Solution to Exercise I.1.4

a) Representing (a,b) by $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ gives the first statements. Regarding the congruence, if $(a,b) \equiv (a',b')$ and $(x,y) \in K \times K$, then for some c, $a+b'+c = a'+b+c$. Thus $a+x+b'+y+c = a'+x+b+y+c$, which shows that $(a+x, b+y) \equiv (a'+x, b'+y)$, that is $(a,b)+(x,y) \equiv (a',b')+(x,y)$. Moreover, let $d = cx+cy$; then

$$\begin{aligned} ax+by+a'y+b'x+d &= (a+b'+c)x + (b+a'+c)y \\ &= (a'+b+c)x + (a+b'+c)y \\ &= a'x+b'y+ay+bx+d\,. \end{aligned}$$

Therefore $(ax+by, ay+bx) \equiv (a'x+b'y, a'y+b'x)$, that is $(a,b)(x,y) \equiv (a',b')(x,y)$. Finally, L is a ring since $(a,b)+(b,a) \equiv (0,0)$.

b) One has $p \circ i(b) = p \circ i(c) \iff p(b,0) \equiv p(c,0) \iff (b,0) \equiv (c,0) \iff \exists a, a+b = a+c$. This proves the injectivity statement. If p is injective, K is embedded in a ring. If K may be embedded in a ring, it must necessarily be regular.

c) Since K is regular, we have $(a,b) \equiv (a',b') \iff a+b' = b+a'$. Suppose now that L has no zero divisor and that $ac+bd = ad+bc$. Then $(a,b)(c,d) = (ac+bd, ad+bc) \equiv (0,0)$. This implies that $a=b$ or $c=d$. The converse is proved similarly.

Now, it is well-known from the construction of the field of fractions, that a commutative ring is embeddable in a field if and only if it without zero divisors.

d) is clear.

e) In K, one has $ac+bd = ad+bc$, but not $a=b$ nor $c=d$ since I has no element of degree 1. Note that K is regular since it is a subsemiring of a ring. Moreover, if $P, Q, R \in \mathbb{N}[a,b,c,d]$ and $PQ \equiv PR \bmod I$, then $(a-b)(c-d)$ divides $P(Q-R)$ in $\mathbb{Z}[a,b,c,d]$. If $P \neq 0$, then $a-b$ cannot divide P, since P has nonnegative coefficients. Thus $(a-b)(c-d)$ divides $Q-R$ and $Q \equiv R \bmod I$. Thus K is simplifiable.