Solution to Exercise I.3.5

a) It is easy to verify that $V \subset V^{\perp\perp}$. Conversely, let $P \in K\langle\langle A\rangle\rangle \setminus V$. Then, by the remark at the beginning of the exercise, we find $S \in K\langle\langle A\rangle\rangle$ such that$S \in V^{\perp}$ and $(S, P) \neq 0$. Hence $P \notin V^{\perp\perp}$.

b) We show that each continuous linear function on $K\langle\langle A\rangle\rangle$ is of the form $S \mapsto (S, P)$ for some unique $P \in K\langle A\rangle$.

Uniqueness is clear. Moreover, such a function is linear and continuous. Indeed, if (S_n) is a sequence in $K\langle\langle A\rangle\rangle$ converging to S, then, for large n, $(S_n, w) = (S, w)$ for any word w of length $\leq \deg(P)$. Then $(S_n, P) = (S, P)$ and therefore $\lim_n (S_n, P) = (S, P)$.

Conversely, let h be some continuous linear function on $K\langle\langle A\rangle\rangle$. Let $n \mapsto w_n$ be some bijection $\mathbb{N} \to A^*$. Considering w_n as an element of $K\langle\langle A\rangle\rangle$, we have $\lim_n w_n = 0$. Thus $\lim_n h(w_n) = 0$. Hence we have $h(w_n) = 0$ for $n \geq N$. Let $P = \sum_{n\in\mathbb{N}} h(w_n)w_n \in K\langle A\rangle$. For any word w, $w = w_k$ for some k, hence $(w, P) = h(w)$. Thus the continuous linear functions h and $S \mapsto (S, P)$ coincide on A^*, hence on $K\langle\langle A\rangle\rangle$ since $K\langle A\rangle$ is dense in $K\langle\langle A\rangle\rangle$.

c) It is easy to verify that $W \subset W^{\perp\perp}$. Now, let $S \in K\langle\langle A\rangle\rangle \setminus W$. Since this latter set is open, it contains, for some n, the set

$$\mathcal{V}_n = \{T \in K\langle\langle A\rangle\rangle \mid \forall w \in A^*, |w| \leq n \Rightarrow (T, w) = (S, w)\}.$$

Indeed, the sets $\mathcal{V}_n$ constitute a fundamental system of neighborhoods of S in $K\langle\langle A\rangle\rangle$. Let π be the projection $K\langle\langle A\rangle\rangle \to E = \sum_{|w|\leq n} Kw$ which maps T onto $\sum_{|w|\leq n}(T, w)w$. Then $\pi(S) \notin \pi(W)$, since otherwise $\mathcal{V}_n$ intersects W. Thus we may find a linear function φ on E such that $\varphi(\pi(W)) = 0$ and $\varphi(\pi(S)) \neq 0$. Since $\{w \in A^* \mid |w| \leq n\}$ is a basis of E, we may find $P \in E$ such that $\varphi(Q) = (Q, P)$ for any Q in E. Finally, for T in $K\langle\langle A\rangle\rangle$, $\varphi(\pi(T)) = (T, P)$ and therefore $(W, P) = 0$ and $(S, P) \neq 0$. We conclude as in a) that $S \notin W^{\perp\perp}$.