Solution to Exercise 8.1.7

a) Assume that $Q(x) = (1-\alpha x)(1-\beta x)$ for some rational numbers α and β. Since $b = \alpha\beta$ and $a = \alpha + \beta$, both α and β are positive. To show that they are integers, set $\alpha = p/q$. Then $b = p/q(a - p/q) = p(qa - p)/q^2$. If $\gcd(p, q) = 1$, then q divides p and therefore $q = 1$. Thus $f(x)$ has star height 1.

b) Assume $f(x)$ is $\mathbb{N}$-rational and has star height 1. Then it is a sum of products series of the form $P(x)/(1 - N(x))$, where $P(x) \in \mathbb{Z}[x]$, $N(x) \in \mathbb{N}[x]$ and $N(0) = 0$. Reducing to the same denominator, $f(x)$ is the quotient of a polynomial by a product of polynomials of the form $1 - N(x)$. Since $Q(x)$ is irreductible and $\mathbb{Z}[x]$ is factorial, it divides one of these polynomials, therefore $Q(x)P(x) = 1 - N(x)$ for some $P(x)$ as required.

c) Similar to (a).

d) Set $M(x) = 1 - N(x)$. Then $M(0) = 1$, $M(x)$ is strictly decreasing for increasing real positive x, and $M(1) < 0$. Therefore $M(x)$ has a positive root. Since the derivative of $M(x)$ is always strictly negative, the root is simple. Thus $Q(x)$ cannot divide $M(x)$.

e) Set $Q(x) = (1 - \alpha x)(1 - \beta x)$ for some rational numbers α and β. Since $b = \alpha\beta$ and $a = \alpha + \beta$, one gets

$$b = \alpha\beta = (\alpha - 1)(\beta - 1) + \alpha + \beta - 1 \geq \alpha + \beta - 1 = a - 1\,,$$

in contradiction avec the condition $a \geq 2 + b$.