Solution to Exercise 8.3.7

Setting $Q(t) = 1 - (n-1)t - \cdots - (n-1)t^{N-1}$, one gets

$$1 - nt = (1-t)(1 - Q(t)) - (n-1)t^N\,,$$

whence $1 - nt + t^N = (1-t)(1 - Q(t) - (n-2)t^N t^*)$. Consequently

$$\frac{1}{1-nt+t^N} = \frac{1}{(1-t)}\,\frac{1}{(1-Q(t)-(n-2)t^N t^*)} = t^*(Q(t) + (n-2)t^N t^*)^*\,.$$

For the second fraction, we start quite similarly with

$$\begin{aligned} 1 - nt + t^N\frac{1-t}{1-t^N} &= (1-t)(1-Q(t)) - (n-1)t^N + t^N\frac{1-t}{1-t^N} \\ &= (1-t)\Big(1 - Q(t) - t^N\Big(\frac{n-1}{1-t} - \frac{1}{1-t^N}\Big)\Big)\,. \end{aligned}$$

It remains to transform $(n-1)/(1-t) - 1/(1-t^N)$. For this, we use the identity

$$1/(1-t) = (1 + t + \cdots + t^{N-1})/(1-t^N)\,,$$

getting

$$\begin{aligned} (n-1)/(1-t) - 1/(1-t^N) &= (n-2)/(1-t^N) + (n-1)t(1 + t + \cdots t^{N-2})/(1-t^N) \\ &= (n-2)(t^N)^* + (n-1)t(1+t+\cdots t^{N-2})(t^N)^*\,. \end{aligned}$$

Altogether, this gives

$$\begin{aligned} &1 - nt + t^N(1-t)/(1-t^N) = \\ &\qquad (1-t)(1 - Q(t) - t^N(n-2)(t^N)^* + (n-1)t(1+t+\cdots t^{N-2})(t^N)^*)\,. \end{aligned}$$

and therefore

$$\frac{1}{1-nt+t^N(1-t)/(1-t^N)} = t^*(Q(t) + t^N(n-2)(t^N)^* + (n-1)t(1+t+\cdots t^{N-2})(t^N)^*)^*\,.$$