Solution to Exercise 4.2.2

We start with the identity to be proven, and transform it by algebraic operations and inversions. The actual proof is obtained by going backwards. Let $x = 1 - a$ and $y = b$. We have to prove that

$$\frac{1}{2}(x+iy)^{-1} + \frac{1}{2}(x-iy)^{-1} = (x+yx^{-1}y)^{-1} .$$

Multiply both sides by 2, by $x + yx^{-1}y$ on the left and by $x + iy$ on the right. This gives

$$x + yx^{-1}y + (x+yx^{-1}y)(x-iy)^{-1}(x+iy) = 2(x+iy) .$$

Writing $x + iy = x - iy + 2iy$, and cancelling x on both sides, we obtain:

$$yx^{-1}y + x + yx^{-1}y + (x+yx^{-1}y)(x-iy)^{-1}2iy = x + 2iy .$$

Now cancel x again and divide by y on the right (NB: backwards, this will be multiplication by y, so we do not invert noninvertible series, as predicted by Theorem 2.1):

$$2yx^{-1} + 2i(x+yx^{-1}y)(x-iy)^{-1} = 2i .$$

Multiply by $x - iy$ on the right:

$$2yx^{-1}(x-iy) + 2i(x+yx^{-1}y) = 2i(x-iy) .$$

That is:

$$2y - 2iyx^{-1}y + 2ix + 2iyx^{-1}y = 2ix + 2y .$$

Formidable!