**Solution to Exercise I.3.4**

Let $(M, +)$ be a commutative monoid, with the subinvariant ultrametric $d$.

Given $a, b \in M$, and a fixed $\epsilon > 0$, $a \sim b$ denotes that $a, b$ are $\epsilon$-near of each other, this means $d(a, b) < \epsilon$. Observe that $\sim$ is an equivalence relation using the fact that $d$ is an ultrametric:

1. Obviously $\sim$ is reflexive and symmetric.
2. If $a \sim b$ and $b \sim c$, then $d(a, c) \leq \max(d(a, b), d(b, c)) < \epsilon$, hence $a \sim c$. Hence $\sim$ is transitive.
3. Also note that if $a \sim b$ then by the fact that $d$ is subinvariant we get $a + c \sim b + c$ for all $c$.
4. Moreover if $c \sim a$ and $c \sim a+b$, then $c \sim c+b$. This is because $d(c, c+b) \leq \max(d(c, a + b), d(a + b, c + b)) \leq \max(d(c, a + b), d(a, c)) < \epsilon$.

Now suppose $\sum a_n = L$, then given any permutation $\pi$ of the elements of this series, we have $\sum a_{\pi(n)} = L$.

*Proof.* Fix $\epsilon > 0$. There exists $K$ such that for all $n, \ell > K$ we have $S_n \sim L$ and $S_n \sim S_\ell$, where $S_n$ is the partial sum of $n$ terms. Fix any $n_1 > K$. Take any $n_3, n_2$ such that $n_3 > n_2 > n_1$. We have the following:

1. We have $S_{n_1} \sim S_{n_2}$ and $S_{n_1} \sim S_{n_3}$. By the above item number 4, we get $S_{n_1} \sim S_{n_1} + a_{n_2+1} + ... + a_{n_3}$.
2. Similarly $S_{n_1} \sim S_{n_2-1}$ and $S_{n_1} \sim S_{n_3}$ entail that $S_{n_1} \sim S_{n_1} + a_{n_2} + ...a_{n_3}$.
3. and thus from $S_{n_1} \sim S_{n_1} + a_{n_2+1} + ...a_{n_3}$ and $S_{n_1} \sim S_{n_1} + a_{n_2} + ...a_{n_3}$ conclude that $S_{n_1} \sim S_{n_1} + a_{n_2}$.

So far we have shown that for any $n_1, n_2$ such that $n_2 > n_1 > K$ we have $S_{n_1} \sim S_{n_1} + a_{n_2}$. We can generalize this iteratively and inductively as follows. Take any finite set $I \subset \mathbb{N}$, such that for all $i \in I$ we have $i > n_1$. Then we have

$$S_{n_1} \sim S_{n_1} + \sum_{i \in I} a_i \, .$$

Let's see this in the simple case $I = \{n_2, n_3\}$, where $n_3 > n_2 > n_1 > K$. By the basis of the induction: $S_{n_1} \sim S_{n_1} + a_{n_2}$. Since $S_{n_1} \sim S_{n_3}$ and $S_{n_1} \sim S_{n_3-1}$, by the transitivity of $\sim$ we get $S_{n_3} \sim S_{n_1} + a_{n_2}$ and $S_{n_3-1} \sim S_{n_1} + a_{n_2}$. Now again by the item number 4 above, we have $S_{n_1} + a_{n_2} + a_{n_3} \sim S_{n_1} + a_{n_2}$. Since $S_{n_1} \sim S_{n_1} + a_{n_2}$, by the transitivity of $\sim$ we get $S_{n_1} \sim S_{n_1} + a_{n_2} + a_{n_3}$.

Now let $M$ be large enough such that $\{\pi(1), \pi(2), \ldots, \pi(M)\} \supseteqq \{1, \ldots, n_1\}$. Take any $\ell > M$. We have:

$$\sum_{1 \leq i \leq \ell} a_{\pi(i)} = S_{n_1} + \sum_{i \in I} a_i$$

for some finite set $I$.Thus

$$\sum_{1 \leq i \leq \ell} a_{\pi(i)} \sim S_{n_1} \,,$$

and since $L \sim S_{n_1}$, we get by transitivity:

$$\sum_{1 \leq i \leq \ell} a_{\pi(i)} \sim L \,.$$

This completes the proof. □