaab, \quad x = aabaabaabab, \quad y = aabaabab,$$

and

$$w = abaabaababaabaaba.$$

One has then $uv = wab$ and $xy = awb$. Moreover,

$$SB(W) = 7/12 = |uv|_b/|uv|_a = |xy|_b/|xy|_a.$$

and

$$Ra(W) = 1/Fa(W) = 11/8 = |u|/|v| = |x|/|y|.$$

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