CHAPTER 2

Sturmian Words

2.0. Introduction

Sturmian words are infinite words over a binary alphabet that have exactly $n + 1$ factors of length n for each $n \geq 0$. It appears that these words admit several equivalent definitions, and can even be described explicitly in arithmetic form. This arithmetic description is a bridge between combinatorics and number theory. Moreover, the definition by factors makes that Sturmian words define symbolic dynamical systems. The first detailed investigations of these words were done from this point of view. Their numerous properties and equivalent definitions, and also the fact that the Fibonacci word is Sturmian, has lead to a great development, under various terminologies, of the research.

The aim of this chapter is to present basic properties of Sturmian words and of their transformation by morphisms. The style of exposition relies basically on combinatorial arguments.

The first section is devoted to the proof of the Morse-Hedlund theorem stating the equivalence of Sturmian words with the set of balanced aperiodic word and the set of mechanical words of irrational slope. We also mention several other formulations of mechanical words, such as rotations and cutting sequences. We next give properties of the set of factors of one Sturmian word, such as closure under reversal, the minimality of the associated dynamical system, the fact that the set depends only on the slope, and we give the description of special words.

In the second section, we give a systematic exposition of standard pairs and standard words. We prove the characterization by the double palindrome property, describe the connection with Fine and Wilf's theorem. Then, standard sequences are introduced to connect standard words to characteristic Sturmian words. The relation to Beatty sequences is in the exercises. This section also contains the enumeration formula for finite Sturmian words. It ends with a short description of frequencies.

The third section starts by proving that the monoid of Sturmian morphisms is generated by three well-known morphisms. Then, standard morphisms are investigated. A description of all Sturmian morphisms in terms of standard morphisms is given next. The section ends with the characterization of those algebraic numbers that yield fixed points by standard morphisms.

Some problems are just exercises, but most contain additional properties of Sturmian words, with appropriate references. It is difficult to trace back many of the properties of Sturmian words, because of the scattered origins, terminology and notation. When we quote a reference in the Notes section, we are only relatively certain that it is the source of the result.

In this chapter, words will be over a binary alphabet $A = \{0, 1\}$.

2.1. Equivalent definitions

This section is devoted to the proof of a theorem (Theorem 2.1.13) stating the equivalence of three properties, all defining what we call Sturmian words. We start by defining Sturmian words to have minimal complexity among aperiodic infinite words. We first prove that Sturmian words are exactly the aperiodic balanced words. We then introduce so called mechanical words and prove that these yield another characterization of Sturmian words. Other formulations of the mechanical definition, by rotation and cutting sequences, are given in the second paragraph. The third paragraph contains several properties concerning the set of factors of a Sturmian word.

2.1.1. Complexity and balance

The *complexity function* of an infinite word x over some alphabet A was defined in Chapter 1. It is the function that counts, for each integer $n \geq 0$, the number $P(x, n)$ of factors of length n in x:

$$P(x, n) = \mathrm{Card}(F_n(x)) .$$

A *Sturmian* word is an infinite word s such that $P(s, n) = n + 1$ for any integer $n \geq 0$. According to Theorem 1.3.13, Sturmian words are aperiodic infinite words of minimal complexity. Since $P(s, 1) = 2$, any Sturmian word is over two letters. A *right special* factor of a word x is a word u such that $u0$ and $u1$ are factors of x. Thus, s is a Sturmian word if and only if it has exactly one right special factor of each length.

A suffix of a Sturmian word is a Sturmian word.

EXAMPLE 2.1.1. We show that the *Fibonacci* word

$$f = 0100101001001010010100100101001001\cdots$$

defined in Chapter 1 is Sturmian. It will be convenient, in this chapter, to start the numeration of finite Fibonacci words differently, and to set $f_{-1} = 1$, $f_0 = 0$.

Since $f = \varphi(f)$, it is a product of words 01 and 0. Thus, the word 11 is not a factor of f and consequently $P(f, 2) = 3$. The word 000 is not a factor of $\varphi(f)$, since otherwise it is a prefix of some $\varphi(x)$ for a factor x of f, and x has to start with 11.

To show that f is Sturmian, we prove that f has exactly one right special factor of each length.

We start by showing that, for no word x, both $0x0$ and $1x1$ are factors of f. This is clear if x is the empty word and if x is a single letter. Arguing by induction on the length, assume that $0x0$ and $1x1$ are in $F(f)$. Then x starts and ends with 0, and $x = 0y0$ for some y. Since $00y00$ and $10y01$ have to be factors of $\varphi(f)$, there exists a factor z of f such that $\varphi(z) = 0y$. Moreover, $00y0 = \varphi(1z1)$ and $010y01 = \varphi(0z0)$, showing that $1z1$ and $0z0$ are factors of f. This is a contradiction because $|z| \leq |\varphi(z)| < |x|$.

We show now that f has at most one right special factor of each length. Assume indeed that u and v are right special factors of the same length, and let x be the longest common suffix of u and v. Then the four words $0x0$, $0x1$, $1x0$, $1x1$ are factors of f, which contradicts our previous observation.

To show that f has at least one right special factor of each length, we use the relation

$$f_{n+2} = g_n \tilde{f}_n \tilde{f}_n t_n \qquad (n \geq 2) \tag{2.1.1}$$

where $g_2 = \varepsilon$ and for $n \geq 3$

$$g_n = f_{n-3} \cdots f_1 f_0, \quad t_n = \begin{cases} 01 & \text{if } n \text{ is odd,} \\ 10 & \text{otherwise.} \end{cases}$$

Observe that the first letter of $\tilde{f}_n$ is the opposite of the first letter of t_n. This proves that $\tilde{f}_n$ is a right special factor for each $n \geq 2$. Since a suffix of a right special factor is itself a right special factor, this proves that right special factors of any length exist.

Equation (2.1.1) is proved by induction. Indeed, $f_4 = \varepsilon(010)(010)10$ and $f_5 = 0(10010)(10010)01$. Next, is it easily checked by induction that

$$\varphi(\tilde{u})0 = 0(\varphi(u))^{\sim} \tag{2.1.2}$$

for any word u. It follows that $\varphi(\tilde{f}_n t_n) = 0\tilde{f}_{n+1} t_{n+1}$ and since $\varphi(g_n)0 = g_{n+1}$, one gets (2.1.1).

We now start to give another description of Sturmian words, namely as balanced words. The *height* of a word x is the number $h(x)$ of letters equal to 1 in x. Given two words x and y of the same length, their *balance* $\delta(x, y)$ is the number

$$\delta(x, y) = \big|h(x) - h(y)\big|$$

A set of words X is *balanced* if

$$x, y \in X, \ |x| = |y| \ \Rightarrow \ \delta(x, y) \leq 1$$

A finite or infinite word is itself balanced if the set of its factors is balanced.

Proposition 2.1.2. *Let X be a factorial set of words. If X is balanced, then for all $n \geq 0$,*

$$\mathrm{Card}(X \cap A^n) \leq n + 1 \,.$$

Proof. The conclusion is clear for $n = 0, 1$, and it holds for $n = 2$ because X cannot contain both 00 and 11. Arguing by contradiction, let $n \geq 3$ be the smallest integer for which the statement is false. Set $Y = X \cap A^{n-1}$ and $Z = X \cap A^n$. Then $\mathrm{Card}(Y) \leq n$ and $\mathrm{Card}(Z) \geq n + 2$. For each $z \in Z$, its suffix of length $n - 1$ is in Y. By the pigeon-hole principle, there exist two distinct words $y, y' \in Y$ such that all four words $0y, 1y, 0y', 1y'$ are in Z. Since $y \neq y'$ there exists a word x such that $x0$ and $x1$ are prefixes of y and y'. But then, both $0x0$ and $1x1$ are words in X, showing that X is unbalanced. ∎

The argument used in the proof can be refined as follows.

PROPOSITION 2.1.3. *Let X be a factorial set of words. The set X is unbalanced if and only if there exists a palindrome word w such that $0w0$ and $1w1$ are in X.*

Proof. The condition is clearly sufficient. Conversely, assume that X is unbalanced. Consider two words $u, v \in X$ of the same length n such that $\delta(u, v) \geq 2$, and take them of minimal length. The first letters of u and v are distinct, and so are the last letters. Assuming that u starts with 0 and v with 1, there are factorizations $u = 0wau'$ and $v = 1wbv'$ for some words w, u', v' and letters $a \neq b$. In fact $a = 0$ and $b = 1$ since otherwise $\delta(u', v') = \delta(u, v)$, contradicting the minimality of n. Thus, again by minimality, $u = 0w0$ and $v = 1w1$.

Assume next that w is not a palindrome. Then there is a prefix z of w and a letter a such that za is a prefix of w, $\tilde{z}$ is a suffix of w but $a\tilde{z}$ is not a suffix of w. Then of course $b\tilde{z}$ is a suffix of w, where b is the other letter. This gives a proper prefix $0za$ of u and a proper suffix $b\tilde{z}1$ of v. If $a = 0$ and $b = 1$, then $\delta(0z0, 1\tilde{z}1) = 2$, contradicting the minimality of n. But then $u = 0z1u''$ and $v = v''1\tilde{z}0$ for two words with $\delta(u'', v'') = \delta(u, v)$, contradicting again the minimality. Thus w is a palindrome. ∎

REMARK 2.1.4. In the proof that the Fibonacci word f is Sturmian given in Example 2.1.1, we actually started by showing that f is balanced.

THEOREM 2.1.5. *Let x be an infinite word. The following conditions are equivalent.*

(i) *x is Sturmian,*
(ii) *x is balanced and aperiodic.*

Proof. If x is aperiodic, then $P(x, n) \geq n + 1$ for all n by Theorem 1.3.13. If x is balanced, then by Proposition 2.1.2, $P(x, n) \leq n + 1$ for all n. Thus x is Sturmian.

To prove the converse, we assume x is Sturmian and unbalanced, and show that x is eventually periodic. Since x is unbalanced, there is a palindrome word w such that $0w0$, $1w1$ are factors of x. This shows that w is right special. Set $n = |w| + 1$. Since x is Sturmian, there is a unique right special factor of length n, which is either $0w$ or $1w$. We suppose that $0w$ is right special, so $1w$ is not, and $0w1$ is a factor of x and $1w0$ is not.

Any occurrence of $1w$ in x is followed by the letter 1. Let v be a word of length $n-1$ such that $u = 1w1v$ is in $F(x)$. The word u has length $2n$. We prove that all factors of length n of u are conservative. In view of Proposition 1.3.14, x is eventually periodic.

To show the claim, it suffices to prove that the only right special factor of length n, that is $0w$, is not a factor of u. Assume the contrary. Then there exist factorizations $w = s0t, v = yz, w = t1y$.

<table>
<tr><td colspan="7">u</td></tr>
<tr><td>1</td><td colspan="3">w</td><td>1</td><td colspan="2">v</td></tr>
<tr><td colspan="2"></td><td>0</td><td colspan="3">w</td><td></td></tr>
<tr><td>1</td><td>s</td><td>0</td><td>t</td><td>1</td><td>y</td><td>z</td></tr>
</table>

Since w is a palindrome, the first factorization implies $w = \tilde{t}0\tilde{s}$, and the letter following the prefix t in w is both a 0 and a 1. ∎

The *slope* of a nonempty word x is the number $\pi(x) = \dfrac{h(x)}{|x|}$.

EXAMPLE 2.1.6. The height of $x = 0100101$ is 3, and its slope is $3/7$. The word x can be drawn on a grid by representing a 0 (resp. a 1) as a horizontal (resp. a diagonal) unit segment. This gives a polygonal line from the origin to the point $(|x|, h(x))$, and the line from the origin to this point has slope $\pi(x)$. See Figure 2.1.

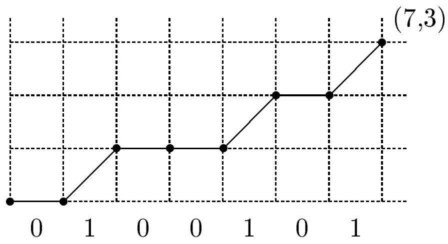

Figure 2.1. Height and slope of the word 0100101.

It is easily checked that

$$\pi(xy) = \frac{|x|}{|xy|}\pi(x) + \frac{|y|}{|xy|}\pi(y)$$

PROPOSITION 2.1.7. *A factorial set of words X is balanced if and only if, for all $x, y \in X$, $x, y \neq \varepsilon$,*

$$\big|\pi(x) - \pi(y)\big| \;<\; \frac{1}{|x|} + \frac{1}{|y|}\;. \tag{2.1.3}$$

Proof. Assume first that (2.1.3) holds. For $x, y \in X$ of the same length, the equation gives

$$\big|h(x) - h(y)\big| < 2$$

showing that X is balanced.

Conversely, assume that X is balanced, and let x, y be in X. If $|x| = |y|$, then (2.1.3) holds. Assume $|x| > |y|$, and set $x = zt$, with $|z| = |y|$. Arguing by induction on $|x| + |y|$, we have

$$\big|\pi(t) - \pi(y)\big| < \frac{1}{|t|} + \frac{1}{|y|}$$

and since X is factorial, $|h(z) - h(y)| \leq 1$, whence $\big|\pi(z) - \pi(y)\big| \leq \dfrac{1}{|y|}$. Next,

$$\begin{aligned}\pi(x) - \pi(y) &= \frac{|z|}{|x|}\pi(z) + \frac{|t|}{|x|}\pi(t) - \pi(y)\\ &= \frac{|z|}{|x|}\Big(\pi(z) - \pi(y)\Big) + \frac{|t|}{|x|}\Big(\pi(t) - \pi(y)\Big)\end{aligned}$$

thus

$$\big|\pi(x) - \pi(y)\big| < \frac{1}{|x|} + \frac{|t|}{|x|}\Big(\frac{1}{|y|} + \frac{1}{|t|}\Big) = \frac{1}{|x|} + \frac{1}{|y|}\ .$$ ∎

Corollary 2.1.8. *Let x be an infinite balanced word, and for each $n \geq 1$, let x_n be the prefix of length n of x. The sequence $(\pi(x_n))_{n\geq 1}$ converges for $n \to \infty$.*

Proof. Indeed, (2.1.3) shows that $(\pi(x_n))_{n\geq 1}$ is a Cauchy sequence. ∎

The limit

$$\alpha = \lim_{n\to\infty} \pi(x_n)$$

is the *slope* of the infinite word x.

Example 2.1.9. To compute the slope of an infinite balanced word, it suffices to compute the limit of the slopes of an increasing sequence of prefixes (or even factors, as shown by the next proposition). For the Fibonacci infinite word, the slopes of the finite Fibonacci words f_n are easily computed. Indeed, $|f_n| = F_n$ and $h(f_n) = F_{n-2}$, whence

$$\pi(f) = \lim_{n\to\infty} \frac{F_{n-2}}{F_n} = \frac{1}{\tau^2}\ ,$$

where $\tau = (1 + \sqrt{5})/2$.

Proposition 2.1.10. *Let x be an infinite balanced word with slope α. For every nonempty factor u of x, one has*

$$\big|\pi(u) - \alpha\big| \leq \frac{1}{|u|}\ . \tag{2.1.4}$$

More precisely, one of the following holds: either

$$\alpha|u| - 1 < h(u) \le \alpha|u| + 1 \qquad \text{for all } u \in F(x) \tag{2.1.5}$$

or

$$\alpha|u| - 1 \le h(u) < \alpha|u| + 1 \qquad \text{for all } u \in F(x) \tag{2.1.6}$$

Of course, the inequalities in (2.1.5) and (2.1.6) are strict if α is irrational.

Proof. Let x_n be the prefix of length n of x. Given some ε, consider n_0 such that for all $n \ge n_0$,

$$\left|\pi(x_n) - \alpha\right| \le \varepsilon .$$

Then, using (2.1.3),

$$\left|\pi(u) - \alpha\right| \le \left|\pi(u) - \pi(x_n)\right| + \left|\pi(x_n) - \alpha\right| < \frac{1}{|u|} + \frac{1}{n} + \varepsilon$$

For $n \to \infty$ and then $\varepsilon \to 0$, the inequality follows. Equation (2.1.4) means that

$$\alpha|u| - 1 \le h(u) \le \alpha|u| + 1$$

If the second claim were wrong, there would exist u, v in $F(x)$ such that $\alpha|u| - 1 = h(u)$ and $\alpha|v| + 1 = h(v)$. But then $|\pi(u) - \pi(v)| = 1/|u| + 1/|v|$, in contradiction with (2.1.3). ■

PROPOSITION 2.1.11. *Let x be an infinite balanced word. The slope α of x is a rational number if and only if x is eventually periodic.*

Proof. If $x = uy^\omega$, then

$$\pi(uy^n) = \frac{h(u) + nh(y)}{|u| + n|y|} \to \pi(y)$$

for $n \to \infty$, showing that the slope is rational.

For the converse, we suppose that (2.1.5) holds. The other case is symmetric. The slope of x is a rational number $\alpha = q/p$ with q and p relatively prime. By (2.1.5), any factor u of x of length p has height q or $q+1$. There are only finitely many occurrences of factors of length p and height $q+1$, since otherwise there is a factor $w = uzv$ of x with $|u| = |v| = p$ and $h(u) = h(v) = q+1$. In view of (2.1.5)

$$2 + 2q + h(z) = h(uzv) \le 1 + \alpha p + \alpha|z| + \alpha p = 1 + 2q + \alpha|z|$$

whence $h(z) \le \alpha|z| - 1$, in contradiction with (2.1.5).

By the preceding observation, there is a factorization $x = ty$ such that every word in $F_p(y)$ has the same height. Consider now an occurrence azb of a factor in y of length $p+1$, with a and b letters. Since $h(az) = h(zb)$, one has $a = b$. This means that y is periodic with period p. Consequently, x is eventually periodic. ■

2.1.2. Mechanical words, rotations

Given two real numbers α and ρ with $0 \le \alpha \le 1$, we define two infinite words

$$s_{\alpha,\rho} : \mathbb{N} \to A, \quad s'_{\alpha,\rho} : \mathbb{N} \to A$$

by

$$\begin{aligned} s_{\alpha,\rho}(n) &= \lfloor \alpha(n+1)+\rho \rfloor - \lfloor \alpha n + \rho \rfloor \\ s'_{\alpha,\rho}(n) &= \lceil \alpha(n+1)+\rho \rceil - \lceil \alpha n + \rho \rceil \end{aligned} \qquad (n \ge 0)$$

It is easy to check that $s_{\alpha,\rho}(n)$ and $s'_{\alpha,\rho}(n)$ indeed are in $\{0,1\}$. The word $s_{\alpha,\rho}$ is the *lower mechanical word* and $s'_{\alpha,\rho}$ is the *upper mechanical word* with *slope* α and *intercept* ρ. (This slope will be shown in a moment to be the same as the slope of a balanced word.) It is clear that if $\rho - \rho'$ is an integer, then $s_{\alpha,\rho} = s_{\alpha,\rho'}$ and $s'_{\alpha,\rho} = s'_{\alpha,\rho'}$. Thus we may assume $0 \le \rho < 1$ or $0 < \rho \le 1$ (both will be useful).

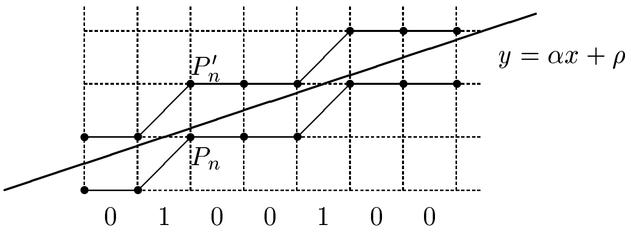

Figure 2.2. Mechanical words associated with the line $y = \alpha x + \rho$.

The terminology stems from the following graphical interpretation (see Figure 2.2). Consider the straight line with equation $y = \alpha x + \rho$. The points with integer coordinates just below this line are $P_n = (n, \lfloor \alpha n + \rho \rfloor)$. Two consecutive points P_n and P_{n+1} are joined by a straight line segment that is horizontal if $s_{\alpha,\rho}(n) = 0$ and diagonal if $s_{\alpha,\rho}(n) = 1$.

The same observation holds for the points $P'_n = (n, \lceil \alpha n + \rho \rceil)$ located just above the line.

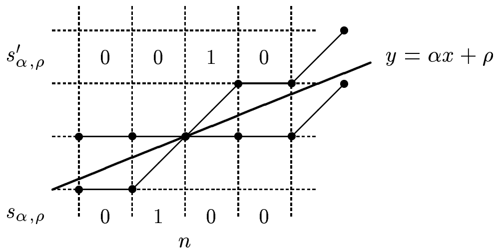

Figure 2.3. Mechanical words with an integral point.

Clearly,

$$s_{0,\rho} = s'_{0,\rho} = 0^{\omega}, \quad s_{1,\rho} = s'_{1,\rho} = 1^{\omega}$$

Let $0 < \alpha < 1$. Since $1 + \lfloor \alpha n + \rho \rfloor = \lceil \alpha n + \rho \rceil$ whenever $\alpha n + \rho$ is not an integer, one has $s_{\alpha,\rho} = s'_{\alpha,\rho}$ excepted when $\alpha n + \rho$ is an integer for some $n \geq 0$. In this case (see Figure 2.3),

$$s_{\alpha,\rho}(n) = 0, \quad s'_{\alpha,\rho}(n) = 1$$

and, if $n > 0$,

$$s_{\alpha,\rho}(n-1) = 1, \quad s'_{\alpha,\rho}(n-1) = 0$$

Thus, if α is irrational, $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ differ by at most one factor of length 2. A mechanical word is *irrational* or *rational* according to its slope is rational or irrational.

A special case deserves consideration, namely when $0 < \alpha < 1$ and $\rho = 0$. In this case, $s_{\alpha,0}(0) = \lfloor \alpha \rfloor = 0$, $s'_{\alpha,0}(0) = \lceil \alpha \rceil = 1$, and if α is irrational

$$s_{\alpha,0} = 0c_{\alpha}, \quad s'_{\alpha,0} = 1c_{\alpha}$$

where the infinite word c_{α} is called the *characteristic* word of α.

Remark 2.1.12. The condition $0 \leq \alpha \leq 1$ in the definition of mechanical words is not a restriction, but a simplification. One could indeed use the same definition of $s_{\alpha,\rho}$ without any condition on α. Since $\lfloor \alpha \rfloor \leq s_{\alpha,\rho}(n) \leq 1 + \lfloor \alpha \rfloor$, the numbers $s_{\alpha,\rho}(n)$ then can have the two values k and $k+1$ where $k = \lfloor \alpha \rfloor$. Thus the words $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are over the two letter alphabet $\{k, k+1\}$. This alphabet can be transformed back into $\{0,1\}$ by using the formula

$$s_{\alpha,\rho}(n) = \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor - \lfloor \alpha \rfloor$$

Mechanical words can be interpreted in several other ways. Consider again a straight line $y = \beta x + \rho$, for some $\beta > 0$ not restricted to be less than 1, and ρ not restricted to be positive. Consider the intersections of this line with the lines of the grid with nonnegative integer coordinates. We get a sequence $Q_0, Q_1, \ldots$ of intersection points. We call $Q_n = (x_n, y_n)$ *horizontal* if y_n is an integer, and *vertical* if x_n is an integer. If both are integers, we insert before Q_n a sibling Q_{n-1} of Q_n with the same coordinates, and we agree that the first is horizontal and the second is vertical (or vice-versa, but we do always the same choice). In Figure 2.4 below, Q_0 is vertical, because ρ is positive.

Writing a 0 for each vertical point and a 1 for each horizontal point, we obtain an infinite word $K_{\beta,\rho}$ that is called the (lower) *cutting sequence* (with the other choice for labeling siblings, one gets an upper cutting sequence $K'_{\beta,\rho}$).

To each $Q_n = (x_n, y_n)$, we associate a point $I_n = (u_n, v_n)$ with integer coordinates. The point I_n is the point below (below and to the right of) Q_n if Q_n is vertical (horizontal). Formally,

$$(u_n, v_n) = \begin{cases} (\lceil x_n \rceil, y_n - 1) & \text{if } Q_n \text{ is horizontal,} \\ (x_n, \lfloor y_n \rfloor) & \text{if } Q_n \text{ is vertical} \end{cases}$$

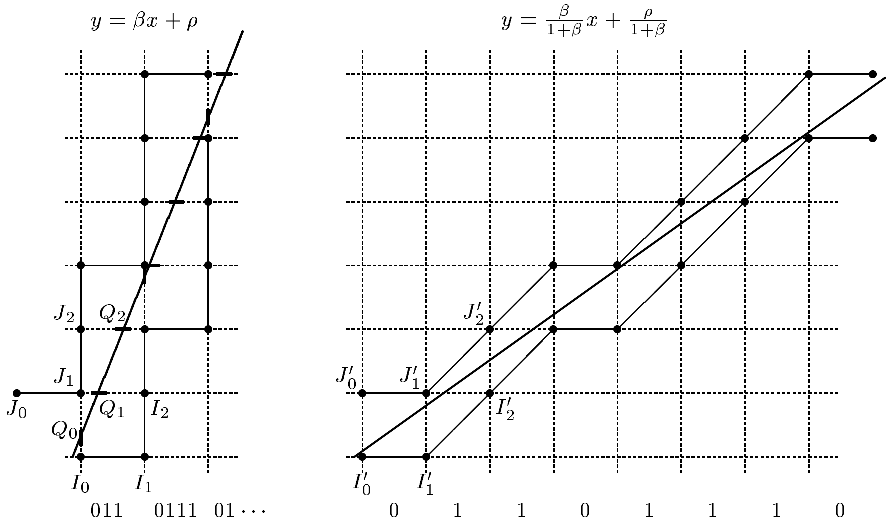

Figure 2.4. Cutting sequence and corresponding mechanical sequence.

Similar points J_n are defined above the line (see Figure 2.4). It is easy to check that $u_n + v_n = n$ for $n \geq 0$, and that

$$K_{\beta,\rho}(n) = v_{n+1} - v_n = 1 + u_n - u_{n+1}$$

In the special case $\rho = 0$ and β irrational, we again get the same infinite word up to the first letter. There is a word C_β such that

$$K_{\beta,0} = 0C_\beta, \qquad K'_{\beta,0} = 1C_\beta$$

Observe that Q_n is horizontal if and only if

$$1 + v_n \leq u_n\beta + \rho < 1 + \rho + v_n \tag{2.1.7}$$

and Q_n is vertical if and only if

$$v_n \leq u_n\beta + \rho < 1 + v_n \tag{2.1.8}$$

We now check that

$$K_{\beta,\rho} = s_{\beta/(1+\beta),\rho/(1+\beta)}$$

Indeed, the transformation $(x, y) \mapsto (x + y, x)$ of the plane maps the line $y = \beta x + \rho$ to $y = \beta/(1+\beta)x + \rho/(1+\beta)$, and a point $I_n = (u_n, v_n)$ to $I'_n = (n, v_n)$. It remains to show that

$$v_n = \left\lfloor \frac{\beta}{1+\beta}n + \frac{\rho}{1+\beta} \right\rfloor \tag{2.1.9}$$

Using $u_n + v_n = n$, we get from (2.1.7) that

$$v_n + 1/(1+\beta) \le \beta/(1+\beta)n + \rho/(1+\beta) < 1 + v_n$$

and from (2.1.8) that

$$v_n \le \beta/(1+\beta)n + \rho/(1+\beta) < v_n + 1/(1+\beta)$$

Thus, (2.1.9) holds for horizontal and for vertical steps. Thus, cutting sequences are just another formulation of mechanical words.

Mechanical words can also be generated by rotations. Let $0 < \alpha < 1$. The *rotation* of angle α is the mapping $R = R_\alpha$ from $[0,1[$ into itself defined by

$$R(z) = \{z + \alpha\}$$

where $\{z\} = z - \lfloor z \rfloor$ is the fractional part of z. Iterating R, one gets

$$R^n(\rho) = \{n\alpha + \rho\}$$

Moreover, a straightforward computation shows that

$$\lfloor (n+1)\alpha + \rho \rfloor = 1 + \lfloor n\alpha + \rho \rfloor \iff \{n\alpha + \rho\} \ge 1 - \alpha$$

Thus, defining a partition of $[0,1[$ by

$$I_0 = [0, 1-\alpha[, \quad I_1 = [1-\alpha, 1[\,,$$

one gets

$$s_{\alpha,\rho}(n) = \begin{cases} 0 & \text{if } R^n(\rho) \in I_0 \\ 1 & \text{if } R^n(\rho) \in I_1 \end{cases} \tag{2.1.10}$$

It will be convenient to identify $[0,1[$ with the torus (or the unit circle). For $0 \le b < a < 1$, the set $[a,1] \cup [0,b[$ is considered as an interval denoted $[a,b[$. Then, for any subinterval I of $[0,1[$, the sets $R(I)$ and $R^{-1}(I)$ are always intervals (even when overlapping the point 0).

As an example of the use of rotations, consider a word $w = b_0 b_1 \cdots b_{m-1}$, with $b_0, b_1, \ldots$ letters. We want to know whether w is a factor of some $s_{\alpha,\rho} = a_0 a_1 \cdots$, with $a_0, a_1, \ldots$ letters. By (2.1.10), $a_{n+k} = b_i$ if and only if $R^{n+i}(\rho) \in I_{b_i}$, or equivalently, if and only if $R^n(\rho) \in R^{-i}(I_{b_i})$. Thus, for $n \ge 0$,

$$w = a_n a_{n+1} \cdots a_{n+m-1} \iff R^n(\rho) \in I_w \tag{2.1.11}$$

where I_w is the interval

$$I_w = I_{b_0} \cap R^{-1}(I_{b_1}) \cap \cdots \cap R^{-m+1}(I_{b_{m-1}})$$

The interval I_w is non empty if and only if w is a factor of $s_{\alpha,\rho}$. Observe that this property is independent of ρ, and thus words $s_{\alpha,\rho}$ and $s_{\alpha,\rho'}$ have the same set of factors. A combinatorial proof will be given later (Proposition 2.1.18).

Mechanical words are quite naturally defined as two-sided infinite words. However, it appears that several properties, such as Theorem 2.1.13 below, only hold with some restrictions (see Problem 2.1.1).

THEOREM 2.1.13. *Let s be an infinite word. The following are equivalent:*
(i) *s is Sturmian;*
(ii) *s is balanced and aperiodic;*
(iii) *s is irrational mechanical.*

The proof will be a simple consequence of two lemmas. In the proofs, we will use several times the formula

$$x' - x - 1 < \lfloor x' \rfloor - \lfloor x \rfloor < x' - x + 1\,.$$

LEMMA 2.1.14. *Let s be a mechanical word with slope α. Then s is balanced of slope α. If α is rational, then s is purely periodic. If α is irrational, then s is aperiodic.*

Proof. Let $s = s_{\alpha,\rho}$ be a lower mechanical word. The proof is similar for upper mechanical words. The height of a factor $u = s(n) \cdots s(n+p-1)$ is the number $h(u) = \lfloor \alpha(n+p) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor$, thus

$$\alpha|u| - 1 < h(u) < \alpha|u| + 1 \tag{2.1.12}$$

This implies $\lfloor \alpha|u| \rfloor \leq h(u) \leq 1 + \lfloor \alpha|u| \rfloor$, and shows that $h(u)$ takes only two consecutive values, when u ranges over the factors of a fixed length of s. Thus, s is balanced. Moreover, by (2.1.12)

$$\left|\pi(u) - \alpha\right| < \frac{1}{|u|}$$

Thus $\pi(u) \to \alpha$ for $|u| \to \infty$ and α is the slope of s as it was defined for balanced words. This proves the first statement.

If α is irrational, the word s is aperiodic by Proposition 2.1.11. If $\alpha = q/p$ is rational, then $\lfloor \alpha(n+p) + \rho \rfloor = q + \lfloor \alpha n + \rho \rfloor$, for all $n \geq 0$. Thus $s(n+p) = s(n)$ for all n, showing that s is purely periodic. ∎

LEMMA 2.1.15. *Let s be a balanced infinite word. If s is aperiodic, then s is irrational mechanical. If s is purely periodic, then s is rational mechanical.*

Proof. In view of Corollary 2.1.8, s has a slope, say α. Denote by h_n the height of the prefix of length n of s.

For every real number τ, one at least of the following holds:
- $h_n \leq \lfloor \alpha n + \tau \rfloor$ for all n;
- $h_n \geq \lfloor \alpha n + \tau \rfloor$ for all n.

Indeed, on the contrary there exist a real number τ and two integers $n, n+k$ such that $h_n < \lfloor \alpha n + \tau \rfloor$ and $h_{n+k} > \lfloor \alpha(n+k) + \tau \rfloor$ (or the symmetric relation). This implies that $h_{n+k} - h_n \geq 2 + \lfloor \alpha(n+k) + \tau \rfloor - \lfloor \alpha n + \tau \rfloor > 1 + \alpha k$, in contradiction with (2.1.4).

Set

$$\rho = \inf\{\tau \mid h_n \leq \lfloor \alpha n + \tau \rfloor \text{ for all } n\}$$

By Proposition 2.1.10, one has $\rho \le 1$, and $\rho < 1$ if α is irrational. Observe that for all $n \ge 0$

$$h_n \le \alpha n + \rho \le h_n + 1 \tag{2.1.13}$$

since otherwise there is an integer n such that $h_n + 1 < \alpha n + \rho$, and setting $\sigma = h_n + 1 - \alpha n$, one has $\sigma < \rho$ and $\alpha n + \sigma = h_n + 1 > h_n$, in contradiction with the definition of ρ.

If s is aperiodic, then α is irrational by Proposition 2.1.11, and $\alpha n + \rho$ is an integer for at most one n. By (2.1.13), either $h_n = \lfloor \alpha n + \rho \rfloor$ for all n, and then $s = s_{\alpha,\rho}$, or $h_n = \lfloor \alpha n + \rho \rfloor$ for all but one n_0, and $h_{n_0} + 1 = \alpha n_0 + \rho$. In this case, one has $h_n = \lceil \alpha n + \rho - 1 \rceil$ for all n and $s = s'_{\alpha,\rho-1}$.

If $s = u^\omega$ is purely periodic with period $|u| = p$, then $\alpha = q/p$ with $q = h(u) = h_p$. Again $h_n = \lfloor \alpha n + \rho \rfloor$ if $\alpha n + \rho$ is never an integer (this depends on ρ).

If $h_n = \alpha n + \rho$ for some n, we claim that $h_n = \lfloor \alpha n + \rho \rfloor$ for all n. Assume the contrary. Then by (2.1.13), $1 + h_m = \alpha m + \rho$, for some m and we may assume $n < m < n + p$. Consider the words $y = s(n+1) \cdots s(m)$ and $z = s(m+1) \cdots s(n+p)$. Then $\pi(y) = (h_m - h_n)/(m - n) = \alpha - 1/|y|$ and $\pi(z) = (h_{n+p} - h_m)/(n + p - m) = \alpha + 1/|z|$, whence $\bigl|\pi(y) - \pi(z)\bigr| = 1/|y| + 1/|z|$, in contradiction with Proposition 2.1.7. Similarly, if $1 + h_n = \alpha n + \rho$ for some n, then $h_n = \lceil \alpha n + \rho \rceil$ for all n. ■

Proof of theorem 2.1.13. We know already by Theorem 2.1.5 that (i) and (ii) are equivalent. Assume that s is irrational mechanical. Then s is balanced aperiodic by Lemma 2.1.14. Conversely, if s is balanced and aperiodic, then by the Lemma 2.1.15 s is irrational mechanical. ■

EXAMPLE 2.1.16. To show that a balanced infinite word is not always mechanical when the slope is rational (so the converse is false in Lemma 2.1.14), consider the infinite balanced word 01^ω. It is not a mechanical word. Indeed, it has slope 1, and all mechanical words $s_{1,\rho}$ are equal to 1^ω.

Let us consider mechanical words with rational slope in some more detail. For a rational number $\alpha = p/q$ with $0 \le \alpha \le 1$ and p, q relatively prime, the infinite words $s_{\alpha,0}$ and $s'_{\alpha,0}$ are purely periodic. Define finite words

$$t_{p,q} = a_0 \cdots a_{q-1}, \quad t'_{p,q} = a'_0 \cdots a'_{q-1}$$

by

$$a_i = \left\lfloor (i+1)\frac{p}{q} \right\rfloor - \left\lfloor i\frac{p}{q} \right\rfloor, \quad a'_i = \left\lceil (i+1)\frac{p}{q} \right\rceil - \left\lceil i\frac{p}{q} \right\rceil$$

Clearly, $t_{p,q}$ and $t'_{p,q}$ have height p. They are primitive words because $(p,q) = 1$. In particular, $t_{0,1} = 0$ and $t_{1,1} = 1$. These words are called *Christoffel words.* In any case, $s_{p/q,0} = t^\omega_{p,q}$ and $s'_{p/q,0} = {t'_{p,q}}^\omega$. Moreover, if $0 < p/q < 1$, the word $t_{p,q}$ starts with 0 and ends with 1 (and $t'_{p,q}$ starts with 1 and ends with 0). There is a word $z_{p,q}$ such that

$$t_{p,q} = 0z_{p,q}1, \quad t'_{p,q} = 1z_{p,q}0 \tag{2.1.14}$$

The word $z_{p,q}$ is easily seen to be a palindrome. Later, we will see that these words, called central words, have remarkable combinatorial properties.

The following result deals with finite words.

PROPOSITION 2.1.17. *A finite word w is a factor of some Sturmian word if and only if it is balanced.*

Proof. Clearly a factor of a Sturmian word is balanced. For the converse, consider a balanced word w, and define

$$\alpha' = \max(\pi(u) - 1/|u|), \qquad \alpha'' = \min(\pi(u) + 1/|u|)$$

where the maximum and the minimum is taken over all non empty factors u of w. Since w is balanced, one gets from Proposition 2.1.10 that

$$\pi(u) - 1/|u| < \pi(v) + 1/|v|$$

for all nonempty factors u and v of w. Thus $\alpha' < \alpha''$.

Take any irrational number α with $\alpha' < \alpha < \alpha''$. Then by construction, for every nonempty factor u of w,

$$\left|\pi(u) - \alpha\right| < 1 \tag{2.1.15}$$

Let w_n be the prefix of length n of w. By (2.1.15), there exists a real ρ_n such that

$$h(w_n) = n\alpha + \rho_n, \quad |\rho_n| < 1$$

Moreover, for $n > m$, setting $w_n = w_m u$, one gets $h(w_n) - h(w_m) = h(u) = (n-m)\alpha + (\rho_n - \rho_m)$, showing that $|\rho_n - \rho_m| < 1$. Set

$$\rho = \max_{1 \le n \le |w|} \rho_n \,.$$

Then

$$n\alpha + \rho \ge h(w_n) = n\alpha + \rho + (\rho_n - \rho) > n\alpha + \rho - 1$$

whence $h(w_n) = \lfloor n\alpha + \rho \rfloor$. This proves that w is a prefix of the Sturmian word $s_{\alpha,\rho}$. ■

2.1.3. The factors of one Sturmian word

The aim of this paragraph is to give properties of the set of factors of a single Sturmian word.

PROPOSITION 2.1.18. *Let s and t be Sturmian words.*

1. *If s and t have same slope, then $F(s) = F(t)$.*
2. *If s and t have distinct slopes, then $F(s) \cap F(t)$ is finite.*

Proof. Let α be the common slope of s and t. By Proposition 2.1.10, every factor u of s verifies

$$|\pi(u) - \alpha| < \frac{1}{|u|}$$

(indeed, equality is impossible because α is irrational). Next, for every factor v of t,

$$|\pi(v) - \alpha| < \frac{1}{|v|}$$

Let $X = F(s) \cup F(t)$. The set X is factorial. It is also balanced since

$$|\pi(u) - \pi(v)| \leq |\pi(u) - \alpha| + |\pi(v) - \alpha| < \frac{1}{|u|} + \frac{1}{|v|}$$

In view of Proposition 2.1.2

$$\mathrm{Card}(X \cap A^n) \leq n + 1$$

for every n. Thus $F(s) = X = F(t)$.

Let now α be the slope of s and β be the slope of t. We may suppose that $\beta > \alpha$. For any factor u of s such that $(\beta - \alpha) \geq 2/|u|$, one has $\pi(u) - \alpha > -1/|u|$ by Proposition 2.1.10 whence $\pi(u) - \beta = (\pi(u) - \alpha) + (\beta - \alpha) \geq 1/|u|$ showing that u is not a factor of t. ■

PROPOSITION 2.1.19. *The set $F(s)$ of factors of a Sturmian word s is closed under reversal.*

Proof. Set $\tilde{F}(s) = \{\tilde{x} \mid x \in F(s)\}$. The set $X = F(s) \cup \tilde{F}(s)$ is balanced. In view of Proposition 2.1.2, $\mathrm{Card}(X \cap A^n) \leq n + 1$, for each n, and since $\mathrm{Card}(F(s) \cap A^n) = n + 1$, one has $X = F(s)$. Thus $\tilde{F}(s) = F(s)$. ■

We now compare Sturmian words, with respect to their slope and intercept. The lexicographic order defined in Chapter 1 extends to infinite words as follows, with the assumption that $0 < 1$. Given two infinite words $x = a_0 \cdots a_n \cdots$ and $y = b_0 \cdots b_n \cdots$, we say that x is *lexicographically less* than y, and we write $x < y$ if there is an integer n such that $a_i = b_i$ for $i = 0, \ldots, n-1$ and $a_n = 0$, $b_n = 1$.

PROPOSITION 2.1.20. *Let $0 < \alpha < 1$ be an irrational number and let ρ, ρ' be real numbers with $0 \leq \rho, \rho' < 1$. Then*

$$s_{\alpha,\rho} < s_{\alpha,\rho'} \iff \rho < \rho' .$$

Proof. Since α is irrational, the set of fractional parts $\{\alpha n\}$ for $n \geq 0$ is dense in the interval $[0, 1[$. Thus $\rho < \rho'$ if and only if there exists an integer $n \geq 1$ such that $1 - \rho' \leq \{\alpha n\} < 1 - \rho$, and this is equivalent to $\lfloor \alpha n + \rho' \rfloor = 1 + \lfloor \alpha n + \rho \rfloor$. If n is the smallest integer for which this equality holds, then $s_{\alpha,\rho}(n-1) = 0$ and $s_{\alpha,\rho'}(n-1) = 1$ and $s_{\alpha,\rho'}(k) = s_{\alpha,\rho}(k)$ for $k < n - 1$. ■

Observe that this proposition does not hold for rational slopes, since indeed $s_{0,\rho} = 0^\omega$ for all ρ.

LEMMA 2.1.21. *Let $0 < \alpha, \alpha' < 1$ be irrational numbers and let ρ, ρ' be real numbers. Any of the equalities $s_{\alpha,\rho} = s_{\alpha',\rho'}$, $s_{\alpha,\rho} = s'_{\alpha',\rho'}$ or $s'_{\alpha,\rho} = s'_{\alpha',\rho'}$ implies $\alpha = \alpha'$ and $\rho \equiv \rho' \bmod 1$.*

Proof. Any of the equalities implies that $\alpha = \alpha'$ because equal words have the same slope. Next, $s_{\alpha,\rho} = s_{\alpha,\rho'}$ implies $\rho \equiv \rho' \bmod 1$ by the previous proposition. Finally, consider the equality $s_{\alpha,\rho} = s'_{\alpha,\rho'}$. If $\alpha n + \rho'$ is not an integer for all $n \geq 1$, then $s'_{\alpha,\rho'} = s_{\alpha,\rho'}$ and the conclusion holds. Otherwise, let n be the unique integer such that $\alpha n + \rho'$ is an integer. Then $s_{\alpha,\rho+(1+n)\alpha} = s'_{\alpha,\rho'+(1+n)\alpha}$, showing again that $\rho \equiv \rho' \bmod 1$. ∎

Sturmian words with intercept 0 have many interesting properties. We observed already that, for an irrational number $0 < \alpha < 1$, the words $s_{\alpha,0}$ and $s'_{\alpha,0}$ differ only by their first letter, and that

$$s_{\alpha,0} = 0c_\alpha, \quad s'_{\alpha,0} = 1c_\alpha$$

where c_α is the *characteristic word* of slope α. Equivalently,

$$c_\alpha = s_{\alpha,\alpha} = s'_{\alpha,\alpha}$$

The following proposition states a combinatorial characterization of characteristic words among Sturmian words.

PROPOSITION 2.1.22. *For every Sturmian word s, either $0s$ or $1s$ is Sturmian. A Sturmian word s is characteristic if and only if $0s$ and $1s$ are both Sturmian.*

Proof. The first claim follows from the fact that $s_{\alpha,\rho-\alpha} = as_{\alpha,\rho}$, for some $a \in \{0, 1\}$.

If $s = s_{\alpha,\alpha} = s'_{\alpha,\alpha}$ is the characteristic word of slope α, then $0s = s_{\alpha,0}$ and $1s = s'_{\alpha,0}$ are Sturmian.

Conversely, the Sturmian words $0s$ and $1s$ have same slope, say α. Denote by ρ and ρ' their intercept. Then their common shift s has intercept $\rho + \alpha = \rho' + \alpha$, and by Lemma 2.1.21, $\rho \equiv \rho' \bmod 1$ and we may take $0 \leq \rho = \rho' < 1$. Thus $0s = s_{\alpha,\rho}$ and $1s = s'_{\alpha,\rho}$. Assume $\rho > 0$. The first letter of $0s$ is gives $0 = \lfloor \alpha + \rho \rfloor - \lfloor \rho \rfloor = \lfloor \alpha + \rho \rfloor$ and the first letter of $1s$ is $1 = \lceil \alpha + \rho \rceil - \lceil \rho \rceil$. Then $2 = \lceil \alpha + \rho \rceil$, a contradiction. Thus $\rho = 0$. ∎

We are now able to describe right special factors.

PROPOSITION 2.1.23. *The set of right special factors of a Sturmian word is the set of reversals of the prefixes of the characteristic word of same slope.*

Call a factor w of a Sturmian word s *left special* if both $0w$ and $1w$ are factors of s. Clearly, w is left special if and only if $\tilde{w}$ is right special. Thus the proposition states that the set of left special factors of a Sturmian word is the set of prefixes of the characteristic word of same slope.

Proof. Let s be a Sturmian word of slope α. By Proposition 2.1.22, the infinite words $0c_\alpha$ and $1c_\alpha$ are Sturmian and clearly have slope α. Thus

$$F(s) = F(c_\alpha) = F(0c_\alpha) = F(1c_\alpha)$$

by Proposition 2.1.18. Consequently, for each prefix p of c_α, $0p$ and $1p$ are factors of s. Since $F(s)$ is closed under reversal, this shows that $\tilde{p}$ is right special. Thus $\tilde{p}$ is the unique right special factor of length $|p|$. ∎

EXAMPLE 2.1.24. Consider again the Fibonacci word f. We have seen in Example 2.1.1 that its right special factors are the reversals of its prefixes. Thus each prefix of f is left special. This shows that $F(f) = F(0f) = F(1f)$. Consequently, f is characteristic of slope $1/\tau^2$.

PROPOSITION 2.1.25. *The dynamical system generated by a Sturmian word is minimal.*

Proof. Let s be a Sturmian word, and let x be an infinite word such that $F(x) \subset F(s)$. Clearly, x is balanced. Also, x has the same irrational slope as s. Thus x is aperiodic and therefore is Sturmian. By Proposition 2.1.18(1), $F(x) = F(s)$. This shows that s and x generate the same dynamical system. ∎

Observe that Proposition 2.1.18(2) is a consequence of Proposition 2.1.25. Indeed, the intersection of two distinct minimal dynamical systems is the trivial system.

2.2. Standard words

This section is concerned with a family of finite words that are basic bricks for constructing characteristic Sturmian words, in the sense that every characteristic Sturmian word is the limit of a sequence of standard words. This will be shown in Section 2.2.2.

2.2.1. Standard words and palindrome words

After basic definitions, we give two characterizations of standard words. The first is by a special decomposition into palindrome words (Theorem 2.2.4), the second (Theorem 2.2.11) by an extremal property on the periods of the word that is closely related to Fine and Wilf's theorem. We give then a "mechanical" characterization of central and standard words (Proposition 2.2.15). We end with an enumeration formula for standard words.

Consider two functions Γ and Δ from $\{0,1\}^* \times \{0,1\}^*$ into itself defined by

$$\Gamma(u,v) = (u, uv), \quad \Delta(u,v) = (vu, v)$$

The set of *standard pairs* is the smallest set of pairs of words containing the pair $(0,1)$ and closed under Γ and Δ. A *standard word* is any component of a standard pair.

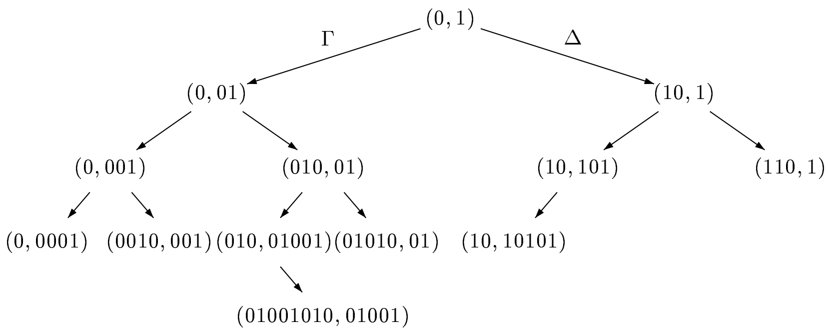

Figure 2.5. The tree of standard pairs.

EXAMPLE 2.2.1. Figure 2.5 shows the beginning of the tree of standard pairs. Considering the leftmost and rightmost paths, one gets the pairs

$$(0, 0^n1),\ (1^n0, 1) \quad (n \geq 1)$$

Next to them are the pairs

$$(0(10)^n, 01),\ (10, (10)^n1) \quad (n \geq 1)$$

These are the pairs with one component of length 1 or 2.

Finite Fibonacci words are standard, since $(f_0, f_{-1}) = (0, 1)$, and for $n \geq 1$, $(f_{2n+2}, f_{2n+1}) = \Delta\Gamma(f_{2n}, f_{2n-1})$.

Every standard word which is not a letter is a product of two standard words which are the components of some standard pair. The next proposition states some elementary facts.

PROPOSITION 2.2.2. *Let $r = (x, y)$ be a standard pair.*

1. *If $r \neq (0, 1)$ then one of x or y is a proper prefix of the other.*
2. *If x (resp. y) is not a letter, then x ends with 10 (resp. y ends with 01).*
3. *Only the last two letters of xy and yx are different.*

Proof. We prove the last claim by induction on $|xy|$. Assume indeed that $xy = p01$ and $yx = p10$. Then $\Gamma(r) = (x, xy)$ and $xxy = xp01$, $(xy)x = x(yx) = xp10$, so the claim is true for $\Gamma(r)$. The same holds for $\Delta(r)$. ■

Every standard pair is obtained in a unique way from $(0, 1)$ by iterated use of Γ and Δ. Indeed, if (x, y) is a standard pair, then it is an image through Γ (resp. Δ) if and only if $|x| < |y|$ (resp. $|x| > |y|$). Thus, there is a unique product $W = \Lambda_1 \circ \ldots \circ \Lambda_n$, with $\Lambda_i \in \{\Gamma, \Delta\}$ such that

$$(x, y) = W(0, 1)$$

Consider two matrices

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and define a morphism μ from the monoid generated by Γ and Δ into the set of 2×2 matrices by

$$\mu(\Gamma) = L, \quad \mu(\Delta) = R,$$

and $\mu(\Lambda_1 \circ \ldots \circ \Lambda_n) = \mu(\Lambda_1) \cdots \mu(\Lambda_n)$. If $(x, y) = W(0,1)$, then a straightforward induction shows that

$$\mu(W) = \begin{pmatrix} |x|_0 & |x|_1 \\ |y|_0 & |y|_1 \end{pmatrix} \tag{2.2.1}$$

Observe that every matrix $\mu(W)$ has determinant 1. Thus if (x, y) is a standard pair,

$$|x|_0 |y|_1 - |x|_1 |y|_0 = 1 \tag{2.2.2}$$

showing that the entries in the same row (column) of $\mu(W)$ are relatively prime. From (2.2.2), one gets

$$h(y)|x| - h(x)|y| = 1\,. \tag{2.2.3}$$

(recall that $h(w) = |w|_1$ is the *height* of w). This shows also that $|x|$ and $|y|$ are relatively prime. A simple consequence is the following property.

PROPOSITION 2.2.3. *A standard word is primitive.*

Proof. Let w be a standard word which is not a letter. Then $w = x$ or $w = y$ for some standard pair (x, y). From (2.2.3), one gets that $h(w)$ and $|w|$ are relatively prime. This implies that w is primitive. ■

The operations Γ and Δ can be explained through three morphisms E, G, D on $\{0,1\}^*$ which we introduce now. These will be used also in the sequel. Let

$$E : \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array}, \quad G : \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 01 \end{array}, \quad D : \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 1 \end{array}$$

It is easily checked that $E \circ D = G \circ E = \varphi$. We observe that, for every morphism f,

$$\Gamma(f(0), f(1)) = (fG(0), fG(1)), \quad \Delta(f(0), f(1)) = (fD(0), fD(1))$$

For $W = \Lambda_1 \circ \ldots \circ \Lambda_n$, with $\Lambda_i \in \{\Gamma, \Delta\}$, define $\hat{W} = \hat{\Lambda}_n \circ \ldots \circ \hat{\Lambda}_1$, with $\hat{\Gamma} = G$, $\hat{\Delta} = D$. Then

$$W(0,1) = (\hat{W}(0), \hat{W}(1))\,. \tag{2.2.4}$$

Standard words have the following description.

THEOREM 2.2.4. *A word w is standard if and only if it is a letter or there exist palindrome words p, q and r such that*

$$w = pab = qr \tag{2.2.5}$$

where $\{a, b\} = \{0, 1\}$. Moreover, the factorization $w = qr$ is unique if $q \neq \varepsilon$.

EXAMPLE 2.2.5. The word 01001010 is standard (see Figure 2.5) and

$$01001010 = (010010)10 = (010)(01010)\,.$$

We start the proof with a lemma of independent interest.

LEMMA 2.2.6. *If a primitive word is a product of two nonempty palindrome words, then this factorization is unique.*

Proof. Let w be a primitive word and assume $w = pq = p'q'$ for palindrome words p, q, p', q'. We suppose $|p| > |p'|$, so that $p = p's(= \tilde{s}p')$, $sq = q'(= q\tilde{s})$ for some nonempty word s. Thus $\tilde{s}p'q = pq = p'q' = p'q\tilde{s}$, showing that $p'q$ and $\tilde{s}$ are powers of some word z. But then $w = pq = \tilde{s}p'q = z^n$ for some $n \geq 2$, contradicting primitivity. ■

Observe that (2.2.5) implies the following relations.

LEMMA 2.2.7. *If $w = pab = qr$ for palindrome words p, q, r, and letters $a \neq b$, then one of the following holds*

(i) $r = \varepsilon$, $p = (ba)^n b$, $q = (ba)^{n+1}b = w$ *for some* $n \geq 0$;
(ii) $r = b$, $p = a^n$, $q = a^{n+1}$, $w = a^{n+1}b$ *for some* $n \geq 0$;
(iii) $r = bab$, $p = b^{n+1}$, $q = b^n$, $w = b^{n+1}ab$ *for some* $n \geq 0$;
(iv) $r = basab$, $p = qbas$, $w = qbasab$ *for some palindrome word* s. ■

We need another lemma.

LEMMA 2.2.8. *Let x, y be words with $|x|, |y| \geq 2$. The pair (x, y) is a standard pair if and only if there exist palindrome words p, q, r such that*

$$x = p10 = qr \quad \text{and} \quad y = q01 \tag{2.2.6}$$

or

$$x = q10 \quad \text{and} \quad y = p01 = qr\,. \tag{2.2.7}$$

Proof. Assume that (2.2.6) holds (the other case is symmetric). If r is the empty word, then by the previous lemma

$$(x, y) = ((01)^{n+1}0, (01)^{n+1}001) = \Gamma((01)^{n+1}0, 01)$$

showing that the pair (x, y) is standard.

If $r = 0$, then $(x, y) = (1^n0, 1^n01) = \Gamma(1^n0, 1)$, and if $r = 010$, then $(x, y) = (0^n10, 0^n1) = \Delta(0, 0^n1)$.

Thus, we may assume that $r = 01s10$ for some palindrome word s. By (2.2.6), if follows that y is a prefix of x, so $x = yz$ for some word z. We show that (z, y) is standard. From $p = q01s = s10q$ it follows that $q \neq s$. Assume $|q| < |s|$ (the other case is symmetric). Then $s = qt$ for some word t, and the equation $p = qt10q$ shows that the word $r' = t10$ is a palindrome. Thus

$$y = q01,\ z = qr' = s10$$

and (z, y) satisfies (2.2.6).

Conversely, let (x, y) be a standard pair, and assume $(x, y) = \Gamma(x, z)$, that is $y = xz$. If z is a letter, then $(x, z) = (1^n0, 1)$ for some $n \geq 1$ and

$$x = q10,\ y = p01 = qr$$

for $q = 1^{n-1}$, $p = 1^n$, $r = 101$.

Thus we may assume that for some palindrome words p, q, r, either

$$x = p10 = qr, \quad z = q01$$

or

$$x = q10, \quad z = p01 = qr\,.$$

In the first case,

$$x = p10, \quad y = xz = (qrq)01 = p(10q01)$$

In the second case,

$$x = q10, \quad y = xz = q(10p01) = (qrq)01$$

because $10p = rq$. Thus (2.2.7) holds. ■

Proof of Theorem 2.2.4. Let w be a standard word, $|w| \geq 2$. Then there exists a standard pair (x, y) such that $w = xy$ (or symmetrically $w = yx$). If $x = 0$, then $y = 0^n1$ for some $n \geq 0$, and $xy = 0^{n+1}1$ has the desired factorization. A similar argument holds for $y = 1$. Otherwise, either (2.2.6) or (2.2.7) of Lemma 2.2.8 holds. In the first case, $xy = p(10q01) = qrq01$ and in the second case, $xy = q(10p01) = qrq01$ because $10p = rq$. The factorization is unique by Lemma 2.2.6 because a standard word is primitive.

Conversely, if $w = p10 = qr$ (or $w = p01 = qr$) for palindrome words p, q, r, then by Lemma 2.2.8, the word w is a component of some standard pair, and thus is a standard word. ■

A word w is *central* if $w01$ (or equivalently $w10$) is a standard word. As we shall see, central words play indeed a central role.

COROLLARY 2.2.9. *A word is central if and only if it is in the set*

$$0^* \cup 1^* \cup (P \cap P10P)$$

where P is the set of palindrome words. The factorization of a central word w as $w = p10q$ with p, q palindrome words is unique.

Observe that $P \cap P10P = P \cap P01P$.

Proof. Let $w \in 0^* \cup 1^* \cup (P \cap P10P)$. By the previous characterization, $w01$ is a standard word, so w is central. Conversely, if $w01$ is standard, then w is a palindrome and $w01 = qr$ for some palindrome words q and r. Either $w \in 0^* \cup 1^*$, or by Lemma 2.2.7, $r = \varepsilon$ and $w = (10)^n1$ for some $n \geq 1$, or $w = q10s$ for some palindrome s, as required. ■

As a simple consequence, we obtain.

Corollary 2.2.10. *A palindrome prefix (suffix) of a central word is central.*

Proof. We consider the case of a prefix. Let p be a central word. If $p \in 0^* \cup 1^*$, the result is clear. Let x be a standard word such that $x = pab$, with $\{a, b\} = \{0, 1\}$. Then $x = yz$ for a standard pair (y, z) or (z, y). Set $y = qba$ and $z = rab$, where q, r are central words. Then $p = qbar = rabq$ and by symmetry we may assume that $|r| < |q|$.

Let w be a palindrome prefix of p. If $|w| \leq |q|$, the result holds by induction. If $w = qb$ then w is a power of b. Thus set $w = qbat$ where t is a prefix of r. Since r is a prefix of q, the word t is a prefix of q, and since $w = \tilde{t}abq$, one has $t = \tilde{t}$. Thus, by Corollary 2.2.9, $w = qbat$ is central. ∎

The next characterization relates central words to periods in words. Recall from Chapter 1 that given a word $w = a_1 \cdots a_n$, where $a_1, \ldots, a_n$ are letters, an integer k is a *period* of w if $k \geq 1$ and $a_i = a_{i+k}$ for all $1 \leq i \leq n - k$. Any integer $k \geq n$ is a period with this definition.

An integer k with $1 \leq k \leq |w|$ is a period of w if and only if there exist words x, y, and z such that

$$w = xy = zx, \qquad |y| = |z| = k\,.$$

Fine and Wilf's theorem states that if a word w has two periods k and ℓ, and $|w| \geq k + \ell - \gcd(k, \ell)$, then $\gcd(k, \ell)$ is also a period of w. In particular, if k and ℓ are relatively prime, and $|w| \geq k + \ell - 1$, then w is the power of a single letter. The bound is sharp, and the question arises to describe the words w of length $|w| = k + \ell - 2$ having periods k and ℓ. This is the object of the next theorem.

Theorem 2.2.11. *A word w is central if and only if it has two periods k and ℓ such that $\gcd(k, \ell) = 1$ and $|w| = k + \ell - 2$. Moreover, if $w \notin 0^* \cup 1^*$, and $w = p10q$ with p, q palindrome words, then $\{k, \ell\} = \{|p| + 2, |q| + 2\}$ and the pair $\{k, \ell\}$ is unique.*

The proof will show that any word w having two periods k and ℓ such that $\gcd(k, \ell) = 1$ and $|w| = k + \ell - 2$ is over an alphabet with at most two letters.

Proof. Let w be a central word. Then $w01$ is a standard word, and there is a standard pair (x, y) such that $w01 = xy$. If $x = 0$ or $y = 1$, then w is a power of 0 resp. of 1, and w has periods $k = 1$ and $\ell = |w| + 1$. Otherwise, $x = p10$ and $y = q01$ for some palindrome words p, q, and $w = p10q = q01p$ has two periods $k = |x|$ and $\ell = |y|$ which are relatively prime by Equation (2.2.3). Assume that w has also periods $\{k', \ell'\}$, with $k' + \ell' - 2 = |w|$. We may suppose $k < k' < \ell' < \ell$. Since $k + \ell' - 1 \leq |w|$, Fine and Wilf's theorem applies. So w has also the period $d = \gcd(k, \ell')$. Similarly, w has also the period $d' = \gcd(k, k')$. So it has the period $\gcd(d, d') = 1$. This proves that the pair $\{k, \ell\}$ is unique.

Conversely, if w is a power of a letter, the result is trivial. Thus we assume that w contains two distinct letters. Since $k, \ell \neq 1$, we assume $2 \leq k < \ell$.

Since w has period k, there is a word x of length $|x| = \ell - 2$ that is both a prefix and a suffix of w. Similarly, there is a word y of length $|y| = k - 2$ that is both a prefix and a suffix of w. Consequently, there exist words u and v, both of length 2, such that

$$w = yux = xvy$$

We prove by induction on $|w|$ that x, y, w are palindrome words, that u and v are composed of distinct letters, and that no other letters than those of u appear in w (that is w is over an alphabet of two letters).

If $k = 2$, then y is the empty word. Thus $ux = xv$, and ℓ is odd. Therefore $u = ab$, $v = ba$, $x = (ab)^n a$, $w = (ab)^{n+1}a$ for letters $a \neq b$ and some $n \geq 0$. The result holds in this case.

If $k = \ell - 1$, then $x = ya = by$ for letters a and b. But then $a = b$ and w is a power of a letter, a case that we have excluded.

Thus we assume $k \leq \ell - 2$. Then yu is a prefix of x. Define z by $yuz = x$. Then

$$x = yuz = zvy$$

showing that x has periods $|yu| = k$ and $|uz| = \ell - k$. Since $\gcd(k, \ell - k) = 1$ and $|x| = k + (\ell - k) - 2$, we get by induction that x is a palindrome, and that its prefix of length $k - 2$, that is y, and its suffix of length $\ell - k - 2$, that is z also are palindromes. Moreover, $u = ab$ for letters $a \neq b$, and $\tilde{u} = v$ because $yuz = z\tilde{u}y = zvy$. Also, the word x (and y, and therefore also w) is composed only of a's and b's. Thus w is central. ■

Theorem 2.2.11 associates, to every central word of length m, a pair $\{k, \ell\}$ of relatively prime integers such that $k + \ell - 2 = m$. We now show that, for each pair $\{k, \ell\}$ of relatively prime integers, there exists indeed a central word of length $k + \ell - 2$ and periods k and ℓ.

Let h, m be relatively prime integers with $1 \leq h < m$. Define a word

$$z_{h,m} = a_1 a_2 \cdots a_{m-2} \qquad (a_n \in \{0, 1\})$$

by

$$a_n = \left\lfloor (n+1)\frac{h}{m} \right\rfloor - \left\lfloor n\frac{h}{m} \right\rfloor .$$

These words have already been mentioned in our discussion of rational mechanical words (Equation 2.1.14). Each word $z_{h,m}$ has length $m - 2$ and height $h - 1$.

Proposition 2.2.12. *For every couple $1 \leq h < m$ of relatively prime integers, the word $z_{h,m}$ is central. It has the periods k and ℓ where $k + \ell = m$ and $kh \equiv 1 \bmod m$.*

Proof. Define k by $1 \leq k \leq m - 1$, and set $kh = 1 + \lambda m$. Observe that k exists because h and m are relatively prime. Let $\ell = m - k$. Then $\ell h \equiv -1 \bmod m$, and ℓ is the unique integer in the interval $[0 \ldots, m-1]$ with this property. Next

$$\left\lfloor (n+k)\frac{h}{m} \right\rfloor = \lambda + \left\lfloor \frac{nh+1}{m} \right\rfloor$$

Since $nh \not\equiv -1 \bmod m$ for $1 \le n \le \ell - 1$, it follows that

$$\left\lfloor \frac{nh+1}{m} \right\rfloor = \left\lfloor \frac{nh}{m} \right\rfloor \qquad (1 \le n \le \ell - 1)$$

Consequently, $a_{n+k} = a_n$ for $1 \le n \le \ell - 2$. A similar argument holds when k is replaced by ℓ and -1 is changed into 1.

Assume that some integer d divides k and ℓ. Then d divides also m. But k and ℓ are relatively prime to m, so $d = 1$ and $\gcd(k, \ell) = 1$. This proves, by Theorem 2.2.11, that $z_{h,m}$ is central. ∎

EXAMPLE 2.2.13. The words $z_{1,m} = 0^{m-2}$ and $z_{m-1,m} = 1^{m-2}$ are central. In particular, $z_{1,2} = \varepsilon$.

EXAMPLE 2.2.14. For $h = 5$, $m = 18$, one gets $z_{5,18} = 0010001001000100$, a word of length 16. By inspection, one finds the periods 7 and 11. The previous proposition allows to compute them, since $11 \cdot 5 \equiv 1 \bmod 18$.

PROPOSITION 2.2.15. *Let h, m be relatively prime integers with $1 \le h < m$. There exist exactly two standard words of height h and length m, namely $z_{h,m}10$ and $z_{h,m}01$. These words are balanced.*

Proof. By Proposition 2.2.12, the words $z_{h,m}10$ and $z_{h,m}01$ are standard words of height h and length m. They are factors of the Sturmian words $s_{h/m,0}$ and $s'_{h/m,0}$ and therefore are balanced. We prove that there exists only one standard word of height h and length m ending in 10. Assume there are two, say w and w'. Then

$$w = xy, \quad w' = x'y'$$

for some standard pairs (x, y), (x', y'). By formula (2.2.3),

$$h(x)|y| - h(y)|x| = 1, \quad h(x')|y'| - h(y')|x'| = 1$$

Since $m = |x| + |y|$ and $h = h(x) + h(y)$, this gives

$$h(x)m - |x|h = 1, \quad h(x')m - |x'|h = 1$$

whence

$$(h(x) - h(x'))m = (|x'| - |x|)h$$

Since $\gcd(m, h) = 1$, m divides $|x'| - |x|$. Thus $|x| = |x'|$, that is $x = x'$ and $y = y'$. ∎

Recall that Euler's *totient function* ϕ is defined for $m \ge 1$ as the number $\phi(m)$ of positive integers less than m and relatively prime to m

COROLLARY 2.2.16. *The number of standard words of length m is $2\phi(m)$, the number of central words of length m is $\phi(m + 2)$, where ϕ is Euler's totient function.* ∎

2.2.2. Standard sequences and characteristic words

In this section, we use particular morphisms that will also be considered in the next section. Three of them, namely E, G, and D, were already introduced earlier. Here, these morphisms are used to relate standard words to characteristic words, and both to the continued fraction expansion of the slope of a characteristic word. Consider the morphisms

$$E : \begin{matrix} 0 \mapsto 1 \\ 1 \mapsto 0 \end{matrix}, \qquad \varphi : \begin{matrix} 0 \mapsto 01 \\ 1 \mapsto 0 \end{matrix}, \qquad \tilde{\varphi} : \begin{matrix} 0 \mapsto 10 \\ 1 \mapsto 0 \end{matrix}$$

From these, we get other morphisms, denoted G, $\tilde{G}$, D, $\tilde{D}$ and defined by

$$G = \varphi \circ E : \begin{matrix} 0 \mapsto 0 \\ 1 \mapsto 01 \end{matrix}, \quad \tilde{G} = \tilde{\varphi} \circ E : \begin{matrix} 0 \mapsto 0 \\ 1 \mapsto 10 \end{matrix}$$

$$D = E \circ \varphi : \begin{matrix} 0 \mapsto 10 \\ 1 \mapsto 1 \end{matrix}, \quad \tilde{D} = E \circ \tilde{\varphi} : \begin{matrix} 0 \mapsto 01 \\ 1 \mapsto 1 \end{matrix}$$

Of course, $\varphi = G \circ E = E \circ D$ and $\tilde{\varphi} = \tilde{G} \circ E = E \circ \tilde{D}$.

LEMMA 2.2.17. *For any real number ρ, the following relations hold: $E(s_{\alpha,\rho}) = s'_{1-\alpha,1-\rho}$ and $E(s'_{\alpha,\rho}) = s_{1-\alpha,1-\rho}$.*

Proof. For $n \geq 0$,

$$\begin{aligned} s'_{1-\alpha,1-\rho}(n) &= \lceil (1-\alpha)(n+1) + 1 - \rho \rceil - \lceil (1-\alpha)n + 1 - \rho \rceil \\ &= 1 - (\lceil -\alpha n - \rho \rceil - \lceil -\alpha(n+1) - \rho \rceil) = 1 - s_{\alpha,\rho}(n) \end{aligned}$$

because $-\lceil -r \rceil = \lfloor r \rfloor$ for every real number r. This proves the first equality, and the second is symmetric. ■

LEMMA 2.2.18. *Let $0 < \alpha < 1$. For $0 \leq \rho < 1$,*

$$G(s_{\alpha,\rho}) = s_{\frac{\alpha}{1+\alpha},\frac{\rho}{1+\alpha}}, \quad \tilde{G}(s_{\alpha,\rho}) = s_{\frac{\alpha}{1+\alpha},\frac{\rho+\alpha}{1+\alpha}}, \quad \varphi(s_{\alpha,\rho}) = s'_{\frac{1-\alpha}{2-\alpha},\frac{1-\rho}{2-\alpha}}$$

and for $0 < \rho \leq 1$,

$$G(s'_{\alpha,\rho}) = s'_{\frac{\alpha}{1+\alpha},\frac{\rho}{1+\alpha}}, \quad \tilde{G}(s'_{\alpha,\rho}) = s'_{\frac{\alpha}{1+\alpha},\frac{\rho+\alpha}{1+\alpha}}, \quad \varphi(s'_{\alpha,\rho}) = s_{\frac{1-\alpha}{2-\alpha},\frac{1-\rho}{2-\alpha}}.$$

Proof. Let $s = a_0 a_1 \cdots a_n \cdots$ be an infinite word, the a_i being letters. An integer n is the index of the k-th occurrence of the letter 1 in s if $a_0 \cdots a_n$ contains k letters 1 and $a_0 \cdots a_{n-1}$ contains $k-1$ letters 1. If $s = s_{\alpha,\rho}$ and $0 \leq \rho < 1$, this means that

$$\lfloor \alpha(n+1) + \rho \rfloor = k, \quad \lfloor \alpha n + \rho \rfloor = k - 1$$

which implies $\alpha n + \rho < k \leq \alpha(n+1) + \rho$, that is

$$n = \left\lceil \frac{k - \rho}{\alpha} - 1 \right\rceil .$$

Similarly, if $s = s'_{\alpha,\rho}$ and $0 < \rho \le 1$, then

$$\lceil \alpha(n+1) + \rho \rceil = k + 1, \quad \lceil \alpha n + \rho \rceil = k$$

and $n = \left\lfloor \frac{k-\rho}{\alpha} \right\rfloor$.

Set $G(s_{\alpha,\rho}) = b_0 b_1 \cdots b_i \cdots$, with $b_i \in \{0,1\}$. Since every letter 1 in $s_{\alpha,\rho}$ is mapped to 01 in $G(s_{\alpha,\rho})$, the prefix $a_0 \cdots a_n$ of $s_{\alpha,\rho}$ (where n is the index of the k-th letter 1) is mapped onto the prefix $b_0 b_1 \cdots b_{n+k}$ of $G(s_{\alpha,\rho})$. Thus the index of the k-th letter 1 in $G(s_{\alpha,\rho})$ is

$$n + k = \left\lceil \frac{k - \frac{\rho}{1+\alpha}}{\frac{\alpha}{1+\alpha}} - 1 \right\rceil$$

This proves the first formula.

Next, we observe that, for any infinite word x, one has

$$G(x) = 0\tilde{G}(x)$$

Indeed, the formula $G(w)0 = 0\tilde{G}(w)$ is easily shown to hold for finite words w by induction. Furthermore, if a Sturmian word $s_{\alpha,\rho}$ starts with 0 and setting $s_{\alpha,\rho} = 0t$, one gets $t = s_{\alpha,\alpha+\rho}$. Altogether $\tilde{G}(s_{\alpha,\rho}) = s_{\alpha/(1+\alpha),(\rho+\alpha)/(1+\alpha)}$ for $0 \le \rho < 1$. The proof of the other formula is similar. Finally, since $\varphi = G \circ E$, $\varphi(s_{\alpha,\rho}) = G(s'_{1-\alpha,1-\rho}) = s'_{(1-\alpha)/(2-\alpha),(1-\rho)/(2-\alpha)}$. ■

Corollary 2.2.19. *For any Sturmian word s, the infinite words $E(s)$, $G(s)$ $\tilde{G}(s)$, $\varphi(s)$, $\tilde{\varphi}(s)$, $D(s)$ $\tilde{D}(s)$ are Sturmian.* ■

Formulas similar to those of Lemma 2.2.18 hold for $\tilde{\varphi}, D, \tilde{D}$ (Problem 2.2.6). Recall that the characteristic word of irrational slope α is defined by

$$c_\alpha = s_{\alpha,\alpha} = s'_{\alpha,\alpha}\,.$$

The previous lemmas imply

Corollary 2.2.20. *For any irrational α with $0 < \alpha < 1$, one has*

$$E(c_\alpha) = c_{1-\alpha}, \quad G(c_\alpha) = c_{\alpha/(1+\alpha)}$$ ■

For $m \ge 1$, define a morphism θ_m by

$$\theta_m : \begin{array}{l} 0 \mapsto 0^{m-1}1 \\ 1 \mapsto 0^{m-1}10 \end{array}$$

It is easily checked that

$$\theta_m = G^{m-1} \circ E \circ G\,.$$

Corollary 2.2.21. *For $m \ge 1$, one has $\theta_m(c_\alpha) = c_{1/(m+\alpha)}$.*

Proof. Since $E \circ G(c_\alpha) = c_{1/(1+\alpha)}$, the formula holds for $m = 1$. Next, $G(c_{1/(k+\alpha)}) = c_{1/(1+k+\alpha)}$, so the claim is true by induction. ∎

We use this corollary for connecting continued fractions to characteristic words. Recall that every irrational number γ admits a unique expansion as a continued fraction

$$\gamma = m_0 + \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\cdots}}} \tag{2.2.8}$$

where $m_0, m_1, \ldots$ are integers, $m_0 \geq 0$, $m_i > 0$ for $i \geq 1$. If (2.2.8) holds, we write

$$\gamma = [m_0, m_1, m_2, \ldots]\,.$$

The integers m_i are called the *partial quotients* of γ. If the sequence (m_i) is eventually periodic, and $m_i = m_{k+i}$ for $i \geq h$, this is reported by overlining the purely periodic part, as in

$$\gamma = [m_0, m_1, m_2, \ldots, m_{h-1}, \overline{m_h, \ldots, m_{h+k-1}}]\,.$$

Let $\alpha = [0, m_1, m_2, \ldots]$ be the continued fraction expansion of an irrational α with $0 < \alpha < 1$. If, for some β with $0 < \beta < 1$,

$$\beta = [0, m_{i+1}, m_{i+2}, \ldots]$$

we agree to write

$$\alpha = [0, m_1, m_2, \ldots, m_i + \beta]\,.$$

COROLLARY 2.2.22. *If $\alpha = [0, m_1, m_2, \ldots, m_i + \beta]$ for some irrational α and $0 < \alpha, \beta < 1$, then*

$$c_\alpha = \theta_{m_1} \circ \theta_{m_2} \circ \cdots \circ \theta_{m_i}(c_\beta)$$ ∎

Let $(d_1, d_2, \ldots, d_n, \ldots)$ be a sequence of integers, with $d_1 \geq 0$ and $d_n > 0$ for $n > 1$. To such a sequence, we associate a sequence $(s_n)_{n \geq -1}$ of words by

$$s_{-1} = 1, \quad s_0 = 0, \qquad s_n = s_{n-1}^{d_n} s_{n-2} \quad (n \geq 1) \tag{2.2.9}$$

The sequence $(s_n)_{n \geq -1}$ is a *standard sequence*, and the sequence $(d_1, d_2, \ldots)$ is its *directive sequence*. Observe that if $d_1 > 0$, then any s_n $(n \geq 0)$ starts with 0; on the contrary, if $d_1 = 0$, then $s_1 = s_{-1} = 1$, and s_n starts with 1 for $n \neq 0$. Every s_{2n} ends with 0, every s_{2n+1} ends with 1.

EXAMPLE 2.2.23. The directive sequence $(1, 1, \ldots)$ gives the standard sequence defined by $s_n = s_{n-1}s_{n-2}$, that is the sequence of finite Fibonacci words. Observe that the directive sequence $(0, 1, 1, \ldots)$ results in the sequence of words obtained from Fibonacci words by exchanging 0 and 1.

Every standard word occurs in some standard sequence, and every word occurring in a standard sequence is a standard word. This results by induction from the fact that, for $s_n = s_{n-1}^{d_n} s_{n-2}$, one has

$$(s_n, s_{n-1}) = \Delta^{d_n}(s_{n-2}, s_{n-1}), \qquad (s_{n-1}, s_n) = \Gamma^{d_n}(s_{n-1}, s_{n-2})$$

Thus

$$\begin{aligned}(s_{2n}, s_{2n-1}) &= \Delta^{d_{2n}} \circ \Gamma^{d_{2n-1}} \circ \cdots \circ \Gamma^{d_1}(0,1)\\ (s_{2n}, s_{2n+1}) &= \Gamma^{d_{2n+1}} \circ \Delta^{d_{2n}} \circ \Gamma^{d_{2n-1}} \circ \cdots \circ \Gamma^{d_1}(0,1)\end{aligned}$$

By Equation 2.2.4, this gives the expressions

$$\begin{aligned} s_{2n} &= G^{d_1} \circ D^{d_2} \circ \cdots \circ D^{d_{2n}}(0) = G^{d_1} \circ \cdots \circ D^{d_{2n}} \circ G^{d_{2n+1}}(0)\\ s_{2n+1} &= G^{d_1} \circ D^{d_2} \circ \cdots \circ D^{d_{2n+2}}(1) = G^{d_1} \circ \cdots \circ D^{d_{2n}} \circ G^{d_{2n+1}}(1)\end{aligned}$$

Proposition 2.2.24. *Let $\alpha = [0, 1 + d_1, d_2, \ldots]$ be the continued fraction expansion of some irrational α with $0 < \alpha < 1$, and let (s_n) be the standard sequence associated to $(d_1, d_2, \ldots)$. Then every s_n is a prefix of c_α and*

$$c_\alpha = \lim_{n\to\infty} s_n \,.$$

Proof. By definition, $s_n = s_{n-1}^{d_n} s_{n-2}$ for $n \geq 1$. Define morphisms h_n by

$$h_n = \theta_{1+d_1} \circ \theta_{d_2} \circ \cdots \circ \theta_{d_n} \,.$$

We claim that

$$s_n = h_n(0), \quad s_n s_{n-1} = h_n(1), \quad n \geq 1$$

This holds for $n = 1$ since $h_1(0) = 0^{d_1}1 = s_1$ and $h_1(1) = 0^{d_1}10 = s_1 s_0$. Next, for $n \geq 2$,

$$h_n(0) = h_{n-1}(\theta_{d_n}(0)) = h_{n-1}(0^{d_n - 1}1) = s_{n-1}^{d_n-1} s_{n-1} s_{n-2} = s_n$$

and

$$h_n(1) = h_{n-1}(0^{d_n-1}10) = s_n s_{n-1}$$

For any infinite word x, the infinite word $h_n(x)$ starts with s_n because both $h_n(0)$ and $h_n(1)$ start with s_n. Thus, setting $\beta_n = [0, d_{n+1}, d_{n+2}, \ldots]$, one has $c_\alpha = h_n(c_{\beta_n})$ by Corollary 2.2.22 and thus c_α starts with s_n. This proves the first claim. The second is an immediate consequence. ∎

It is easily checked that

$$\begin{aligned}\theta_{1+d_1} \circ \theta_{d_2} \circ \cdots \circ \theta_{d_r} &= G^{d_1} \circ E \circ G^{d_2} \circ E \circ \cdots \circ G^{d_r} \circ E \circ G\\ &= \begin{cases} G^{d_1} \circ D^{d_2} \circ \cdots \circ D^{d_r} \circ G & \text{if } r \text{ is even,}\\ G^{d_1} \circ D^{d_2} \circ \cdots \circ D^{d_r} \circ D \circ E & \text{otherwise.}\end{cases}\end{aligned}$$

Example 2.2.25. The directive sequence for the Fibonacci word is $(1, 1, \ldots)$. The corresponding irrational is $1/\tau^2 = [0, 2, 1, 1, \ldots]$, and indeed the infinite Fibonacci word is the characteristic word of slope $1/\tau^2$.

EXAMPLE 2.2.26. Since $1/\tau = [0, 1, 1, 1, \ldots]$, the corresponding standard sequence is $s_1 = 1$, $s_2 = 10$, $s_3 = 101, \ldots$. The sequence is obtained from the Fibonacci sequence by exchanging 0's and 1's, in concordance with Lemma 2.2.17, since indeed $1/\tau + 1/\tau^2 = 1$.

EXAMPLE 2.2.27. Consider $\alpha = (\sqrt{3} - 1)/2 = [0, 2, 1, 2, 1, \ldots]$. The directive sequence is $(1, 1, 2, 1, 2, 1, \ldots)$, and the standard sequence starts with $s_1 = 01$, $s_2 = 010$, $s_3 = 01001001, \ldots$, whence

$$c_{(\sqrt{3}-1)/2} = 0100100101001001001010010010 01\cdots$$

Due to the periodicity of the development, we get for $n \geq 2$ that $s_{n+2} = s_{n+1}^2 s_n$ if n is odd, and $s_{n+2} = s_{n+1} s_n$ if n is even.

COROLLARY 2.2.28. *Every standard word is a prefix of some characteristic word.* ■

Thus, every standard word is left special.

COROLLARY 2.2.29. *A word is central if and only if it is a palindrome prefix of some characteristic word.*

Proof. A central word is a prefix of some standard word, so also of some characteristic word. Conversely, a palindrome prefix of a characteristic word is a prefix of any sufficiently long word in its standard sequence, so also of some sufficiently long central word. Thus the result follows from Proposition 2.2.10. ■

Proposition 2.2.24 has several interesting consequences. The relation to fixpoints is left to section 2.3.6. We focus on two properties, first the powers that may appear in a Sturmian word, and then the computation of the number of factors of Sturmian words.

Let x be an infinite word. For $w \in F(x)$, the *index* of w in x is the greatest integer d such that $w^d \in F(x)$, if such an integer exists. Otherwise, w is said to have infinite index.

PROPOSITION 2.2.30. *Every nonempty factor of a Sturmian word s has finite index in s.*

Proof. Assume the contrary. There exist a Sturmian word s and a nonempty factor u of s such that u^n is a factor of s for every $n \geq 1$. Consequently, the periodic word u^ω is in the dynamical system generated by s. Since this system is minimal, $F(s) = F(u^\omega)$, a contradiction. ■

An infinite word x has *bounded index* if there exists an integer d such that every nonempty factor of x has an index less than or equal to d.

THEOREM 2.2.31. *A Sturmian word has bounded index if and only if the continued fraction expansion of its slope has bounded partial quotients.*

We start with a lemma.

Lemma 2.2.32. *Let $(s_n)_{n\geq -1}$ be the standard sequence of the characteristic word c_α, with $\alpha = [0, 1+d_1, d_2, \ldots]$. For $n \geq 3$, the word $s_n^{1+d_{n+1}}$ is a prefix of c_α, and $s_n^{2+d_{n+1}}$ is not a prefix. If $d_1 \geq 1$, this holds also for $n = 2$.*

Example 2.2.33. For the Fibonacci word $f = 0100101001001\cdots$, we have $s_n = f_n$ and $d_n = 1$ for all n. The lemma claims that for $n \geq 2$, the word f_n^2 is a prefix of the infinite word f, and that f_n^3 is not. As an example, $f_2^2 = 010010$ is a prefix and $f_2^3 = 010010010$ is not. Observe also that $f_1^2 = 0101$ is not a prefix of f.

Proof. We show that for $n \geq 3$ (and for $n \geq 2$ if $d_1 \geq 1$), one has

$$s_{n-1}s_n = s_n t_{n-1}, \quad \text{with } t_n = s_{n-1}^{d_n-1} s_{n-2} s_{n-1}$$

Indeed

$$\begin{aligned} s_{n-1}s_n &= s_{n-1}s_{n-1}^{d_n}s_{n-2} = s_{n-1}^{d_n}s_{n-2}^{d_{n-1}}s_{n-3}s_{n-2} \\ &= s_{n-1}^{d_n}s_{n-2}s_{n-2}^{d_{n-1}-1}s_{n-3}s_{n-2} = s_n t_{n-1} \end{aligned}$$

provided $d_{n-1} \geq 1$. Observe that t_{n-1} is not a prefix of s_n, since otherwise $s_n = t_{n-1}u$ for some word u, and $s_{n-1}s_n u = s_n^2$ and s_n is not primitive.

Clearly, $s_{n+1}s_n$ is a prefix of the characteristic word c_α. Since

$$s_{n+1}s_n = s_n^{d_{n+1}} s_{n-1} s_n = s_n^{1+d_{n+1}} t_{n-1}$$

the word $s_n^{1+d_{n+1}}$ is a prefix of c_α, and since t_{n-1} is not a prefix of s_n, the word $s_n^{2+d_{n+1}}$ is not a prefix of c_α. ∎

Proof of Theorem 2.2.31. Since a Sturmian word has the same factors as the characteristic word of same slope, it suffices to prove the result for characteristic words. Let c be the characteristic word of slope $\alpha = [0, 1+d_1, d_2, \ldots]$. Let $(s_n)_{n\geq -1}$ be the associated standard sequence.

To prove that the condition is necessary, observe that $s_n^{d_{n+1}}$ is a prefix of c for each $n \geq 1$. Consequently, if the sequence (d_n) of partial quotients is unbounded, the infinite word c has factors of arbitrarily great exponent.

Conversely, assume that the partial quotients (d_n) are bounded by some D and arguing by contradiction, suppose that c has unbounded index. Let r be some integer such that $F(c)$ contains a primitive word of length r with index greater than $D+4$. Among those words, let w be a word of length r of maximal index. Let $d+1$ be the index of w. Then $d \geq D+3$. The proof is in three steps.

(1) The characteristic word c has prefixes of the form w^d, with $d \geq D+3$. Indeed, if w^{d+1} is a prefix of c, we are done. Otherwise, consider an occurrence of w^{d+1}. Set $w = za$ with a a letter, and let b be the letter preceding the occurrence of w^{d+1}. If $b = a$, replace w by az and proceed. The process will stop after at most $|w|-1$ steps because either a prefix of c is obtained, or because otherwise w would occur in c at the power $d+2$. Thus, we may assume $b \neq a$.

Thus $b(za)^{d+1}$ is a factor of c. This implies that $a(za)^d$ and $b(za)^d$ are factors, so w^d is a right special factor, and therefore it is a prefix of c.

(2) If w^d is a prefix of the characteristic word c, then w is one of the standard words s_n. Indeed, set $e = d-2$, so that $e \geq D+1$. Let n be the greatest integer such that s_n is a prefix of w^{e+1}. Then w^{e+1} is a prefix of $s_{n+1} = s_n^{d_{n+1}} s_{n-1}$, thus also of $s_n^{1+d_{n+1}}$. This shows that

$$(1+D)|w| \leq (1+e)|w| \leq (1+d_{n+1})|s_n| \leq (1+D)|s_n|$$

whence $|w| \leq |s_n|$. Now, since both w^{e+2} and $s_n^{1+d_{n+1}}$ are prefixes of c, one is a prefix of the other. If w^{e+2} is the shorter one, then $|w^{e+2}| = |w^{e+1}| + |w| \geq |s_n| + |w|$. Thus, w^{e+2} and $s_n^{1+d_{n+1}}$ share a common prefix of length $\geq |s_n| + |w|$. Consequently, w and s_n are powers of the same word, and since they are primitive, they are equal.

If $s_n^{1+d_{n+1}}$ is the shorter one then, since $(1+e)|w| \leq (1+d_{n+1})|s_n|$,

$$\left|s_n^{1+d_{n+1}}\right| = |s_n| + d_{n+1}|s_n| \geq |s_n| + \frac{d_{n+1}}{1+d_{n+1}}(1+e)|w| \geq |s_n| + |w|$$

and the same conclusion holds.

(3) If follows that s_n^{1+e} is a prefix of c and, since $e \geq D+1 \geq d_{n+1}+1$, also $s_n^{2+d_{n+1}}$ is a prefix of c, contradicting Lemma 2.2.32. ∎

We conclude this section with the computation of the number of factors of Sturmian words. Another characterization of central words will help. Recall that a finite word is balanced if and only if it is a factor of some Sturmian word. Moreover, every balanced word w, as a factor of some uniformly recurrent infinite word, can be extended to the right and to the left, that is wa and bw are balanced for some letters a, b.

PROPOSITION 2.2.34. *For any word w, the following are equivalent:*
(i) *the word w is central;*
(ii) *the words $0w0$, $0w1$, $1w0$, $1w1$ are balanced;*
(iii) *the words $0w1$ and $1w0$ are balanced.*

Proof. $(i) \Rightarrow (ii)$. The words $w01$ and $w10$ are standard, and therefore are prefixes of some characteristic words c and c'. By Proposition 2.1.22 the four infinite words $0c$, $1c$, $0c'$ and $1c'$ are Sturmian, and consequently their prefixes $0w0$, $0w1$, $1w0$, $1w1$ are balanced. $(ii) \Rightarrow (iii)$ is trivial.

$(iii) \Rightarrow (i)$. We prove first that w is a palindrome word. Assume the contrary. Then there are words u,v, v' and letters $a \neq b$ such that $w = uav = v'b\tilde{u}$. But then $awb = auavb = av'b\tilde{u}b$ has factors aua and $b\tilde{u}b$ with height satisfying $|h(aua) - h(b\tilde{u}b)| = 2$, contradiction.

Let c be a characteristic word such that $0w1 \in F(c)$. Since $F(c)$ is closed under reversal (Proposition 2.1.19), and w is a palindrome, $1w0 \in F(c)$, showing that w is a right special factor of c. Thus its reversal (that is w itself) is a prefix of c. In view of Corollary 2.2.29, the word w is central. ∎

Words satisfying condition (ii) are sometimes called *strictly bispecial*.

We now want to count the number of balanced words of length n. We need a lemma.

LEMMA 2.2.35. *Let w be a word. If $w0$ and $w1$ are balanced, then there is a letter a such that $aw0$ and $aw1$ are balanced.*

Before giving the proof, let us observe that there seems to be a difference, for a word w, to be right special or have both extensions $w0$ and $w1$ balanced. Indeed, a word w can only be right special with respect to some Sturmian word s that contains both factors $w0$ and $w1$. On the contrary, if $w0$ and $w1$ are balanced, then there exist Sturmian words x an y such that $w0 \in F(x)$ and $w1 \in F(y)$, but x and y need not be the same. In fact, one can show (Problem 2.2.7) that both notions coincide.

Proof of Lemma 2.2.35. Since $w0$ and $w1$ are factors of Sturmian words, there exist letters a and b such that $aw0$ and $bw1$ are balanced. If $a = b$, we get the claim. If $a = 1$ and $b = 0$, then w is central by Proposition 2.2.34, and therefore is balanced. Thus suppose $a = 0$, $b = 1$. Then $0w0$ and $1w1$ are balanced, but neither $1w0$ nor $0w1$ are. According to Proposition 2.1.3, there exists a palindrome word u such that $1u1$ and $0u0$ are factors of $1w0$. However, since $1w$ and $w0$ are balanced, $1u1$ is a prefix of $1w0$ and $0u0$ is a suffix of $1w0$. Thus there exist words p, s such that $1w0 = 1u1s0 = 1p0u0$, whence $w = u1s = p0u$. Similarly, there exist words u', p', s' such that $w = u'0s' = p'1u'$. We may assume $|u| < |u'|$ and set $u' = u1x = y0u$ for some words x, y. Then $w = y0u0s' = p'1u1x$, showing that w is unbalanced, a contradiction. ∎

THEOREM 2.2.36. *The number of balanced words of length n is*

$$1 + \sum_{i=1}^{n} (n + 1 - i)\phi(i)$$

where ϕ is Euler's totient function.

Proof. Let $R(n)$ be the set of words w of length n such that $0w$ and $1w$ are balanced, and set $r(n) = \text{Card } R(n)$. Then $r(0) = 1 = \phi(1)$ and

$$r(n + 1) = r(n) + \phi(n + 2)$$

Indeed, for each $w \in R(n)$, one has $0w \in R(n + 1)$ or $1w \in R(n + 1)$ by Lemma 2.2.35, and both $0w, 1w \in R(n + 1)$ if and only if $w \in R(n)$ and $0w1$ and $1w0$ are balanced, that is if and only if w is central, by Proposition 2.2.34. Thus $r(n + 1) - r(n)$ is the number of central words of length n, which in turn is $\phi(n + 2)$ by Corollary 2.2.16. It follows that

$$r(n) = \sum_{i=1}^{n+1} \phi(n) \,.$$

Let $g(n)$ be the number of balanced words of length n. Then

$$g(n+1) = g(n) + r(n)$$

since for each balanced word w, the word $w0$ or $w1$ is balanced, and both are balanced if and only if $w \in R(n)$. Since $g(0) = 1$, it follows that

$$g(n) = 1 + \sum_{k=0}^{n-1} r(k) = 1 + \sum_{k=0}^{n-1}\sum_{i=1}^{k+1} \phi(i) = 1 + \sum_{k=1}^{n}\sum_{i=1}^{k} \phi(i) = 1 + \sum_{i=1}^{n} (n+1-i)\phi(i)$$

as required. ∎

2.2.3. Frequencies

Let x be an infinite word. Recall from Chapter 1 that the *factor graph* $G_n(x)$ of order n is the graph with vertex set $F_n(x)$ and domain $F_{n+1}(x)$. A triple (p, a, s) is an edge if and only if $pa = bs \in F_{n+1}(x)$ for some letter b.

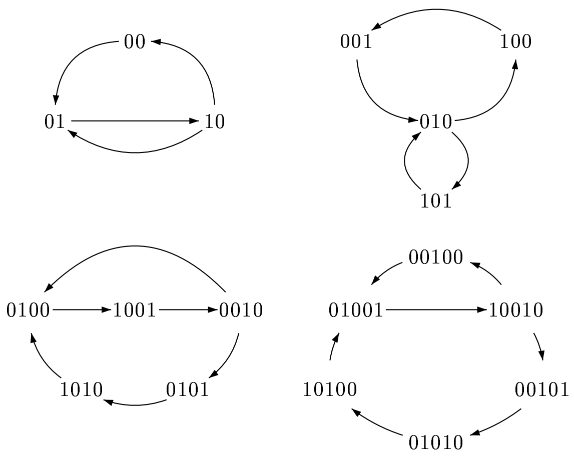

Figure 2.6. Factor graphs for the Fibonacci word.

If x is a Sturmian word, then there is exactly one vertex in $G_n(x)$ with out-degree 2. This is the right special factor d_n of length n. The edges leaving d_n are $(d_n, 0, d_{n-1}0)$ and $(d_n, 1, d_{n-1}1)$, because d_{n-1} is a suffix of d_n. Similarly, there is exactly one vertex with in-degree 2. This is the left special factor g_n of length n. Let a be the letter such that $g_n = g_{n-1}a$. Then the edges entering g_n are $(0g_{n-1}, a, g_n)$ and $(1g_{n-1}, a, g_n)$. Observe that $d_n = g_n$ if and only if d_n is a palindrome word. See Figure 2.6 for the word graphs of the Fibonacci word.

The factor graph of order n of a Sturmian word x is composed of three paths: the first is from g_n to d_n, both vertices included. This path is never empty. There are two other paths, from d_n to g_n, one through vertex $d_{n-1}0$ the other through $d_{n-1}1$. We consider that the endpoints d_n and g_n are not part of these paths. Then such a path may be empty. This happens if and only if $d_{n-1}0 = g_n$ or $d_{n-1}1 = g_n$ which in turn is the case if and only if $d_{n-1} = g_{n-1}$ because g_{n-1} is a prefix of g_n.

Let $s = s_{\alpha,\rho}$ be a Sturmian word of slope α. We have seen how to associate to s a rotation R on the unit circle. Also (Equation 2.1.11), a word w is a factor of s if and only if the interval I_w of the unit circle is non empty. Moreover, an integer $n \geq 0$ is the starting index of an occurrence of w in s if and only if $R^n(\rho) \in I_w$.

Let $\mu_N(w)$ be the number of occurrences of w in the prefix of length $N + |w| - 1$ of s. This is exactly the number of integers n, with $0 \leq n < N$, such that $R^n(\rho) \in I_w$. It is known from number theory that the numbers $R^n(\rho)$, $(n \geq 1)$ are uniformly distributed in the interval $[0, 1[$. As a consequence, the limit

$$\mu(w) = \lim_{N\to\infty} \mu_N(w)$$

always exists and is equal to the length of the interval I_w. The number $\mu(w)$ is the *frequency* of w in s. Of course, $\mu(w) = 0$ if and only if $w \notin F(s)$. It is easily seen that, for any word w, one has $\mu(0w) + \mu(1w) = \mu(w)$ and symmetrically $\mu(w) = \mu(w0) + \mu(w1)$.

THEOREM 2.2.37. *Let s be a Sturmian word. For each n, the frequencies of the factors of length n take at most three values. If they take three values, then one is the sum of the two others.*

LEMMA 2.2.38. *Let s be a Sturmian word. Let (p, a, q) be an edge in $G_n(s)$. If p is not right special and q is not left special, then $\mu(p) = \mu(q)$.*

Proof. There exists a letter b such that $pa = bq \in F_{n+1}(s)$. Since $pb, aq \notin F_{n+1}$, one has $\mu(p) = \mu(pa) = \mu(bq) = \mu(q)$. ∎

Proof of Theorem 2.2.37. By the lemma, the frequencies are constant on each of the three paths in the factor graph $G_n(s)$. Thus there are at most three frequencies. Assume that none of the three paths in the factor graph is empty. According to our discussion, this happens if and only if $d_{n-1} \neq g_{n-1}$. Moreover, the frequencies are those of any set of vertices taken in the paths, e.g. $\mu(d_n)$, $\mu(d_{n-1}0)$, and $\mu(d_{n-1}1)$. Set $d_n = 0d_{n-1}$. Since d_{n-1} is not left special, $1d_{n-1}$ is not a factor of s. Thus

$$\mu(d_n) = \mu(0d_{n-1}) = \mu(d_{n-1}) = \mu(d_{n-1}0) + \mu(d_{n-1}1)$$

showing the second part of the theorem. ∎

2.3. Sturmian morphisms

All morphisms will be endomorphisms of $\{0, 1\}^*$. The identity morphism Id and the morphism E that exchanges the letters 0 and 1 will be called *trivial* morphisms.

A morphism f is *Sturmian* if $f(s)$ is a Sturmian word for every Sturmian word s. Since an erasing morphism can never be Sturmian, all morphisms considered here are assumed to be nonerasing. The trivial morphisms Id and E are Sturmian. The set of Sturmian morphisms is closed under composition, and consequently is a submonoid of the monoid of endomorphisms of $\{0,1\}^*$.

2.3.1. A set of generators

The main result of this section is the characterization of Sturmian morphisms (Theorem 2.3.7). Consider the morphisms

$$\varphi : \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 0 \end{array} \qquad \tilde{\varphi} : \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 0 \end{array}$$

Recall from Chapter 1 that the morphism φ generates the infinite Fibonacci word $f = \varphi(f) = 0100101001001010010100\cdots$.

PROPOSITION 2.3.1. *The morphisms E, φ and $\tilde{\varphi}$ are Sturmian.*

Proof. This follows from Corollary 2.2.19. ∎

We shall see below that every Sturmian morphism is a composition of these three morphisms. The following property gives a converse of Proposition 2.3.1.

PROPOSITION 2.3.2. *Let x be an infinite word.*
(i) *If $\varphi(x)$ is Sturmian then x is Sturmian.*
(ii) *If $\tilde{\varphi}(x)$ is Sturmian and x starts with the letter 0, then x is Sturmian.*

Proof. Let x be an infinite word. If $\varphi(x)$ or $\tilde{\varphi}(x)$ is Sturmian, then x is clearly aperiodic. Arguing by contradiction, let us suppose that x is not balanced and suppose that $0v0$ and $1v1$ are both factors of x.

Clearly, $\varphi(0v0) = 01\varphi(v)01$, $\varphi(1v1) = 0\varphi(v)0$ and every occurrence of $\varphi(1v1)$ in $\varphi(x)$ is followed by the letter 0. Consequently $1\varphi(v)01$ and $0\varphi(v)00$ are both factors of $\varphi(x)$ which is not balanced.

Next, if x does not start with 1, then either $01v1$ or $11v1$ is a factor of x. But $\tilde{\varphi}(0v0)$ contains the factor $10\tilde{\varphi}(v)1$, and $\tilde{\varphi}(01v1)$ and $\tilde{\varphi}(11v1)$ both contain the factor $00\tilde{\varphi}(v)0$. Consequently, $\tilde{\varphi}(x)$ is not balanced. ∎

COROLLARY 2.3.3. *Let x be an infinite word and let f be a morphism that is a composition of E and φ. If $f(x)$ is Sturmian then x is Sturmian.* ∎

EXAMPLE 2.3.4. We give an example of a non Sturmian word x starting with 1 and such that $\tilde{\varphi}(x)$ is Sturmian. Let f be the Fibonacci word. The infinite word $11f$ is not Sturmian because it contains both 00 and 11 as factors. However, since f is a characteristic word, the infinite word $0f$ is Sturmian. Consequently $\tilde{\varphi}(\varphi(0f)) = \tilde{\varphi}(01f) = 100\tilde{\varphi}(f)$ is Sturmian. Thus $00\tilde{\varphi}(f)$ also is Sturmian and, since $00 = \tilde{\varphi}(11)$, $\tilde{\varphi}(11f)$ is Sturmian.

Let us denote St the submonoid of the monoid of endomorphisms obtained by composition of E, φ and $\tilde{\varphi}$ in any number and order. St is called the *monoid of Sturm* and by Proposition 2.3.1 all its elements are Sturmian. A first step to the converse is the following.

LEMMA 2.3.5. *Let f and g be two morphisms and let x a Sturmian word. If $f \in St$ and $f \circ g(x)$ is a Sturmian word, then $g(x)$ is a Sturmian word.*

Proof. Let x be a Sturmian word and g a morphism. It suffices to prove the conclusion for $f = E$, $f = \varphi$ and $f = \tilde{\varphi}$.

Set $y = g(x)$. If $E(y)$ is a Sturmian word then y is also a Sturmian word too and, by Proposition 2.3.2, this also holds if $\varphi(y)$ is a Sturmian word. It remains to prove that if $\tilde{\varphi}(y)$ is a Sturmian word then so is y.

Suppose that y is not a Sturmian word. Observe that y is aperiodic, since otherwise $\tilde{\varphi}(y)$ is eventually periodic thus it is not Sturmian. Thus $y = g(x)$ is not balanced and contains two factors $0v0$ and $1v1$ which are factors of images of some factors of x. The Sturmian word x is recurrent, thus $1v1$ occurs infinitely often in y, which implies that $01v1$ or $11v1$ is a factor of y. Since $\tilde{\varphi}(0v0) = 10\tilde{\varphi}(v)10$ and $\tilde{\varphi}(1v1) = 0\tilde{\varphi}(v)0$, both $10\tilde{\varphi}(v)1$ and $00\tilde{\varphi}(v)0$ are factors of $\tilde{\varphi}(y)$ and thus $\tilde{\varphi}(y)$ is not balanced. A contradiction. ∎

COROLLARY 2.3.6. *Let $f \in St$ and g be a morphism. The morphism $f \circ g$ is Sturmian if and only if g is Sturmian.*

Proof. Assume first that g is Sturmian. Since f is a composition of E, φ and $\tilde{\varphi}$, the morphism $f \circ g$ is Sturmian by Proposition 2.3.1.

Conversely, if $f \circ g$ is Sturmian, then for every Sturmian word x, the infinite word $f \circ g(x)$ is Sturmian and, by Lemma 2.3.5, the infinite word $g(x)$ is Sturmian. This means that g is Sturmian. ∎

A morphism f is *locally Sturmian* if there exists at least one Sturmian word x such that $f(x)$ is a Sturmian word.

THEOREM 2.3.7. *Let f be a morphism. The following three conditions are equivalent:*
(i) *$f \in St$;*
(ii) *f is Sturmian;*
(iii) *f is locally Sturmian.*

The equivalence of (i) and (ii) means that the monoid of Sturm is exactly the monoid of Sturmian morphisms.

The *length* of a morphism f is the number $\|f\| = |f(0)| + |f(1)|$. The proof of Theorem 2.3.7 is based on the following fundamental lemma.

LEMMA 2.3.8. *Let f be a non trivial morphism. If f is locally Sturmian then $f(0)$ and $f(1)$ both start or end with the same letter.*

Proof. Let f be a non trivial morphism and suppose that $f(0)$ and $f(1)$ do not start nor end with the same letter.

Suppose $f(0)$ starts with the letter 0. Then $f(1)$ starts with the letter 1. If $f(0)$ ends with 1 then $f(1)$ ends with 0. But in this case $f(01)$ contains a factor 11 and $f(10)$ contains a factor 00. Thus the image of any Sturmian word contains the two factors 00 and 11 which means that f is not locally Sturmian.

Otherwise $f(0) \in 0A^*0 \cup \{0\}$ and $f(1) \in 1A^*1 \cup \{1\}$, and we prove the result by induction on $\|f\|$.

If $\|f\| = 3$, then $f(a) = cc$ and $f(b) = d$ for letters a, b, c, d, $a \neq b$, and since any Sturmian word x contains the two factors a^{n+1} and $ba^n b$ for some integer n, $f(x)$ contains $(cc)^{n+1}$ and $d(cc)^n d$ and thus is not Sturmian.

Arguing by contradiction, suppose that $\|f\| \geq 4$ and f is locally Sturmian. Let x be a Sturmian word such that $f(x)$ is Sturmian (such a word exists because f is locally Sturmian) and suppose that x contains the factor 00 (the case where x contains 11 is clearly the same). Since $f(0)$ starts and ends with 0, $f(x)$ contains also 00. Consequently, since the infinite word $f(x)$ is balanced, neither $f(0)$ nor $f(1)$ contains the factor 11.

Since x is Sturmian, x does not contain 11 and there is an integer $m \geq 1$ such that every block of 0 between two consecutive occurrences of 1 is either 0^m or 0^{m+1}.

The word $f(0)$ does not contain the factor 00. Indeed, otherwise $f(0) = u00v$ and $f(1) = r1 = 1s$ for some words u, v, r, s. Since 0^{m+1} and 10^m1 are factors of w, the words $f(0^{m+1})$ and $f(10^m1)$ are factors of $f(x)$. But

$$f(0^{m+1}) = u00vf(0^{m-1})u00v = uw_1v, \quad f(10^m1) = r1f(0^{m-1})u00v1s = rw_2s$$

for suitable w_1, w_2, and one has $|w_1| = |w_2|$ and $\delta(w_1, w_2) = 2$, a contradiction.

Consequently $f(0) = (01)^n0$ for some integer $n \geq 0$.

Since 10^m1 and $10^{m+1}1$ are factors of x, the infinite word $f(x)$ contains the two factors 10^m1 and $10^{m+1}1$ if $n = 0$, and the two factors 101 and 1001 if $n \neq 0$. Set $p = m$ if $n = 0$, and $p = 1$ if $n \neq 0$. Then in both cases, $f(x)$ contains the factors 10^p1 and $10^{p+1}1$, and in both cases $1 \leq p \leq m$.

Since $f(1)$ does not contain the factor 11, there exist an integer $k \geq 0$, and integers $m_1, \ldots, m_k \in \{0, 1\}$ such that

$$f(1) = 10^{p+m_1}10^{p+m_2}1 \cdots 10^{p+m_k}1$$

Consider a new alphabet $B = \{a, b\}$ and two morphisms $\rho, \eta : B^* \to A^*$

$$\rho : \begin{array}{l} a \mapsto 0 \\ b \mapsto 0^p1 \end{array} \qquad \eta : \begin{array}{l} a \mapsto (01)^n0 \\ b \mapsto 0^p1 \end{array}$$

We show that there exists a word u over B such that $f(\rho(b)) = \eta(bub)$.

(i) If $n = 0$, set $u = a^{m_1}ba^{m_2}b \ldots ba^{m_k}$. Since $f(1) \neq 1$, one has $f(1) = 1\eta(u)0^p1$. Thus $f(\rho(b)) = f(0^p1) = \eta(bub)$.

(ii) If $n \neq 0$ and $m_1 = \ldots = m_k = 0$, set $u = b^{k+n-1}$. Since $f(1) = (10)^k1$, one gets $\eta(u) = (01)^{k+n-1}$ and $f(\rho(b)) = f(01) = \eta(bub)$.

(iii) Otherwise $n \neq 0$ and $m_i = 1$ for at least one integer $i, 1 \leq i \leq k$. Thus there exist integers $t \geq 2$, $n_1, \ldots, n_t$ such that

$$f(1) = 1(01)^{n_1}0(01)^{n_2}0 \ldots (01)^{n_{t-1}}0(01)^{n_t}$$

Since $f(01)$ starts with $(01)^{n+1}$, one has $n_1 \geq 0$, $n_i \geq n$ for $2 \leq i \leq t-1$ and $n_t \geq 1$. Set $u = b^{n_1}ab^{n_2-n}a \ldots b^{n_{t-1}-n}ab^{n_t-1}$. Then, again, $f(\rho(b)) = f(01) = \eta(bub)$.

Define a morphism $g : B^* \to B^*$ by

$$g : \begin{array}{l} a \mapsto a \\ b \mapsto bub \end{array}$$

Then $f \circ \rho = \eta \circ g$. Since $m \geq p$, by deleting if necessary some letters at the beginning of x, one may suppose that x starts with 0^p1. It follows that there exists a (unique) infinite word x' over B such that $\rho(x') = x$.

Thus there exists a (unique) infinite word y' over B such that

$$\begin{tikzcd} x & x' \arrow[l, "\rho"'] \arrow[d, "g"] \\ f(x) \arrow[from=u, "f"'] & y' \arrow[l, "\eta"] \end{tikzcd}$$

Identifying a with 0 and b with 1, one has $\rho = (\varphi \circ E)^p$. If $n = 0$ then $\eta = \rho$. If $n \neq 0$ then $p = 1$, so $\eta = \varphi \circ E \circ (E \circ \varphi)^n$. Thus since x and $f(x)$ are Sturmian, the words x' and y' are Sturmian by Corollary 2.3.3. Consequently the morphism g is locally Sturmian.

However, the words $g(0)$ and $g(1)$ do not start nor end with the same letter and $3 \leq \|g\| < \|f\|$. By induction, g is not locally Sturmian, a contradiction. The lemma is proved. ∎

Proof of Theorem 2.3.7. It is easily seen that $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$.

So let us suppose that f is a locally Sturmian morphism. The property is straightforward if $f = Id$ or $f = E$. Thus we assume $\|f\| \geq 3$.

Let x be a Sturmian word such that $f(x)$ is also a Sturmian word. Since $f(x)$ is balanced, it contains only one of the two words 00 or 11.

Suppose that $f(x)$ contains 00. From Lemma 2.3.8, the words $f(0)$ and $f(1)$ both start or end with 0. Consider first the case where $f(0)$ and $f(1)$ both start with 0. Then $f(0), f(1) \in \{0, 01\}^+$ and there exists two words u and v such that $f(0) = \varphi(u)$ and $f(1) = \varphi(v)$. Define g a morphism by $g(0) = u$ and $g(1) = v$. Then $f = \varphi \circ g$ and, by Lemma 2.3.5, $g(x)$ is a Sturmian word. Next, $\|f\| = \|g\| + |uv|_0$ and $|uv|_0 > 0$. Otherwise, $f(0) = \varphi(u)$ and $f(1) = \varphi(v)$ would contain only 0 and $f(x) = 0^\omega$ would not be Sturmian. Thus $\|g\| < \|f\|$ and the result follows by induction.

If $f(0)$ and $f(1)$ both end with 0, the same argument holds with $\tilde{\varphi}$ instead of φ, and if $f(x)$ contains 11 then $E \circ f$ is of the same height and contains 00. ∎

We give here only one property of the monoid St which shows how decide whether a morphism is Sturmian by trying to decompose it over $\{E, \varphi, \tilde{\varphi}\}$. Other properties will be seen in section 2.3.3 and in the problem section.

COROLLARY 2.3.9. *The monoid of Sturm is left and right unitary, i.e. for all morphisms f and g:*

1. *If $f \circ g \in St$ and $f \in St$ then $g \in St$.*
2. *If $f \circ g \in St$ and $g \in St$ then $f \in St$.*

Proof. Let f and g be two morphisms such that $f \circ g \in St$. Let x be a Sturmian word. Then $f \circ g(x)$ is a Sturmian word.

1. If $f \in St$ then by Lemma 2.3.5, $g(x)$ is a Sturmian word. Consequently g is locally Sturmian and, by Theorem 2.3.7, $g \in St$.

2. If $g \in St$ then $g(x)$ is a Sturmian word. Thus f is locally Sturmian and by Theorem 2.3.7, $f \in St$. ∎

From this property we deduce an algorithm to decide whether a morphism is Sturmian. Indeed, if f is a non trivial Sturmian morphism then f decomposes as $f = g \circ \sigma$, where g is Sturmian by Corollary 2.3.9 and where σ is one of the eight morphisms in $\{\varphi, \varphi \circ E, E \circ \varphi, E \circ \varphi \circ E, \tilde{\varphi}, \tilde{\varphi} \circ E, E \circ \tilde{\varphi}, E \circ \tilde{\varphi} \circ E\}$. According to σ, one gets the following factorizations of $f(0)$ and $f(1)$.

$g(0) = f(1)$ and $f(0) = f(1)u$ with $u = g(1)$ if $\sigma = \varphi$;
$g(0) = f(1)$ and $f(0) = uf(1)$ with $u = g(1)$ if $\sigma = \tilde{\varphi}$;
$g(1) = f(1)$ and $f(0) = f(1)u$ with $u = g(0)$ if $\sigma = E \circ \varphi$;
$g(1) = f(1)$ and $f(0) = uf(1)$ with $u = g(0)$ if $\sigma = E \circ \tilde{\varphi}$;
$g(0) = f(0)$ and $f(1) = f(0)u$ with $u = g(1)$ if $\sigma = \varphi \circ E$;
$g(0) = f(0)$ and $f(1) = uf(0)$ with $u = g(1)$ if $\sigma = \tilde{\varphi} \circ E$;
$g(1) = f(0)$ and $f(1) = f(0)u$ with $u = g(0)$ if $\sigma = E \circ \varphi \circ E$;
$g(1) = f(0)$ and $f(1) = uf(0)$ with $u = g(0)$ if $\sigma = E \circ \tilde{\varphi} \circ E$.

PROPOSITION 2.3.10. *A morphism f is Sturmian if and only if, with f as input, the algorithm below ends with $g = Id$ or E. In this case, the output h is a decomposition of f over $\{E, \varphi, \tilde{\varphi}\}$.* ∎

```
Algorithm:
    input:   f morphism;
    output:  h morphism;
    local:   g morphism;
begin
    g ← f;
    h ← Id;
    while one of the two words g(0) and g(1) is a proper prefix
            or a proper suffix of the other
    do if g(1) = g(0)u then
```

```
            g(1) ← u;  h ← φ ∘ E ∘ h
        else if g(1) = ug(0) then
            g(1) ← u;  h ← φ̃ ∘ E ∘ h
        else if g(0) = g(1)u then
            g(0) ← u;  h ← E ∘ φ ∘ h
        else {g(0) = ug(1)}
            g(0) ← u;  h ← E ∘ φ̃ ∘ h;
    if g = E then h ← E ∘ h
end.
```

Observe that $f(0)$ may be both a proper prefix and a proper suffix of $f(1)$ (or vice versa). In this case, there are two decompositions of f over $\{E, \varphi, \tilde{\varphi}\}$. These are obtained in the algorithm by inverting the order in the tests. We shall see in Section 2.3.3, that these are all decompositions (not containing E^2) of a given Sturmian morphism over $\{E, \varphi, \tilde{\varphi}\}$.

2.3.2. Standard morphisms

In this section it will be convenient to consider unordered standard pairs. An *unordered standard pair* is a set $\{x, y\}$ such that either (x, y) or (y, x) is a standard pair.

In particular, if $\{x, y\}$ is a unordered standard pair then $\{E(x), E(y)\}$ is a unordered standard pair. On the contrary, $\{\tilde{\varphi}(x), \tilde{\varphi}(y)\}$ is never a unordered standard pair because $\tilde{\varphi}(x)$ and $\tilde{\varphi}(y)$ both end with the same letter (Proposition 2.2.2).

Consequently, Sturmian morphisms that are compositions of E and φ are an interesting special case. Because of the following proposition, a morphism is called *standard* if it is a composition of E and φ.

PROPOSITION 2.3.11. *A morphism f is standard if and only if $\{f(0), f(1)\}$ is an unordered standard pair.*

Proof. Assume first that f is standard and, arguing by induction on $\|f\|$, suppose that $\{f(0), f(1)\}$ is an unordered standard pair. If $g = f \circ E$, then $\{g(0), g(1)\} = \{f(0), f(1)\}$ is an unordered standard pair. If $g = f \circ \varphi$, then $\{g(0), g(1)\} = \{f(0)f(1), f(0)\}$ is also an unordered standard pair.

Conversely, assume that $\{f(0), f(1)\}$ is an unordered standard pair, and that $|f(0)| > |f(1)|$. Then $f(0) = f(1)v$ for some word v, and $\{v, f(1)\}$ is an unordered standard pair. By induction, there is a standard morphism g such that $\{g(0), g(1)\} = \{v, f(1)\}$. If $g(0) = f(1)$ and $g(1) = v$ then $f = g \circ \varphi$, in the other case $f = g \circ E \circ \varphi$. Thus f is standard. ∎

The set of standard morphisms is interesting because these morphisms are closely related to characteristic words (recall that an infinite word x is characteristic if and only if $0x$ and $1x$ are Sturmian words), as it will appear in a moment.

A morphism f is *characteristic* if $f(x)$ is a characteristic word for every characteristic word x, and it is *locally characteristic* if there exists a characteristic word x such that $f(x)$ is a characteristic word.

The following theorem is an analogue of Theorem 2.3.7 for standard morphisms.

THEOREM 2.3.12. *Let f be a morphism. The following conditions are equivalent:*
(i) *f is standard;*
(ii) *f is characteristic;*
(iii) *f is locally characteristic.*

To prove this result we need the following lemma.

LEMMA 2.3.13. *Let x be an infinite word.*
1. *x is characteristic if and only if $E(x)$ is characteristic.*
2. *x is characteristic if and only if $\varphi(x)$ is characteristic.*

Proof. This is a consequence of Corollary 2.2.20 and Proposition 2.3.2. ∎

Proof of Theorem 2.3.12. The implication $(ii) \Rightarrow (iii)$ is obvious and the implication $(i) \Rightarrow (ii)$ is an immediate consequence of Lemma 2.3.13.

Let f be a locally characteristic morphism. Then f is locally Sturmian and by Theorem 2.3.7, it is a composition of E, φ and $\tilde{\varphi}$. We show that no occurrence of $\tilde{\varphi}$ appears in the decomposition of f, by induction on $\|f\|$.

If $\|f\| = 2$ then $f = Id$ or $f = E$ and the result holds.

Assume $\|f\| \geq 3$ and let x be a characteristic word such that $f(x)$ is characteristic.

If x contains 11 as a factor then we can replace x by $E(x)$ which is also a characteristic word (Lemma 2.3.13) and consider $f \circ E$ instead of f, and if $f(x)$ contains 11 as a factor then we can consider $E \circ f$ instead of f. Since $\|f\| = \|f \circ E\| = \|E \circ f\|$, we may suppose that x and $f(x)$ both contain the factor 00 (and thus none contains the factor 11).

Since x and $f(x)$ are characteristic, both $1x$ and $1f(x)$ are Sturmian, and thus both x and $f(x)$ start with the letter 0, and thus $f(0)$ also starts with 0.

If $f(1)$ starts with 1 then, by Lemma 2.3.8, $f(0)$ and $f(1)$ both end with the same letter. If this letter is a 1 then 11 is a factor of $f(01)$ and thus of $f(x)$ which is impossible. So $f(0)$ and $f(1)$ both end with the letter 0. Let $r \geq 1$ be such that x starts with 0^r1. Since $0x$ is Sturmian, x contains $0^{r+1}1$ and then 10^{r+1} as a factor. Consequently $1f(0^r)1$ is a prefix of $1f(x)$ and $0f(0^r)0$ is a factor of $f(x)$. A contradiction.

Thus, $f(1)$ starts with 0 and since $f(0)$ and $f(1)$ do not contain 11 as a factor, $f(0) \in \{01, 0\}^+$ and $f(1) \in \{01, 0\}^+$. Consequently there exists a morphism g such that $f = \varphi \circ g$ with $\|g\| < \|f\|$. But $\varphi \circ g(x)$ is characteristic thus $g(x)$ is characteristic (Lemma 2.3.13) and, by induction, $g \in \{E, \varphi\}^*$. So f is standard. ∎

2.3.3. A presentation of the monoid of Sturm

In this section, it will be convenient to write the composition of morphisms as a concatenation (so we will write fg instead of $f \circ g$).

Let $G = \varphi E$ and $\tilde{G} = \tilde{\varphi} E$. Clearly, the monoid of Sturm St is also generated by E, G and $\tilde{G}$.

THEOREM 2.3.14. *The monoid of Sturm has the presentation*

$$E^2 = Id, \tag{2.3.1}$$
$$GEG^k E\tilde{G} = \tilde{G}E\tilde{G}^k EG, \quad k \geq 0\,. \tag{2.3.2}$$

Formula (2.3.2) can be rewritten, in terms of the generators φ and $\tilde{\varphi}$, as

$$\varphi(\varphi E)^k E\tilde{\varphi} = \tilde{\varphi}(\tilde{\varphi} E)^k E\varphi, \quad k \geq 0\,.$$

Proof. We consider words over the alphabet $\{E, G, \tilde{G}\}$. For each word W over $\{E, G, \tilde{G}\}$, denote by f_W the Sturmian morphism defined by composing the letters of W. Two words W and W' are *equivalent* if $f_W = f_{W'}$. The words W and W' are *congruent* ($W \sim W'$) if one can obtain one from the other by a repeated application of (2.3.1) and (2.3.2) viewed as rewriting rules (i.e. if W and W' are in the same equivalence class of the congruence generated by (2.3.1) and (2.3.2)).

We prove that equivalent words are congruent (the converse is clear). Let W, W' be equivalent words. The proof is by induction on $|WW'|$. We may assume that W and W' do not contain E^2. Since $E, G, \tilde{G}$ are injective, we may also assume that W and W' do not start with the same letter. Observe that if W starts with φ or $\tilde{\varphi}$, then $|f_W(01)|_1 < |f_W(01)|_0$ and if W starts with $E \circ \varphi$ or $E \circ \tilde{\varphi}$, then $|f_W(01)|_1 > |f_W(01)|_0$. Consequently W starts with E if and only if W' starts with E, so we suppose that none does. Finally, since $G\tilde{G} \sim \tilde{G}G$, we may assume that one of W and W' starts with $G^n E$ and the other with $\tilde{G}^p E$ with $n \neq 0$ and $p \neq 0$. Thus

$$W = \tilde{G}^{r_1} E\tilde{G}^{r_2} G^{s_2} E \cdots E\tilde{G}^{r_q} G^{s_q}$$
$$W' = G^{s'_1} E\tilde{G}^{r'_2} G^{s'_2} E \cdots E\tilde{G}^{r'_{q'}} G^{s'_{q'}}$$

with $r_1, s'_1 \geq 1$, $r_i, s_i, r'_i, s'_i \geq 0$, and $r_i + s_i \geq 1$ for $2 \leq i < q$, $r'_j + s'_j \geq 1$for $2 \leq j < q'$.

Observe first that $f_{W'}(0)$ and $f_{W'}(1)$ both start with the letter 0 (because G does).

Next, $s_2 = 0$. Indeed, otherwise W is congruent to a word starting with $\tilde{G}^{r_1} EG$, and since $\tilde{G}^{r_1} EG(0)$ and $\tilde{G}^{r_1} EG(1)$ both start with the letter 1, W' is not equivalent to W.

If $s_i = 0$ for $i = 3, \ldots, q$, then $W = \tilde{G}^{r_1} E\tilde{G}^{r_2} E \cdots E\tilde{G}^{r_q}$, and $f_W(0)$ or $f_W(1)$ starts with the letter 1, according to whether q is even or odd. Thus, there is a smallest $i \geq 3$ such that $s_i \geq 1$. Then W is congruent to a word starting with

$$U = \tilde{G}^{r_1} E\tilde{G}^{r_2} E \cdots E\tilde{G}^{r_{i-2}} E\tilde{G}^{r_{i-1}} EG$$

If i is even, then $f_U(0)$ and $f_U(1)$ start with the letter 1. Thus i is odd, and using (2.3.2), U is congruent to

$$U' = \tilde{G}^{r_1} E \tilde{G}^{r_2} E \cdots E \tilde{G}^{r_{i-2}-1} G E G^{r_{i-1}} E \tilde{G}$$

and eventually U is congruent to

$$G \tilde{G}^{r_1-1} E G^{r_2} E \tilde{G}^{r_3} E \cdots E \tilde{G}^{r_{i-2}} E G^{r_{i-1}} E \tilde{G}$$

Thus W' and some word congruent to W start with the same letter. By induction, they are congruent. ∎

As a corollary, we obtain a presentation of the monoid of standard morphisms.

COROLLARY 2.3.15. *The only nontrivial identity in the monoid of standard morphisms generated by E and φ is $E^2 = Id$.* ∎

2.3.4. Conjugate morphisms

In this section, we characterize Sturmian morphisms by standard morphisms. The main notion is a special kind of *conjugacy* relation for morphisms.

Let f and g be morphisms. The morphism g is a *right conjugate* of f, in symbols $f \triangleleft g$ if there is a word w such that

$$f(x)w = wg(x), \quad \text{for all words } x \in A^* \tag{2.3.3}$$

This implies that the words $f(x)$ and $g(x)$ are conjugate, and moreover all pairs $(f(x), g(x))$ share the same "sandwich" word w. It suffices, for (2.3.3) to hold, that

$$f(a)w = wg(a), \quad \text{for all letters } a \in A \tag{2.3.4}$$

since by induction $f(xa)w = f(x)f(a)w = f(x)wg(a) = wg(xa)$. Observe that if (2.3.4) holds for a nonempty word w, then all words $f(a)$ for $a \in A$ start with the same letter. Right conjugacy is a preorder over the set of all morphisms over A. Indeed, if $f(x)w = wg(x)$ and $g(x)v = vh(x)$, then $f(x)wv = wg(x)v = wvh(x)$.

EXAMPLE 2.3.16. The morphism $\tilde{\varphi}$ is a right conjugate of φ since $\varphi(0)0 = 010 = 0\tilde{\varphi}(0)$ and $\varphi(1) = \tilde{\varphi}(1) = 0$. Observe that φ is not a right conjugate of $\tilde{\varphi}$ since $\tilde{\varphi}(0)$ and $\tilde{\varphi}(1)$ do not start with the same letter.

This example shows that right conjugacy is not a symmetric relation. However, one has the following formulas.

LEMMA 2.3.17. *Let f, g, f', g' be morphisms.*
(i) *If $f \triangleleft g$ and $f' \triangleleft g$, then $f \triangleleft f'$ or $f' \triangleleft f$,*
(ii) *If $f \triangleleft g$ and $f \triangleleft g'$, then $g \triangleleft g'$ or $g' \triangleleft g$,*
(iii) *If $f \triangleleft g$ and $f' \triangleleft g'$, then $f \circ f' \triangleleft g \circ g'$.*

Proof. We start with the first implication. If $f(x)w = wg(x)$ and $f'(x)v = vg(x)$, then for convenient x, the word $g(x)$ is longer than v and w. Thus w is a suffix of v or vice-versa. Assume $v = zw$. Then $zf(x) = f'(x)z$. The second is symmetric.

For the third, assume $f(x)w = wg(x)$ for all words x. For any morphism h, $h(f(x)w) = h(f(x))h(w) = h(w)h(g(x))$, and consequently $h \circ f \triangleleft h \circ g$. Also $f(h(x))w = wg(h(x))$, showing that $f \circ h \triangleleft g \circ h$. Thus, if $f \triangleleft g$ and $f' \triangleleft g'$, then $f \circ f' \triangleleft g \circ f' \triangleleft g \circ g'$. ■

The next result states that the monoid of Sturm is the closure under right conjugacy of the monoid of standard morphisms.

PROPOSITION 2.3.18. *A morphism is Sturmian if and only if it is a right conjugate of some standard morphism.*

Proof. We show first that a Sturmian morphism is a right conjugate of some standard morphism. Let g be a Sturmian morphism, and consider a decomposition

$$g = h_1 \circ h_2 \circ \cdots \circ h_n$$

with $h_1, \ldots, h_n \in \{E, \varphi, \tilde{\varphi}\}$. If none of the h_i is equal to $\tilde{\varphi}$, then g is standard. Otherwise, consider the smallest i such that $h_i = \tilde{\varphi}$. Then $g = g' \circ \tilde{\varphi} \circ g''$, for $g' = h_1 \circ \cdots \circ h_{i-1}$ and $g'' = h_{i+1} \circ \cdots \circ h_n$. By induction, g'' is a right conjugate of some standard morphism f'', and since $\varphi \triangleleft \tilde{\varphi}$ and by Lemma 2.3.17, $g' \circ \varphi \circ f'' \triangleleft g$, with $g' \circ \varphi \circ f''$ a standard morphism.

Conversely, let f be a standard morphism, and let g be a right conjugate of f. Then there is a word w such that $f(x)w = wg(x)$ for every word x. It follows that, for any infinite word s, one has $f(s) = wg(s)$. If s is a Sturmian word, then $g(s)$ is a Sturmian word, and g is a Sturmian morphism. ■

We start an explicit description of the right conjugates of a standard morphism by the following observation.

PROPOSITION 2.3.19. *Right conjugate standard morphisms are equal.*

Proof. Let f and f' be two standard morphisms, and assume $f \triangleleft f'$. There is a word w such that

$$f(0)w = wf'(0), \ f(1)w = wf'(1) \tag{2.3.5}$$

Set $x = f(0)$, $y = f(1)$, and $x' = f'(0)$, $y' = f'(1)$. Then $|x| = |x'|$ and $|y| = |y'|$. Next, by Proposition 2.3.11, $\{x, y\}$ and $\{x', y'\}$ are unordered standard pairs. If $\{x, y\} = \{0, 1\}$, then $\{x, y\} = \{x', y'\}$ and $f = f'$. Otherwise, the words xy, yx, $x'y'$ and $y'x'$ are standard words with same height and length by (2.3.5), and moreover $xy \neq yx$, $x'y' \neq y'x'$ by Proposition 2.2.2. In view of Proposition 2.2.15, there exist exactly two standard words of this height and length. Thus $xy = x'y'$ or ($xy = y'x'$ and $yx = x'y'$). In the first case, $f = f'$. In the second case, assume $|x| \leq |y|$. Then x is a prefix of y, and the equation $yx = x'y'$ shows that $x = x'$. Thus $f = f'$ in this case also. ■

We now show a way to construct all Sturmian morphisms from standard morphisms.

As in Lothaire (1983) Section 1.3, we use the permutation γ over A^+ defined by $\gamma(ax) = xa$, $a \in A$, $x \in A^*$. Two words x, y are conjugate if and only if $y = \gamma^i(x)$ for some $0 \le i < |x|$.

Let f be a standard morphism. For $0 \le i \le \|f\| - 1$, define a morphism f_i by $f_i(01) = \gamma^i(f(01))$ and $|f_i(0)| = |f(0)|$.

Example 2.3.20. Let f be the morphism defined by $f(0) = 01010$, $f(1) = 01$. The corresponding 7 morphisms are

$$\begin{aligned}
f_0 &: 0 \mapsto 01010\,, & 1 &\mapsto 01\\
f_1 &: 0 \mapsto 10100\,, & 1 &\mapsto 10\\
f_2 &: 0 \mapsto 01001\,, & 1 &\mapsto 01\\
f_3 &: 0 \mapsto 10010\,, & 1 &\mapsto 10\\
f_4 &: 0 \mapsto 00101\,, & 1 &\mapsto 01\\
f_5 &: 0 \mapsto 01010\,, & 1 &\mapsto 10\\
f_6 &: 0 \mapsto 10101\,, & 1 &\mapsto 00
\end{aligned}$$

It is easily checked that all morphisms except f_6 are Sturmian and are right conjugates of f.

Proposition 2.3.21. *Let f be a non trivial standard morphism. The right conjugates of f are the morphisms f_i, for $0 \le i \le \|f\| - 2$.*

This means that the morphism $f_{\|f\|-1}$ is never Sturmian (in the example above, this was f_6).

Proof. Let g be a right conjugate of f. Then $f(01)w = wg(01)$ for some word w, so $g = f_i$ for some i.

For the converse, we show first that $f_i(0)$ and $f_i(1)$ start with the same letter if and only if $0 \le i \le \|f\| - 3$. Indeed, set $x = f(0)$, $y = f(1)$, $x' = f_i(0)$ and $y' = f_i(1)$, and set $n = |x| = |x'|$. The word $x'y'$ is a factor of $xyxy$, thus there exists a non empty word t of length i such that $xyxy$ starts with $tx'y'$. The first letter of x' is the $(i+1)$th letter of xy. The first letter of y' is the $(n+i+1)$th letter of xyx, i.e. the $(i+1)$th letter of yx. Since $\{x, y\}$ is an unordered standard pair, only the last two letters of the words xy and yx are different by Proposition 2.2.2. Consequently the first letter of x' is equal to the first letter of y' if and only if $i + 1 \le \|f\| - 2$.

For any i with $0 \le i \le \|f\| - 3$, set $f_i(0) = au$, $f_i(1) = av$ for a letter a and words u, v. Then $f_{i+1}(0) = ua$, $f_{i+1}(1) = va$. Thus $f_i(0)a = af_{i+1}(0)$, $f_i(1)a = af_{i+1}(1)$, showing that $f_i \triangleleft f_{i+1}$, whence $f \triangleleft f_{i+1}$. ∎

Proposition 2.3.22. *Let g be a Sturmian morphism. There exists a unique standard morphism f such that $f \triangleleft g$. This standard morphism is obtained from any decomposition of g in elements of $\{E, \varphi, \tilde{\varphi}\}$ by replacing all the occurrences of $\tilde{\varphi}$ by φ.*

Proof. Let g be a Sturmian morphism, and let f be obtained from a decomposition of g in elements of $\{E, \varphi, \tilde{\varphi}\}$ by replacing all the occurrences of $\tilde{\varphi}$ by φ. Since f is a composition of E and φ, f is standard. Moreover, since $\varphi \triangleleft \tilde{\varphi}$, one has $f \triangleleft g$ by repeated application of Lemma 2.3.17(iii).

Moreover if there exists a standard morphism f' such that $f' \triangleleft g$ then by Lemma 2.3.17, one has $f' \triangleleft f$ or $f \triangleleft f'$. By Proposition 2.3.19, $f = f'$ which proves that f is unique. ∎

2.3.5. Automorphisms of the free group

Consider two letters $\bar{0}, \bar{1}$ not in $A = \{0,1\}$. The free monoid $A^\bullet = \{0,1,\bar{0},\bar{1}\}^*$ is equipped with an involution by defining $\bar{\bar{a}} = a$ for $a \in A$, and $\overline{uv} = \bar{v}\bar{u}$. The *free group* $F(A)$ over $A = \{0,1\}$ is the quotient of the free monoid $A^\bullet$ under the congruence relation generated by $0\bar{0} \equiv \bar{0}0 \equiv 1\bar{1} \equiv \bar{1}1 \equiv \varepsilon$. A word in $A^\bullet$ without factors of the form $0\bar{0}, \bar{0}0, 1\bar{1}, \bar{1}1$ is *reduced.* Every word in $A^\bullet$ is equivalent to a unique reduced word. If w is reduced, so is $\bar{w}$. The free group can be viewed as the set of reduced words. The product of two elements in $F(A)$ is the reduced word equivalent to the concatenation of the reduced words corresponding to the group elements, and the inverse of an element in $F(A)$ represented by w is $\bar{w}$. An element in $F(A)$ has a *length.* It is the length of its corresponding reduced word.

In this section, we give a characterization of Sturmian morphisms in terms of automorphisms of the free group $F(A)$.

Any morphism f on A is extended in a natural way to an endomorphism on $F(A)$, by defining $f(\bar{0}) = \overline{f(0)}$, $f(\bar{1}) = \overline{f(1)}$. It follows that $f(\bar{w}) = \overline{f(w)}$ for any $w \in F(A)$. Conversely, consider an endomorphism f of $F(A)$. It is called *positive* if the (reduced) words $f(0)$ and $f(1)$ are words over A, that is do not contain any barred letter. An endomorphism f that is a bijection is an *automorphism.* Its inverse is denoted f^{-1}.

The morphisms E, φ and $\tilde{\varphi}$ are extended to $F(A)$ by

$$E : \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0 \\ \bar{0} \mapsto \bar{1} \\ \bar{1} \mapsto \bar{0} \end{array} \qquad \varphi : \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 0 \\ \bar{0} \mapsto \bar{1}\bar{0} \\ \bar{1} \mapsto \bar{0} \end{array} \qquad \tilde{\varphi} : \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 0 \\ \bar{0} \mapsto \bar{0}\bar{1} \\ \bar{1} \mapsto \bar{0} \end{array}$$

They are automorphisms, and their inverses are given by

$$E^{-1} = E \qquad \varphi^{-1} : \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto \bar{1}0 \end{array} \qquad \tilde{\varphi}^{-1} : \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0\bar{1} \end{array}$$

It follows that every Sturmian morphism is a (positive) automorphism of $F(A)$. The converse also holds.

THEOREM 2.3.23. *The positive automorphisms of $F(A)$ are exactly the Sturmian morphisms.*

The theorem states that the three morphisms $E, \varphi, \tilde{\varphi}$ are a set of generators of the monoid of positive automorphisms. The full automorphism group of a free group is a well-known object (see Notes). In particular, sets of generators can be expressed in terms of so-called Nielsen transformations. In the present case, the morphisms

$$\begin{array}{cccc} 0 \mapsto 0 & 0 \mapsto \bar{0} & 0 \mapsto 01 & 0 \mapsto 0 \\ 1 \mapsto \bar{1} & 1 \mapsto 1 & 1 \mapsto 1 & 1 \mapsto 10 \end{array}$$

generate the automorphism group of $F(A)$. The two last morphisms are $E \circ \tilde{\varphi}$ and $\tilde{\varphi} \circ E$.

We first prove a special case of the theorem.

PROPOSITION 2.3.24. *Let f be a positive automorphism of $F(A)$. If the words $f(0)$ and $f(1)$ do not end with the same letter, then f is a standard Sturmian morphism.*

Proof. Let f be a positive automorphism of $F(A)$. We may assume $|f(0)| \leq |f(1)|$. We suppose first that $f(0)$ is not a prefix of $f(1)$. There exist words u, v_0, v_1 over A such that v_0 and v_1 start with different letters and $f(0) = uv_0$ and $f(1) = uv_1$. Since $f(0)$ and $f(1)$ do not end with the same letter, the words v_0 and v_1 also end with different letters. The images of reduced words of length 2 under f are $uv_a uv_b$, $uv_a \bar{v}_b \bar{u}$, $\bar{v}_a v_b$, $\bar{v}_a \bar{u} \bar{v}_b \bar{u}$. Each of these words is reduced because v_0 and v_1 start and end with different letters. It follows that for any reduced word w of length at least 2, the reduced word $f(w)$ has length at least 2. Consider now any letter $a \in A$. Since $|f(f^{-1}(a))| = 1$, it follows that $|f^{-1}(a)| = 1$, that is f is either the identity or E. Thus f is Sturmian.

Next, if $f(0)$ is a prefix of $f(1)$, there exists a word u such that $f(1) = f(0)u$. Define a morphism g by $g(0) = f(0)$ and $g(1) = u$. Then $f = g \circ \varphi \circ E$. Since f is a bijection, g is also a bijection. By induction on $\|g\|$, the morphism g is a standard Sturmian morphism, and so is f. ∎

Proof of Theorem 2.3.23. Let g be a positive automorphism. The words $g(01)$ and $g(10)$ are different because g is a bijection. They have same length. Let u be their longest common suffix. There exist words v_0, v_1 over A of same length such that $g(01) = v_0 u$, $g(10) = v_1 u$ and v_0, v_1 do not end with the same letter. Since for letters $a \neq b$, $g(aba) = v_a u g(a) = g(a) v_b u$, the words $ug(a)$ end with u. Define a morphism f by $f(a) = ug(a)\bar{u}$ for $a \in \{0, 1\}$. Then $f(w) = ug(w)\bar{u}$ for all w in $F(A)$. Since $ug(a)$ ends with u for $a \in \{0, 1\}$, the morphism f is positive.

Since g is a bijection, f is also a bijection. Moreover $f(01) = uv_0$ and $f(10) = uv_1$ end with different letters and since f is positive, also $f(0)$ and $f(1)$ end with different letters. By Proposition 2.3.24, f is a standard Sturmian morphism. Now $f(0)u = ug(0)$ and $f(1)u = ug(1)$ which means that g is a right conjugate of f. Consequently, by Proposition 2.3.18, g is a Sturmian morphism. ∎

2.3.6. Fixpoints

In this section, we make use of Theorem 2.3.12 to describe those characteristic words that are fixpoints of standard morphisms. As an example, we know from Chapter 1 that the morphism φ fixes the infinite Fibonacci word f.

We say that a morphism h *fixes* an infinite word x if $h(x) = x$. In this case, x is a *fixpoint* of h. Every infinite word is fixed by the identity, and no infinite word is fixed by E.

For the description of characteristic words which are fixpoints of morphisms, we introduce a special set of irrational numbers. A *Sturm number* is a number α that has a continued fraction expansion of one of the following kinds:

(i) $\alpha = [0, 1, a_0, \overline{a_1, \ldots, a_k}]$, with $a_k \geq a_0$,
(ii) $\alpha = [0, 1 + a_0, \overline{a_1, \ldots, a_k}]$, with $a_k \geq a_0 \geq 1$.

Observer that (i) implies $\alpha > 1/2$, and (ii) implies $\alpha < 1/2$. More precisely, α has an expansion of type (i) if and only if $1 - \alpha$ has an expansion of type (ii). Consequently, α is a Sturm number if and only $1 - \alpha$ is a Sturm number.

As an example, $1/\tau = [0, \overline{1}]$ is covered by the first case (for $k = 1$ and $a_k = a_0 = 1$), and $1/\tau^2 = [0, 2, \overline{1}]$ is covered by the second case.

We shall give later (Theorem 2.3.26) a simple algebraic description of Sturm numbers. There is also a simple combinatoric characterization of these numbers (Problem 2.3.4).

Theorem 2.3.25. *Let $0 < \alpha < 1$ be an irrational number. The characteristic word c_α is a fixpoint of some non trivial morphism if and only if α is a Sturm number.*

Proof. Let

$$\alpha = [0, m_1, m_2, \ldots]$$

be the continued fraction expansion of α, and suppose that $f(c_\alpha) = c_\alpha$ for some morphism f. In view of Theorem 2.3.12, the morphism f is standard. Thus, f is a product of E and G, and is not a power of E. Also, f is not a proper power of G, because a morphism G^n with $n \geq 1$ fixes only the infinite word 0^ω. Thus (we write composition as concatenation), f has the form

$$f = G^{n_1} E G^{n_2} \cdots E G^{n_k} E G^{n_{k+1}}$$

for some $k \geq 1$, $n_1, n_{k+1} \geq 0$, and $n_2, \ldots, n_k \geq 1$. We use the morphisms $\theta_m = G^{m-1} E G$ for $m \geq 1$ and the fact (Corollary 2.2.21) that

$$\theta_m(c_\alpha) = c_{1/(m+\alpha)} \, .$$

There are three cases.
(a) Suppose first that $n_{k+1} > 0$. Then

$$f = \theta_{n_1+1} \theta_{n_2} \cdots \theta_{n_k} G^{n_{k+1}-1}$$

Since f fixes c_α, this implies

$$[0, m_1, m_2, \ldots] = [0, 1 + n_1, n_2, \ldots, n_k, n_{k+1} - 1 + m_1, m_2, \ldots]$$

which in turn gives $m_1 = 1+n_1, m_2 = n_2, \ldots, m_k = n_k, m_{k+1} = n_{k+1}-1+m_1 = n_{k+1} + n_1$, and $m_j = m_{j+k}$ for $j \geq 2$. Thus

$$\alpha = [0, 1+n_1, \overline{n_2, \ldots, n_{k+1} + n_1}], \quad \text{with } n_1 \geq 0, n_2, \ldots, n_{k+1} \geq 1 \qquad (2.3.6)$$

(b) Suppose now that $n_{k+1} = 0$, and consider the morphism $f' = EfE$. From $c_\alpha = f(c_\alpha)$, it follows that $f'(Ec_\alpha) = Ec_\alpha$, that is $f'(c_\beta) = c_\beta$ for $\beta = 1 - \alpha$. Now

$$f' = EG^{n_1}EG^{n_2}\cdots EG^{n_k}$$

where $n_1 \geq 0$ and $n_2, \ldots, n_k \geq 1$. There are two sub-cases.
(b.1) If $n_1 = 0$, then $k \geq 3$ and

$$f' = G^{n_2}\cdots EG^{n_k} = \theta_{n_2+1}\cdots\theta_{n_{k-1}}G^{n_k-1}$$

whence, as above, $\beta = [0, 1+n_2, \overline{n_3, \ldots, n_{k-1}, n_2 + n_k}]$ and since $n_2 \geq 1$,

$$\alpha = 1 - \beta = [0, 1, n_2, \overline{n_3, \ldots, n_{k-1}, n_2 + n_k}] \quad \text{with } n_2, \ldots, n_k \geq 1 \qquad (2.3.7)$$

(b.2) If $n_1 \geq 1$, then

$$f' = EG^{n_1}\cdots EG^{n_k} = \theta_1\theta_{n_1}\cdots\theta_{n_{k-1}}G^{n_k-1}$$

whence as above $\beta = [0, 1, \overline{n_1, \ldots, n_{k-1}, n_k}]$ and

$$\alpha = 1 - \beta = [0, 1+n_1, \overline{n_2, n_3, \ldots, n_k, n_1}] \quad \text{with } n_1, \ldots, n_k \geq 1 \qquad (2.3.8)$$

To show that Equations (2.3.6)–(2.3.8) describe exactly Sturm numbers, observe that Equation (2.3.6) with $n_1 = 0$ corresponds, in the definition of Sturm numbers, to case (i) with $a_k = a_0$, that Equation (2.3.6) with $n_1 > 0$ corresponds to case (ii) with $a_k > a_0$, that Equation (2.3.7) is equivalent to case (i) with $a_k > a_0$ and that Equation (2.3.8) is case (ii) with $a_k = a_0$.

The proof that a Sturm number indeed yields a fixpoint is exactly the reverse of the previous one. ■

Sturm numbers have a simple algebraic description. Clearly, a Sturm number α is quadratic irrational, that is solution of some equation

$$x^2 + px + q = 0$$

with rational coefficients p, q. The other solution of this equation is the *conjugate* of α, denoted by $\bar{\alpha}$, and satisfies $\alpha\bar{\alpha} = q$. It is easy to prove that the conjugate of $1 - \alpha$ is $1 - \bar{\alpha}$, and that the conjugate of $1/\alpha$ is $1/\bar{\alpha}$.

THEOREM 2.3.26. *A quadratic irrational α with $0 < \alpha < 1$ is a Sturm number if and only if $1/\bar{\alpha} < 1$.*

We need some facts from number theory. A quadratic irrational number γ is said to be *reduced* if $\gamma > 1$ and $-1 < \bar{\gamma} < 0$. This is equivalent to $1 > 1/\gamma > 0$ and $1/\bar{\gamma} < -1$. It is known that

1. the continued fraction of a quadratic irrational γ is purely periodic if and only if γ is reduced.
2. if γ is reduced and $\gamma = [\overline{a_1, \ldots, a_n}]$, then $-1/\bar{\gamma} = [\overline{a_n, \ldots, a_1}]$.

Proof of Theorem 2.3.26. The condition $1/\bar{\alpha} < 1$ is equivalent to $\bar{\alpha} \notin [0,1]$. This in turn is equivalent to $1 - \bar{\alpha} \notin [0,1]$. Thus $\bar{\alpha}$ verifies the condition if and only if $1 - \bar{\alpha}$ does. Consequently, it suffices to prove the equivalence for $0 < \alpha < 1/2$. We have to prove that $1/\bar{\alpha} < 1$ if and only if

$$\alpha = [0, 1 + a_0, \overline{a_1, \ldots, a_k}], \quad \text{with} \quad a_k \geq a_0 \geq 1\,.$$

Let first α be a Sturm number with $0 < \alpha < 1/2$. Then

$$\alpha = \cfrac{1}{1 + a_0 + \cfrac{1}{\gamma}}, \quad \text{with} \quad \gamma = [\overline{a_1, \ldots, a_k}], \quad a_k \geq a_0 \geq 1 \tag{2.3.9}$$

Thus γ is reduced, and since $-1/\bar{\gamma} = [\overline{a_k, \ldots, a_1}] > a_k$, it follows from (2.3.9) that

$$1/\bar{\alpha} = 1 + a_0 + 1/\bar{\gamma} < 1 + a_0 - a_k \leq 1\,.$$

Conversely, let $0 < \alpha < 1/2$ be a quadratic irrational with $1/\bar{\alpha} < 1$. Since $2 < 1/\alpha$, write

$$1/\alpha = 1 + a_0 + 1/\gamma \tag{2.3.10}$$

where $a_0 = \lfloor 1/\alpha - 1 \rfloor \geq 1$ and $1 < 1/\gamma < 1$. From $1/\bar{\alpha} < 1$ and the conjugate of (2.3.10), one gets

$$1/\bar{\gamma} < -a_0 \leq -1$$

Thus γ is reduced, and writing $\gamma = [\overline{a_1, \ldots, a_k}]$, one gets

$$a_0 < -1/\bar{\gamma} = [\overline{a_k, \ldots, a_1}] < a_k + 1$$

whence $a_k \geq a_0 \geq 1$ and

$$\alpha = \cfrac{1}{1 + a_0 + \cfrac{1}{\gamma}} = [0, 1 + a_0, \overline{a_1, \ldots, a_k}]\,. \qquad \blacksquare$$

Problems

Section 2.1

2.1.1 We consider two-sided infinite words over $\{0,1\}$ of complexity $n+1$.
1. Show that the word x defined by $x(k) = 1$ for $k \geq 0$, and $x(k) = 0$ for $k < 0$ has $n+1$ factors of length n for each $n \geq 0$.
2. Let $z \notin 0^* \cup 1^*$ be a central word with period k and ℓ, and set $w = p10q$ where p and q are palindrome words with $k = |p|$, $\ell = |q|$. Define two (onesided) infinite words $x = (10q)^\omega$ and $y = (01p)^\omega$. Then the two-sided infinite word $\tilde{y}zx$ has $n+1$ factors of length n for each $n \geq 1$. (These are the only two-sided infinite words with complexity $n+1$, see Coven and Hedlund 1973.)

2.1.2 Let x be an infinite word which contains infinitely many occurrences of 0 and of 1. The *cell*-condition for x is the following: for any words w, w' such that $|w|_0 = |w'|_0$ and $0w0, 0w'0 \in F(x)$, one has $\big||w| - |w'|\big| \leq 1$, and the same condition with 0 and 1 exchanged. Show that x is balanced if and only if x satisfies the cell-condition. (Morse and Hedlund 1940. A proof consists in considering the word y such that $x = G(y)$.)

2.1.3 Let x be an infinite word. For $n \geq 1$, let X_n be the set of factors of x starting with 0, ending with 0, and containing exactly n occurrences of the letter 0. Define similarly Y_n, replacing 0 by 1. Show that x is Sturmian if and only if $\mathrm{Card}(X_n) = \mathrm{Card}(Y_n) = n$ for every n (Richomme 1999a).

2.1.4 Show that a word w is unbalanced if and only if it admits a factorization $w = xauayb\tilde{u}bz$ for words u, x, y, z and letters $a \neq b$. Use this characterization to prove that the set of unbalanced words is a context-free language. (Dulucq and Gouyou-Beauchamps 1990, see also Mignosi 1991, 1990)

Section 2.2

2.2.1 Show that for any standard word $w \neq 0, 1$, there is only one standard pair (x, y) such that $w = xy$ or $w = yx$.

2.2.2 Define sequences of words $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ by

$$A_0 = a, \quad B_0 = b$$

and

$$R_1 : \begin{array}{l} A_{n+1} = A_n \\ B_{n+1} = A_n B_n \end{array} \quad \text{and} \quad R_2 : \begin{array}{l} A_{n+1} = B_n A_n \\ B_{n+1} = B_n \end{array}$$

The R_i's are called *Rauzy's rules* (see Rauzy 1985).

1. Show that, provided each of the rules R_i is applied infinitely many often, the sequences A_n and B_n converge to the same infinite word which is characteristic.

2. Show that conversely every characteristic word is obtained in this way.

2.2.3 Let $0 \leq h \leq m$ be integers with $(h, m) = 1$. The lower and upper *Christoffel words* $t_{h,m}$ and $t'_{h,m}$ are defined by $t_{0,1} = t'_{0,1} = 0$, $t_{1,1} = t'_{1,1} = 1$, and $t_{h,m} = 0z_{h,m}1$, $t'_{h,m} = 1z_{h,m}0$ if $m \geq 2$. These are exactly the words defined in Section 2.1.2.

1. Show that if $h'm - m'h = 1$, then

$$t_{h,m}t_{h',m'} = t_{h+h',m+m'}, t'_{h',m'}t'_{h,m} = t'_{h+h',m+m'}$$

2. For $1 \leq h < m$ and $(h, m) = 1$, show that there exist integers m', h' with $0 \leq h' \leq m' < m$, $h' < h$ such that $m'h - h'm = 1$, and

$$t_{h,m} = t_{h',m'}t_{h-h',m-m'}$$

3. Define $\sigma_{h,m} = z_{h,m}10$, $\sigma'_{h,m} = z_{h,m}01$. Show that

$$\sigma'_{h,m}\sigma_{h',m'} = \sigma_{h+h',m+m'}, \sigma_{h,m}\sigma'_{h',m'} = \sigma'_{h+h',m+m'}.$$

Show that the pairs of standard words are $(0,1)$ and all the pairs $(\sigma_{h,m}, \sigma'_{h,m})$, for $h'm - hm' = 1$.

2.2.4 Consider a function Δ' from $\{0,1\}^*$ into itself defined by $\Delta'(u,v) = (uv, v)$. The family of *Christoffel pairs* is the smallest set of pairs of words containing $(0,1)$ and closed under Γ and Δ'. A standard pair and a Christoffel pair are *corresponding* if they are obtained by the same sequence of Γ and Δ (resp. Γ and Δ').
1. Let (u,v) be a standard pair and let (u',v') be the corresponding Christoffel pair. Show that if $u = p10$, then $u' = 0p1$ and if $v = q01$, the $v' = 0q1$.
2. Show that the components of Christoffel pairs are exactly the lower Christoffel words. (see Borel and Laubie 1993.)

2.2.5 Christoffel words and Lyndon words.
1. Show that every lower Christoffel word is a Lyndon word.
2. Show that a balanced word is a Lyndon word if and only if it is a Christoffel word (Berstel and De Luca 1997).
3. Any lower Christoffel word w which is not a letter admits a unique factorization $w = xy$, where (x,y) is a Christoffel pair. Show that this factorization is the standard Lyndon factorization (Borel and Laubie 1993).

2.2.6 Show that, for $0 \le \rho < 1$,

$$\tilde{\varphi}(s_{\alpha,\rho}) = s'_{\frac{1-\alpha}{2-\alpha},\frac{2-\alpha-\rho}{2-\alpha}},\ D(s_{\alpha,\rho}) = s_{\frac{1}{2-\alpha},\frac{1-\alpha+\rho}{2-\alpha}},\ \tilde{D}(s_{\alpha,\rho}) = s_{\frac{1}{2-\alpha},\frac{\rho}{2-\alpha}}.$$

Show that for $0 < \rho \le 1$,

$$\tilde{\varphi}(s'_{\alpha,\rho}) = s_{\frac{1-\alpha}{2-\alpha},\frac{2-\alpha-\rho}{2-\alpha}},\ D(s'_{\alpha,\rho}) = s'_{\frac{1}{2-\alpha},\frac{1-\alpha+\rho}{2-\alpha}},\ \tilde{D}(s'_{\alpha,\rho}) = s'_{\frac{1}{2-\alpha},\frac{\rho}{2-\alpha}}.$$

(see Parvaix 1997)

2.2.7 The aim of this problem is to prove that if w is a word such that $w0$ and $w1$ are balanced, then w is a right special factor of some Sturmian word.
Let w be a word such that $w0$ and $w1$ are balanced.
1. Show that if w is a palindrome, then w is central.
2. Show that if $w = uap$, with a a letter and p a palindrome, then pa is a prefix of some characteristic word.
3. Show that w is always a suffix of a central word.
4. Show that w is a right special factor of some Sturmian word.
(see De Luca 1997c)

2.2.8 Let $\alpha = [0, 1+d_1, d_2, \ldots]$ be the continued fraction expansion of the irrational α, let (s_n) be the associated standard sequence, and define $(t_n)_{n\geq -1}$ by

$$t_{-1} = 1, \quad t_0 = 0, \quad t_n = t_{n-1}^{d_n - 1} t_{n-2} t_{n-1}, \ (n \geq 1)\,.$$

1. Show that $t_0 t_1 \cdots t_n = s_n \cdots s_1 s_0$.
2. Show the follow product formula: $c_\alpha = t_0 t_1 \cdots t_n \cdots$.(Brown 1993)

2.2.9 Let $\alpha = [0, 1+d_1, d_2, \ldots]$ be the continued fraction expansion of the irrational α, let (s_n) be the associated standard sequence. Let w be a standard word that is a prefix of the characteristic word c_α. Show that there is an integer n such that $w = s_n^k s_{n-1}$ for some $1 \leq k \leq d_{n+1}$.

2.2.10 Let $\alpha = [0, 1+d_1, d_2, \ldots]$ be the continued fraction expansion of the irrational α, let (s_n) be the associated standard sequence. Define three sequences of words by $(u_n)_{n\geq -1}$, $(v_n)_{n\geq -1}$ and $(w_n)_{n\geq -1}$

$$u_{-1} = v_{-1} = w_{-1} = 1, \quad u_0 = v_0 = w_0 = 0$$

and

$$\begin{aligned} u_{2n} &= u_{2n-2}(u_{2n-1})^{d_{2n}} \quad (n \geq 1) \\ u_{2n+1} &= (u_{2n})^{d_{2n+1}} u_{2n-1} \quad (n \geq 0) \\ v_{2n} &= (v_{2n-1})^{d_{2n}} v_{2n-2} \quad (n \geq 1) \\ v_{2n+1} &= v_{2n-1}(v_{2n})^{d_{2n+1}} \quad (n \geq 0) \\ w_n &= w_{n-2}(w_{n-1})^{d_n} \quad (n \geq 1) \end{aligned}$$

1. Show that

$$\begin{aligned} 0c_\alpha &= \lim_{n\to\infty} u_n, & 1c_\alpha &= \lim_{n\to\infty} v_n \\ 01c_\alpha &= \lim_{n\to\infty} w_{2n} & 10c_\alpha &= \lim_{n\to\infty} w_{2n+1}\,. \end{aligned}$$

2. Define a sequence $(p_n)_{n\geq -1}$ by $p_{-1} = 0^{-1}$, $p_0 = 1^{-1}$ and

$$\begin{aligned} p_{2n} &= p_{2n-2}(10\pi_{2n-1})^{d_{2n}} \quad & n \geq 1 \\ p_{2n+1} &= (p_{2n}10)^{d_{2n+1}} p_{2n-1} \quad & n \geq 0 \end{aligned}$$

Show that the words p_n, for $n \geq 1$ are palindromes, and

$$\begin{aligned} s_{2n} &= p_{2n}10, & u_n &= 0p_n1, & w_{2n} &= 01p_{2n}, \\ s_{2n+1} &= p_{2n+1}01, & v_n &= 1p_n0, & w_{2n+1} &= 10p_{2n+1}. \end{aligned}$$

2.2.11 A number system associated with a directive sequence.
Let $\alpha = [0, 1+d_1, d_2, \ldots]$ be the continued fraction of the irrational α, and (s_n) be the associated standard sequence. Define integers by

$$q_{-1} = 1, \quad q_0 = 1, \qquad q_n = d_n q_{n-1} + q_{n-2}, \qquad (n \geq 1)\,.$$

Then of course $|s_n| = q_n$.

1. Show that any integer $m \geq 0$ can be written in the form

$$m = z_h q_h + \cdots + z_0 q_0, \qquad (0 \leq z_i \leq d_{i+1}) \tag{2.4.1}$$

2. Show that every integer $0 \leq m \leq q_{h+1} - 1$ admits a unique such representation provided

$$z_i = d_{i+1} \implies z_{i-1} = 0 \qquad (1 \leq i \leq h)$$

3. Show that if $m = z_h q_h + \cdots + z_0 q_0$ is as in eq. (2.4.1), then the prefix of c_α of length m has the form $s_h^{z_h} \cdots s_0^{z_0}$ (see Fraenkel 1985, 1982, Brown 1993 and the references cited there).

2.2.12 A *Beatty* sequence is a set $B = \{\lfloor rn \rfloor \mid n \geq 1\}$ for some irrational number $r > 1$ (it is a spectrum).
1. Let $\alpha = 1/r$, and let $c_\alpha = a_1 a_2 \cdots$ be the characteristic word of slope α. Show that $B = \{k \mid a_k = 1\}$.
2. Two Beatty sequences B and B' are *complementary* if B and B' form a partition of $\{1, 2, \ldots\}$. Show that the sets $\{\lfloor rn \rfloor \mid n \geq 1\}$ and $\{\lfloor r'n \rfloor \mid n \geq 1\}$ are complementary if and only if $1/r + 1/r' = 1$. (Use 1., see Beatty 1926)

2.2.13 Write $x < y$ if x is lexicographically less that y. Show that for any irrational characteristic word c, the word $0c$ is lexicographically smaller than all its proper suffixes, and $1c$ is lexicographically greater than all its proper suffixes. (Borel and Laubie 1993)

2.2.14 Define a mapping $C : \{0,1\}^* \to \{0,1\}^*$ by $C(\varepsilon) = \varepsilon$ and $C(ax) = xa$ for $a \in \{0,1\}$. This is just a cyclic permutation. Let $\alpha = [0, 1 + d_1, d_2, \ldots]$ be the continued fraction of the irrational α, and (s_n) be the associated standard sequence.
1. Show that for $n \geq 0$, the words $C^{-1}(s_{2n})$ and $C^{|s_{2n}-1|}(s_{2n+1})$ are Lyndon words. (Borel and Laubie 1993, Melançon 1996)
2. Set $\ell_n = C^{|s_{2n}-1|}(s_{2n+1})$. Show that $c_\alpha = \ell_0^{d_2} \ell_1^{d_4} \cdots \ell_n^{d_{2n+2}} \cdots$ and that the sequence ℓ_n is a lexicographically strictly decreasing sequence.

2.2.15 Let $\alpha = [0, 1 + d_1, d_2, \ldots]$ be the continued fraction of the irrational α, and (s_n) be the associated standard sequence.
1. Show that s_n^2 is a factor of c_α for every $n \geq 1$.
Since s_n is primitive, every factor of c_α of length $|s_n|$ excepted one is a conjugate of s_n. This is the *singular word*, denoted w_n. For the Fibonacci word, the singular words are 00, 101, 00100, 10100101, ...
2. Let p_n be the palindrome prefix of s_n of length $|s_n| - 2$. Show that $w_n = a_n p_n a_n$, where $a_n = 0$ if n is odd, and $a_n = 1$ if n is even.
3. Show that the Fibonacci word is the product of 01 and its singular words: $f = 01(00)(101)(00100)\cdots$. (see Wen and Wen 1994b)

2.2.16 To compute all conjugates of s_n, define sequences $(w_h)_{0 \leq h \leq n}$ of words parameterized by sequences of integers $z_0, \ldots, z_{n-1}$ with $0 \leq z_h \leq d_{h+1}$ by $w_{-1} = 1$, $w_0 = 0$ and $w_{h+1} = w_h^{d_{h+1} - z_h} w_{h-1} w_h^{z_h} \qquad 0 \leq h < n$.
1. Show that $w_n = C^k(s_n)$, where $k = \sum_{h=0}^{n-1} q_h z_h$.
2. Show that one gets all conjugates exactly once. (see Chuan 1997)

2.2.17 Sturmian words and palindromes.
1. Let s be a Sturmian word. Show that $F(s)$ contains exactly one palindrome word of even length, and two palindrome words of odd length for each nonnegative integer.
2. Show that conversely, if $F(s)$ contains exactly one palindrome word of even length, and two palindrome words of odd length for each nonnegative integer, then s is Sturmian (Droubay and Pirillo 1999).

2.2.18 Sturmian words and decimation.
Let $1 \leq k \leq m$ be integers with $m \geq 2$. Let x be an infinite word with infinitely many 0's and 1's. The transformation $M_{k,m}$ deletes in x every 0 excepted those occurring at position congruent to k modulo m. The transformation $D_{k,m}$ operates in the same way on 1's. For example, $M_{3,4}$, applies to

$$010\mathit{0}101001\mathit{0}01010\mathit{0}101001\mathit{0}01010\mathit{0}1001\cdots$$

keeps only the italicized letter 0, and gives the word

$$101110110111011011\cdots$$

1. Give a geometric argument (by cutting sequences) showing that $M_{k,m}(s)$ and $D_{k,m}(s)$ are Sturmian for Sturmian words.
2. Give explicit formulas for $M_{k,m}(s_{\alpha,\rho})$ and $D_{k,m}(s_{\alpha,\rho})$ similar to those of Problem 2.2.6.
3. Show that $M_{m,m} \circ D_{m,m}(c) = c$ for every characteristic word c.
4. Show that conversely, if $M_{m,m} \circ D_{m,m}(s) = s$ for every m, then the infinite word s is balanced. (Justin and Pirillo 1997, the explicit formulas are in Parvaix 1998)

Section 2.3

2.3.1 For integers $m \geq 1, r \geq 1$, set

$$\begin{aligned} w_{m,r} &= 0^{m-1}1(0^{m+1}1)^{r+1}0^m1(0^{m+1}1)^r0^m1 \\ w'_{m,r} &= 0^m1(0^m1)^{r+1}0^{m+1}1(0^m1)^r0^{m+1}1 \end{aligned}$$

In particular, $w_{1,1} = 10^210^21010^2101$ is a word of length 14. Any Sturmian word contains one and only one word from the set

$$\Omega = \{w_{m,r}, w'_{m,r}, E(w_{m,r}), E(w'_{m,r}) \mid m \geq 1, r \geq 1\}\,.$$

1. Prove that a morphism f is Sturmian if and only if f is acyclic and there exists a word $w \in \Omega$ such that $f(w)$ is a balanced word (in particular, an acyclic morphism f is Sturmian if and only if $f(w_{1,1})$ is a balanced word) (Berstel and Séébold 1994a).
2. Prove that no word of length less or equal to 13 has the above property. (Richomme 1999b)

2.3.2 Let C be the set of morphic Sturmian characteristic words. Prove that, for any $c \in C$, the words $0c, 1c, 01c$ and $10c$ are morphic (Berstel and Séébold 1994a).

2.3.3 Prove that a morphism f is standard if and only if $f(0)$, $f(1)$ and $f(01)$ are standard words (De Luca 1997b).

2.3.4 Let $\alpha = [0, 1 + d_1, d_2, \ldots]$ be the continued fraction of an irrational number α. Define an infinite word δ_α over $\{0,1\}$ by

$$\delta_\alpha = 0^{d_1} 1^{d_2} 0^{d_3} 1^{d_4} \cdots$$

Show that α is a Sturm number if and only if δ_α is purely periodic (Droubay, Justin, and Pirillo 2001).

Notes

The history of Sturmian words goes back to the astronomer J. Bernoulli III (Bernoulli 1772). The book of Venkov (1970) describes early work by Christoffel (1875) and Markoff (1882). The first in depth study is by Morse and Hedlund (1940). They also introduce the term "Sturmian", more precisely Sturmian trajectories, named after the mathematician Charles François Sturm (1803–1855), born in Geneva, and who taught at the École Polytechnique in Paris since 1840. He is famous for his rule to compute the roots of an algebraic equation. As described by Hedlund and Morse, Sturmian words are obtained in considering the zeroes of solutions $u(x)$ of linear homogeneous second order differential equations

$$y'' + \phi(x)y = 0\,,$$

where $\phi(x)$ is continuous of period 1. If k_n is the number of zeros of u in the interval $[n, n+1[$, then the infinite word $01^{k_0}0^{k_1}0^{k_2}\cdots$ is Sturmian (or eventually periodic). The papers by Coven and Hedlund (1973) and Coven (1974) contain many combinatorial properties (in particular the description of two-sided infinite words of minimal complexity), and the paper by Stolarsky (1976) shows the relation with continued fractions, fixpoints, and Beatty sequences. The last twenty years have seen large developments, from the point of view of arithmetics, dynamical systems and combinatorics on words. Surveys are by T. C. Brown (1993), Berstel (1996), Ziccardi (1995), partly De Luca (1997a) and for finite factors of Sturmian words Bender, Patashnik, and Rumsey (1994). Sturmian words are known under many other names. Each reflects the emphasis on a particular property. Thus, they are called two-distance sequences (see e.g. Lunnon and Pleasants 1992), Beatty sequences (de Bruijn 1989, 1981), characteristic sequences (Christoffel 1875), spectra (Boshernitzan and Fraenkel 1981, 1984, the *spectrum* of a number α is the multiset $\{\lfloor n\alpha \rfloor \mid n \geq 1\}$ in the book Graham, Knuth, and Patashnik 1989), digitized straight lines, cutting sequences and even musical sequence in a special case (Series 1985).

Sturmian words are of lowest possible complexity. For an overview on complexity of infinite words, see Allouche (1994). Two-sided infinite words of complexity $P(n) = n + 1$ include strictly mechanical words (Problem 2.1.1, Coven and Hedlund 1973). There is a large literature on infinite words with slightly more than minimal complexity (Coven 1974, Alessandri 1996, Cassaigne 1996, Ferenczi 1995, Rote 1994, Hubert 1995, 1996, Rauzy 1988). An extension to 3 letters has been initiated by Arnoux and Rauzy (1991), Arnoux, Mauduit, Shiokawa, and Tamura (1994), Castelli, Mignosi, and Restivo (1999) (the last paper relates Arnoux-Rauzy words to central words over 3 letters). Several properties have been extended to larger alphabets by Droubay et al. (2001). The property of balance and Theorem 2.1.5 are due to Morse and Hedlund (1940), our exposition benefits from Coven and Hedlund (1973). In particular, Proposition 2.1.3 is there. Theorem 2.1.13 is also from Morse and Hedlund (1940). The argument of the proof of Lemma 2.1.15 is from Tijdeman (1996). Christoffel words were investigated in Christoffel (1875). A systematic geometric study is in Borel and Laubie (1991, 1993). Several propositions of Section 2.1.3 Propositions 2.1.18, 2.1.19, 2.1.23 are from Mignosi (1989). He uses rotations (in a slightly different setting).

Mechanical words are also known as digitized straight lines. They have been considered for a long time in pattern recognition, where the problem is to compute the slope and the intercept of a finite Sturmian word as fast as possible, to test whether a word is a finite Sturmian word and, if not, to get the polygonal decomposition (see Bruckstein 1991, Dorst and Smeulders 1991 and the literature quoted there, also Berstel and Pocchiola 1996). Words generated by rotations are in fact more general than Sturmian words when the partition of $[0,1[$ is defined independently from the angle of rotation (see Alessandri 1996, Gambaudo, Lanford, and Tresser 1984, Iwanik 1994, Rauzy 1988, Sidorov and Vershik 1993). Interval exchange is even more general, because the exchange functions are piecewise rotations (see e.g. Rauzy 1979, Didier 1997).

Standard pairs were introduced in a slightly different form in Rauzy (1985). His construction is known as *Rauzy's rules* (see also Problem 2.2.2).

Theorem 2.2.4 and its corollaries are from De Luca and Mignosi (1994). Theorem 2.2.11 is from De Luca and Mignosi (1994). It appears in a similar form in Coven and Hedlund (1973), see also Pedersen 1988.

Lemmas 2.2.17 and 2.2.18 are from Parvaix (1997). Proposition 2.2.24 has been proved by Fraenkel, Mushkin, and Tassa (1978), see also Brown (1993). Theorem 2.2.31 is from Mignosi (1991), although the present proof is different. The proof of Theorem 2.2.36 given here is from De Luca and Mignosi (1994). There are several other proofs, in Mignosi (1991), Berstel and Pocchiola (1993). The formula also appeared in Koplowitz, Lindenbaum, and Bruckstein (1990).

The proof of Theorem 2.2.37 by the factor graphs is from Berthé (1996). The result is also known as the *three distance theorem*. There is a large literature on this subject (see Berthé 1996 and the survey paper Alessandri and Berthé 1998).

Sturmian morphisms were investigated in Séébold (1991). The equivalence (i) and (ii) of Theorem 2.3.7 is due to Mignosi and Séébold (1993), the third is adapted from Berstel and Séébold (1994a). Proposition 2.3.11 is from Berstel and Séébold (1994b). Theorem 2.3.12 appears in De Luca (1997c). The results of Section 2.3.4 are from Séébold (1998). The relation to automorphisms of free groups is from Wen and Wen 1994a. The proof given here is simpler than the original one. For results on free groups and their automorphisms, see e.g. Magnus, Karrass, and Solitar 1966 or Lyndon and Schupp 1977. Theorem 2.3.25 is from Crisp, Moran, Pollington, and Shiue (1993). Several weak versions of this theorem were known earlier (see Brown 1993 for a discussion). Our proof is adapted from Berstel and Séébold (1994a). A self-contained proof exists by Komatsu and van der Poorten (1996). The characterization of Sturm numbers is from Allauzen (1998). Several generalizations to non characteristic Sturmian words were proposed (see e.g. Komatsu 1996, Arnoux, Ferenczi, and Hubert 2000).