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1: The category $\mathcal{C}$

We shall consider sets X equipped with a (partial) order (written: $x' \subset x$) satisfying the following two properties

(1.1) X has a smallest element, denoted by 0. Thus $0 \subset x$ holds for all $x \in X$

(1.2) Every ascending sequence

$$x_0 \subset x_1 \subset \dots \subset x_n \subset \dots$$

of elements of X has a least upper bound. This l.u.b. will be denoted by

$$\lim_{n\to\infty} x_n$$

A set equipped with such a partial order will be called a <u>continuous set</u>. Given a function

$$f : X \longrightarrow Y$$

where X and Y are continuous sets

we shall say that f is continuous provided it satisfies the following two properties

(1.3) $\quad x' \subset x$ implies $x'f \subset xf$

(1.4) $\quad \lim_{n\to\infty}(x_n f) = (\lim_{n\to\infty} x_n) f$

Condition (1.4) should be interpreted as follows: given an ascending sequence

$$x_0 \subset x_1 \subset \dots \subset x_n \subset \dots$$

in $\mathfrak{X}$, it follows from (1.3) that

$$x_0 f \subset x_1 f \subset \dots \subset x_n f \subset \dots$$

also is an ascending sequence. Thus both sides of (1.4) are defined and the axiom asserts that they are equal. Note that we do not require a continuous ~~mapping~~ function to satisfy $0f = 0$.

The composition fg of two continuous functions

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

obviously is continuous. So is the identity function $X \to X$. Thus continuous sets and continuous functions form a category that we shall denote by $\mathcal{C}$.

We shall need some further properties of the category $\mathcal{C}$. Given continuous sets X_1, X_2 the cartesian product $X_1 \times X_2$ equipped with the partial order

$$(x_1', x_2') \subset (x_1, x_2)$$

defined by

$$x_1' \subset x_1 \text{ and } x_2' \subset x_2$$

is easily seen to be continuous set. The projections

$$\pi_i : X_1 \times X_2 \longrightarrow X_i, \quad i=1,2$$

defined by $(x_1, x_2)\pi_i = x_i$ obviously are continuous. Further, a function

$$f : Y \to X_1 \times X_2$$

is continuous iff the two compositions

$$f\pi_i : Y \longrightarrow X_i, \quad i=1,2$$

are continuous.

If Y is a continuous set and X is any set, we denote by Y^X the set of all functions $\varphi : X \to Y$. Until order

$$\varphi' < \varphi$$

defined by

$$x\varphi' \leq x\varphi \quad \text{for all} \quad x \in X$$

it is easy to see that Y^X is a continuous set. The projections

$$\pi_x : Y^X \to Y$$

defined by

$$\varphi\pi_x = x\varphi$$

are continuous. Further, a function $f : Z \to Y^X$ is continuous iff $f\pi_x$ is continuous for all $x \in X$.

Let X in Y be continuous sets. In the set Y^X of all functions $X \to Y$ we then may consider the subset, denoted by (X, Y), of all continuous functions [illegible]. We assert that (X, Y) is a closed subset of Y^X, i.e. that the limit of continuous functions is continuous. For this consider an ascending sequence

$$\varphi_0 \subset \varphi_1 \subset \dots \subset \varphi_n \subset \dots$$

of continuous functions $\varphi_n : X \to Y$ and let $\varphi = \lim_{n\to\infty} \varphi_n$. Thus

$$x\varphi = \lim_{n\to\infty} x\varphi_n$$

To show that φ is continuous, we consider an ascending sequence $\{x_m\}$ in X with limit x. We must show that

$$(1.5) \qquad x\varphi = \lim_{m\to\infty}(x_m\varphi)$$

The left hand side is

$$x\varphi = \lim_{n\to\infty}(x\,\varphi_n) = \lim_{n\to\infty}\left(\lim_{m\to\infty} x_n\,\varphi_m\right)$$

while the right hand side is

$$\lim_{m\to\infty}(x_m\,\varphi) = \lim_{m\to\infty}\left(\lim_{n\to\infty} x_n\,\varphi_m\right)$$

Thus equality (5.5) follows from

Lemma 1.1. Given a doubly indexed family $\{x_{m,n},\ m\geq 0, n\geq 0\}$ in a continuous set $\mathfrak{X}$, such that

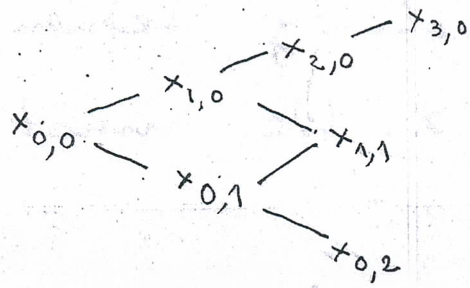
$$x_{m,n} \subset x_{m+1,n}\ , \quad x_{m,n} \subset x_{m,n+1}$$

the equality

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} x_{m,n}\right) = \lim_{n\to\infty}\left(\lim_{m\to\infty} x_{m,n}\right)$$

holds.

Indeed, both sides are the least upper bound of $\{x_{m,n}\}$ ▮



7. Let X, Y, Z be continuous sets and let

(1.6) $$f : X \times Y \longrightarrow Z$$

be a continuous function. For a fixed $x \in X$ this defines a continuous function $Y \to Z$, so that we obtain a function

(1.7) $$\bar{f} : X \longrightarrow (Y, Z)$$

Explicitely, $\bar{f}$ is defined by

(1.8) $$y(x\bar{f}) = (x, y)f$$

The reader should verify that $\bar{f}$ is continuous. Conversely, given a continuous $\bar{f}$ as in (1.7), formula (1.8) defines f as in (1.6) which is continuous. The correspondence $f \longleftrightarrow \bar{f}$ is a bijection and $f' \subset f$ holds iff $\bar{f}' \subset \bar{f}$. We thus obtain an isomorphism

$$(X \times Y, Z) \approx (X, (Y, Z))$$

of continuous sets. This isomorphism will frequently be treated as an identification

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The facts about the category $\mathcal{C}$ developped above, may be summarized by saying that $\mathcal{C}$ is a <u>cartesian closed category</u>.

Exercise 1.1. Establish the isomorphism.

$$Z^{X \times Y} \approx (Z^Y)^X$$

where X and Y are sets and Z is a continuous set.

Exercise 1.2. Show that the evaluation mapping

$$e : X \times (X, Y) \longrightarrow Y$$
$$(x, f)e = xf$$

is continuous.

Exercise 1.3. Show that the composition mapping

$$c : (X, Y) \times (Y, Z) \to (X, Z)$$
$$(f, g)c = fg$$

is continuous.

2. The fixed-point theorem

Let X be a continuous set and

$$f : X \to X$$

a continuous function. Since

$$0 \subset 0f$$

and since f preserves the order it follows that $0f^n \subset 0f^{n+1}$ holds for all $n \geq 0$. Thus an ascending sequence

(2.1) $$0 \subset 0f \subset 0f^2 \subset \dots \subset 0f^n \subset \dots$$

is obtained. The limit of this sequence is denoted by

$$f^\# \in X$$

Applying f to (3.1) we obtain the ascending sequence $0f^1 \subset 0f^2 \subset \dots \subset 0f^{n+1} \subset \dots$ which has the same limit as (3.1). Since f is continuous we have

$$f^\# f = (\lim_{n\to\infty} 0f^n) f = \lim_{n\to\infty} (0f^{n+1}) = f^\#$$

so that

$$f^{\#} f = f$$

Thus $f^{\#}$ is a fixed-point for the transformation f.

Next we prove

(2.2) If $x \in \mathfrak{X}$ and $xf \subset x$, then $f^{\#} \subset x$.

~~Indeed we have $0 \subset x$. Thus $0f \subset xf \subset x$.~~

Indeed, to prove $f^{\#} \subset x$ it suffices to prove that $0f^n \subset x$ for all $n \geq 0$. ~~Since $0 \subset x$ we have $0f \subset xf \subset x$.~~ Assuming $0f^n \subset x$ we have $0f^{n+1} \subset xf \subset x$. Thus by induction $0f^n \subset x$ for all $n \geq 0$.

It follows from (5.2) that

(2.3) $f^{\#}$ is the smallest element in $\mathfrak{X}$ satisfying $xf \subset x$

In particular,

(2.4) $f^{\#}$ is the minimal fixed-point of f.

This minimal fixed point (abbreviated: mfp) will also be referred to as the minimal solution of the equation $x = xf$.

Proposition 2.1. The function

$$\# : (X, X) \to X$$

which to each continuous function $f : X \to X$ assigns its mfp $f^{\#}$ is continuous.

Indeed, let $f = \lim_{m\to\infty} f_m$. Then

$$f^{\#} = \lim_{n\to\infty} (0f^n) = \lim_{n\to\infty} \left(\lim_{m\to\infty} (0f_m^n) \right)$$

$$\lim_{m\to\infty} f_m^{\#} = \lim_{m\to\infty} \left(\lim_{n\to\infty} (0f_m^n) \right)$$

Thus the conclusion follows from Lemma 1.1 ∎

(Coherence Theorem)

Theorem 2.2. Given a commuting diagram in $\mathcal{C}$

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\
Y \arrow[r, "g"'] & Y
\end{tikzcd}
\tag{2.5}
$$

such that $0h = 0$, the equality

$$f^{\#} h = g^{\#}$$

holds.

Indeed, we have

$$f^{\#} h = (\lim_{n\to\infty} 0 f^n) h = \lim_{n\to\infty} (0 f^n h)$$

$$= \lim_{n\to\infty} (0 h g^n) = \lim_{n\to\infty} (0 g^n) = g^{\#} \qquad \blacksquare$$

The Coherence Theorem is of crucial importance for the applications of the continuous fixed-point theorem. In fact it can be used to describe $f^{\#}$ axiomatically. This is stated in

Proposition 2.3. Suppose that for each continuous set X and each continuous transformation $f : X \to X$ an element $f^\flat \in X$ is give satisfying

(2.6) $f^\flat f = f^\flat$ i.e. $f^\flat$ is a fixedpoint for f

(2.7) $f^\flat h = g^\flat$ holds in each commuting diagram (5.5) in $\mathcal{C}$ such that $Oh = C$

Then $f^\flat = f^\#$

Proof. Given $g : Y \to Y$ in $\mathcal{C}$, we shall show that a commuting diagram (2.5) can be found in $\mathcal{C}$ such that $Oh = C$ and such that f has exactly one fixed point. Assume that this is done. Then

$$g^\flat = f^\flat h = f^\# h = g^\#$$

as required.

We define X to consist of all

the ascending sequences

$$x = \{ y_0 < y_1 < \dots < y_n < \dots \}$$

in $\underline{Y}$. If $x' = \{ y_0' < y_1' \dots < y_n' < \dots \}$ is another element of X, then we set $x' < x$ provided $y_n' < y_n$ for all $n \geq 0$. The verification that X is a continuous set is immediate. The function $h : X \to \underline{Y}$ is defined by setting $xh = \lim y_n$. The continuity of h is easily verified. The function $f : X \to \underline{Y}$ is defined by

$$xf = \{ 0, y_0 f, y_1 f, \dots, y_{n-1} f, \dots)$$

The continuity of f and the fact that (2.5) commutes are easily established.

Now suppose that $xf = x$. This holds iff

$$0 = y_0 \quad \text{and} \quad y_n = y_{n-1} f \quad \text{for all } n \geq 1$$

Thus x is a fixed-point for f iff $y_n = 0 f^n$ for all $n \geq 0$. Consequently f has only one fixed-point ∎

It is worth observing that the analogue of the Coherence Theorem in the filtered case is completely trivial and holds without any assumptions on h. Indeed if X and Y are filtered and f and g are shrinkings then

$$f^{\#} h g = f^{\#} f h = f^{\#} h$$

Thus $f^{\#} h$ is a fixed-point for g, and since g has only one fixed-point it follows that $f^{\#} h = g^{\#}$. Note that h need not even be filtration preserving.

3. $\mathcal{C}$-semirings

Let K be a semiring which is simultaneously a continuous set. We shall postulate the following connections between the two structures:

(3.1) The zero element of K as a semiring also is the smallest element of K.

(3.2) The functions $K \times K \to K$ defined by addition and multiplication are continuous.

We shall then say that K is a $\mathcal{C}$-semiring. The semiring $\mathbb{B}$ and $\mathcal{N}$ equipped with their natural ordering are $\mathcal{C}$-semirings. The semiring $\underline{N}$ is not.

Consider an at most countable set I and an indexed family

$$\{ x_i \mid i \in I \}$$

of elements of K (equivalently $x: I \to K$ is a function). Let

(3.3) $I_0 \subset I_1 \subset \dots \subset I_n \subset \dots$

be a sequence of finite subsets of I such that $I = \bigcup_{n=0}^{\infty} I_n$. Define

$$s_n = \sum_{i \in I_n} x_i$$

Since

$$s_{n+1} = s_n + \sum_{i \in I_{n+1} - I_n} x_i$$

it follows from (3.1) and (3.2) that $s_n \subset s_{n+1}$. Thus $s_0 \subset s_1 \subset \dots \subset s_n \subset \dots$ is an ascending sequence in K and the limit

$$s = \lim_{n \to \infty} s_n \in K$$

is well defined. If J is any finite subset of I then $J \subset I_n$ for some n sufficiently large and therefore

$$\sum_{i \in J} x_i \subset s_n \subset s$$

It follows that s is the least upper bound of all elements

$$s_J = \sum_{i \in J} x_i$$

where J ranges over all finite subsets of I. In particular, s is independent of the choice of the sequence (3.3) exhausting I. We shall write

$$s = \sum_{i \in I} x_i .$$

We now slightly generalize the above definition by replacing the condition that I is at most countable by the condition that the function $x : I \to K$ is of countable type. Setting

$$J = \{ i \mid i \in I,\ x_i \neq 0 \}$$

we find that J is at most countable. We define

$$\text{(3.4)} \qquad \sum_{i \in I} x_i = \sum_{i \in J} x_i$$

We ask the reader to verify the following formal rules governing the summation (3.4)

(3.5) If I has a single extent i, then

$$\sum_{i\in I} x_i = x_i$$

(3.6) If $I = \bigcup_{j\in J} I_j$ is a disjoint partition of I, then

$$\sum_{i\in I} x_i = \sum_{j\in J}\left(\sum_{i\in I_j} x_i\right)$$

(3.7) $$z\left(\sum_{i\in I} x_i\right) = \sum_{i\in I} z x_i$$

(3.8) $$\left(\sum_{i\in I} x_i\right) z = \sum_{i\in I} x_i z$$

(3.9) If $I = \emptyset$, then $\Sigma_{i\in I}\, x_i = 0$.

These are exactly the rules A, VI (2.5)–(2.9) that were used to define the notion of a complete semiring. The only difference is that here we assume that the functions $I \to K$ are of countable type. We may thus say that K is <u>countably complete</u>.

Given a fixed set I, we shall denote by $\langle I, K\rangle$ the set of all functions $x : I \to K$ of <u>countable type</u>, i.e. the set of K-subsets of I of countable type. We ask the reader to verify the following facts

(3.10) $\langle I, K\rangle$ is a closed subset of K^I, i.e. the limit of an ascending sequence of functions $I \to K$ of countable type again is of countable type

(3.11) The function

$$S : \langle I, K\rangle \longrightarrow I$$

given by

$$xS = \sum_{i\in I} x_i$$

is continuous

(3.12) If $x, y \in \langle I, K\rangle$ then $x+y$ defined by $(x+y)_i = x_i + y_i$ again is in $\langle I, K\rangle$. The function

$$\langle I, K\rangle \times \langle I, K\rangle \to \langle I, K\rangle$$

given by $(x,y) \to x+y$ is continuous.

If M is a monoid and K is a $\mathcal{C}$-semiring, then we may introduce a multiplication in $\langle M, K\rangle$ by setting

$$m(xy) = \sum_{m=m_1 m_2} (m_1 x)(m_2 y)$$

In this formula, $x, y \in \langle M, K\rangle$ are K-subsets of M of countable type, and for convenience mx is written instead of xm. Since both x and y are of countable type the set of all pairs $(m_1, m_2) \in M \times M$ such that $(m_1 x)(m_2 y) = 0$ is at most countable, so that the summation above has meaning and $xy \in \langle M, K\rangle$. We ask the reader to verify

(3.13) The function

$$\langle M, K\rangle \times \langle M, K\rangle \to \langle M, K\rangle$$

defined by $(x,y) \to xy$ is continuous.

Summarizing the above we obtain

(3.14) If K is a $\mathcal{C}$-semiring and M is any monoid then $\langle M, K\rangle$ again is a $\mathcal{C}$-semiring.

Now let $G = (\Xi, \xi_0, R)$ be a grammar in the monoid M with weights in a commutative $\mathcal{C}$-semiring K. Since there is only at most a countable set of derivation $d : \xi_0 \to m$ with $m \in M$, the summation

$$mL = \sum d\mu$$

is well defined, so that L becomes a K-subset of M of countable type, i.e. $L \in \langle M, K\rangle$. The K-subsets of M thus obtained are called algebraic and their class is denoted by

$$M\mathcal{A}lg_K$$

Thus $M\mathcal{A}lg_K$ is a subset of $\langle M, K\rangle$.

4. The Main Theorem

We now consider the case when $G = (\Xi, R)$, $\Xi = \{\xi_1, \dots, \xi_n\}$ is an (M, K)-grammar where M is an arbitrary monoid and K is a commutative $\mathcal{C}$-semiring. Then $V = \langle M, K \rangle$ is a $\mathcal{C}$-semiring. Let $w \in M[\Xi]$ be a monomial. It follows from (3.13) that the associated monomial transformation $w : V^n \longrightarrow V$ is continuous. This implies that $\underline{P}_i : V^n \rightarrow V$ and $\underline{P} : V^n \rightarrow V^n$ are continuous. As before L_i designates the language defined by G with ξ_i as start variable. Thus $L_i \in V$.

Theorem 4.1. The vector

$$L = (L_1, \dots, L_n) \in V$$

is the minimal fixed point of the polynomial transformation

$$\underline{P} : V^n \longrightarrow V^n$$

defined by the grammar G.

Proof. For each $r \in R$ let

$$\underline{r} = m_0 \, \xi_{i_1} \, m_1 \cdots m_{p-1} \, \xi_{i_p} \, m_p \tag{4.1}$$

with $p = \underline{r}\delta$. We consider an alphabet Σ made up of letters $\sigma_{i,r}$ with $0 \le i \le \underline{r}\delta$, $r \in R$. A morphism $\varphi : \Sigma^* \to M$ is defined by setting $\sigma_{i,r}\varphi = m_i$ and is extended to a morphism $\varphi : \Sigma^+[\Xi] \to M[\Xi]$ by setting $\xi_i \varphi = \xi_i$.

We define the (Σ^+, K)-grammar $\hat{G} = (\Xi, \hat{R})$ as follows. For each rule ~~$r \in R$~~ ~~with $\underline{r}$ given by (6.1) we introduce a rule~~ ~~$\hat{r} : \bar{r} \to$~~

$$r : \xi_i \to \underline{r} \quad \text{in } R$$

with $\underline{r}$ given by (4.1), we introduce a rule

$$\hat{r} : \xi_i \to \hat{\underline{r}} \quad \text{in } \hat{R}$$

with

$$\hat{\underline{r}} = \sigma_{0r} \, \xi_{i_1} \, \sigma_{1r} \cdots \sigma_{p-1,r} \, \xi_{i_p} \, \sigma_{pr}$$

Thus $\hat{\underline{r}}\varphi = \underline{r}$. The rule $\hat{r}$ receives the same weight as r i.e. $\hat{r}\mu = r\mu$.

Let $\hat{L}_i$, $i=1,\ldots,n$ be the languages defined by the grammar $\hat{G}$. We assert that

(4.2) $$L_i = \hat{L}_i \varphi$$

Indeed, let $d : \xi_0 \to m$ be a derivation in G with $d = r_1 \cdots r_q$. Then $\hat{d} = \hat{r}_1 \cdots \hat{r}_q$ is a derivation $\hat{d} : \xi_0 \to s$ is a derivation in $\hat{G}$ with $s \in \Sigma^+$, $s\varphi = m$. Further any derivation $\xi_0 \to s$, $s\varphi = m$, in $\hat{G}$ has the form $\hat{d}$ for a unique derivation $d : \xi_0 \to m$ in G. Thus

$$\sum_{\xi_i d = m} d\mu = \sum_{s\varphi = m} \sum_{\xi_i \hat{d} = s} \hat{d}\mu$$

This yields

$$mL_i = \sum_{s\varphi = m} s\hat{L}_i$$

which proves (4.2)

~~Next we consider the diagram~~

Next we consider the polynomial transformation

$$\hat{P} : \hat{V}^n \longrightarrow \hat{V}^n$$

defined by the grammar $\hat{G}$. We note that the grammar $\hat{G}$ is positive. Therefore by Theorem II.4.1, $1\hat{L}_i = 0$ and $\hat{L}$ is the only fixed-point of $\hat{P}$ with that property. Let $\hat{P}^\#$ be the minimal fixed-point of $\hat{P}$. Then $\hat{P}^\# \subset L$ and this implies $1\hat{P}^\#_i = 0$ for $i = 1, \ldots, n$. Thus

(4.3) $$\hat{P}^\# = \hat{L}$$

The morphism $\varphi : \Sigma^* \longrightarrow M$ extends to a morphism $\varphi : \langle \Sigma^*, K \rangle \longrightarrow \langle M, K \rangle$. The reader should verify the continuity of this extension (actually a proof of continuity in a more general situation will appear in IV,). In view of the definition of $\hat{G}$ we obtain a commuting diagram

$$\begin{tikzcd} \hat{V}^n \arrow[r, "\hat{\underline{P}}"] \arrow[d, "\varphi"'] & \hat{V}^n \arrow[d, "\varphi"] \\ V^n \arrow[r, "\underline{P}"'] & V^n \end{tikzcd}$$

with the vertical mappings defined by $\varphi: \hat{V} \to V$. Since $O\varphi = O$, the Coherence Theorem implies

$$\underline{P}^{\#} = \hat{\underline{P}}^{\#} \varphi$$

This combined with (4.3) and (4.2) yields

$\underline{P}^{\#} = L$ as required ∎

5. Lattice semirings; fuzzy sets

Let K be a commutative semiring satisfying the following conditions

$$(5.1) \qquad xx = x = x + x$$

$$(5.2) \qquad 1 + x = 1$$

If we define $y \subset x$ iff $x + y = x$, then a partial order in K is obtained. Indeed, we ~~clearly~~ have $x \subset x$ since $x + x = x$. If $y \subset x$ and $x \subset y$ then $y = x + y = x$. If $z \subset y$ and $y \subset x$ then $y + z = y$ and $x + y = x$. Thus $x + z = = x + y + z = x + y = x$ so that $z \subset x$. We also have $0 \subset x \subset 1$ for all $x \in K$. Further $y \subset x$ implies $z + y \subset z + x$ and $zy \subset zx$.

We assert that

$$(5.3) \qquad x + y = \mathrm{l.u.b.}(x, y)$$

$$(5.4) \qquad xy = \mathrm{glb}(x, y)$$

Indeed we have $x \subset x + y$ since $x + y + x = x + y$. Assume $x \subset z$ and $y \subset z$. Then $x + y \subset z + z = z$. This proves (5.3). Since $1 + x = x$ we have $x + xy =$

and thus $xy < x$. Suppose $z < x$ and $z < y$. Then $z = zz < xy$ and this proves (5.4).

We thus find that K becomes a distributive lattice with 0 as smallest element and 1 as largest element. The converse also holds. Any such distributive lattice becomes a semiring using (5.3) and (5.4) as definitions. Further, (5.1) and (5.2) hold. Because of this we shall call a commutative semiring satisfying conditions (5.1) and (5.2) a <u>lattice semiring</u>.

Let K be a lattice semiring and let F be a finite subset of K of cardinality n. Because $xx = x$, it is easy to see that at most 2^n distinct monomials can be built using the elements of F. Because $x + x = x$, it follows that there are at most 2^{2^n} ~~polynomials th~~ distinct polynomials that can be built using the elements of F. This implies that F generates a finite subsemiring $\overline{F}$ of K

~~Now let $G=(\Xi, \xi_0, R)$ be a grammar in a monoid M with weights in K.~~ Since $\overline{F}$ is a complete lattice it follows easily that $\overline{F}$ is a $\mathcal{C}$-semiring.

Now let $G=(\Xi, \xi_0, R)$ be an algebraic grammar in a monoid M with weights in K. Let

$$F=\{ r\mu \mid r\in R \}$$

be the set of all weights of the rules of G. Then F is finite. For each derivation d in G we must have $d\mu \in \overline{F}$. Thus even though there may be infinitely many derivations, they can have only a finite number of weights. Since $\overline{F}$

is a $\mathcal{C}$-semiring, the summation

$$mL = \sum_{\xi_0 d = m} d\mu$$

for $m \in M$ is well defined. Thus the language L defined by G becomes a $\bar{F}$-subset of M and thus also ~~a~~ K-subset of M. Thus even though K is not necessarily a $\mathcal{C}$-semiring in many ways it behaves like one.

The most interesting example of a lattice semiring is the closed unit interval $\mathcal{F} = [0, 1]$ with the usual ordering. Thus in $\mathcal{F}$ we have

$$x + y = \sup(x, y)$$
$$xy = \inf(x, y)$$

The reader will easily verify that $\mathcal{F}$ is a $\mathcal{C}$-semiring. The $\mathcal{F}$-subsets of a set X are called _fuzzy subsets_ of X.

??

If we return to the grammar G above assuming that the weights are in Ξ we find that if $d = r_1 \cdots r_n$ then

$$d\mu = \min_{1 \le i \le n} (r_i \mu)$$

If we consider all the derivations $d : \xi_0 \to$ for a fixed m, then the set of weights $d\mu$ is finite. Thus

$$mL = \sum_{\xi_0 d = m} d\mu = \max_{\xi_0 d = m} (d\mu)$$

It follows in particular, that there exists a derivation $d : \xi_0 \to m$ such that $d\mu = mL$, i.e. a derivation of maximal weight. We shall see later (- - -) that one can always choose d to be "reasonably" short.