CHAPTER 3

Unavoidable patterns

3.0. Introduction

In Chapter 1, avoidable and unavoidable sets of words have been defined. The focus was then on the case of finite sets of words. In the present chapter, we turn to particular infinite sets of words, defined as *pattern languages*. A *pattern* is a word that contains special symbols called *variables*, and the associated pattern language is obtained by replacing the variables with arbitrary non-empty words, with the condition that two occurrences of the same variable have to be replaced with the same word.

The archetype of a pattern is the square, $\alpha\alpha$. The associated pattern language is $L = \{uu \mid u \in A^+\}$, and it is now a classical result that L is an avoidable set of words if A has at least three elements, whereas it is an unavoidable set of words if A has only one or two elements. Indeed, an infinite square-free word on three letters can be constructed, and it is easy to check that every binary word of length 4 contains a square. For short, we will say that the pattern $\alpha\alpha$ is 3-avoidable and 2-unavoidable.

General patterns can contain more than just one variable. For instance, $\alpha\beta\alpha$ represents words of the form uvu, with $u, v \in A^+$ (this pattern is unavoidable whatever the size of the alphabet, see Proposition 3.1.2). They could also be allowed to contain constant letters, that unlike variables are never replaced with arbitrary words, but this is not very useful in the context of avoidability, so we will consider here only "pure" patterns, constituted only of variables.

There are in fact two separate notions of avoidability for patterns. The difference is in how the alphabet is specified. This may seem a minor point, but it results in completely different problems. In Section 3.2, we study "absolute" avoidability, where a pattern is said to be avoidable when there exists one alphabet for which the corresponding pattern language is avoidable, and unavoidable if the language is unavoidable whatever the size of the alphabet. This part of the theory is well advanced, the main result being the existence of an algorithm deciding whether a given pattern is avoidable or not.

In Section 3.3, on the contrary, the alphabet is fixed. This gives a hierarchy of avoidability notions, depending on the size of this alphabet, and which is nicely expressed by associating an *avoidability index* to every pattern. Here, no general decidability theorem is known, but some bounds can be given, and an exhaustive classification is possible in certain simple cases.

3.1. Definitions and basic properties

3.1.1. Patterns and avoidability

Let us first fix notation and give some formal definitions. Throughout the chapter, we will mainly make use of two distinct alphabets. The first one, A, which is always assumed to be finite, is the usual alphabet on which ordinary words are constructed, and its elements, denoted a, b, c, etc., are just called *letters*. The second alphabet, E, is used in patterns. Its elements are denoted α, β, γ, etc. and are called *variables*, and words in E^* are called *patterns*. This distinction is meant to help the understanding of the roles of the different words used, but in some occasions it may be necessary to treat a pattern as an ordinary word, which amounts to taking $A = E$.

The *pattern language* associated to a pattern $p \in E^*$ is the language on A containing all the words $h(p)$, where h is a non-erasing morphism from E^* to A^* that substitutes an arbitrary non-empty word to every variable. It is denoted $p(A^+)$. A word $w \in A^*$ is said to *encounter* the pattern p if it contains an element of the pattern language as a factor, i.e. if $\mathrm{Fact}(w) \cap p(A^+) \neq \emptyset$. Equivalently, we say that p *occurs* or *appears* in w, otherwise w is said to *avoid* p. These definitions also apply to infinite words $w \in A^\omega$.

For example, consider the pattern $p = \alpha\alpha\beta\beta\alpha$. The pattern language associated to p on the alphabet A is $p(A^+) = \{uuvvu \mid u, v \in A^+\}$. The word 1011011000111 contains p (through $h{:}\,\alpha \mapsto 011, \beta \mapsto 0$), whereas the word 0000100010111 avoids p.

Given two patterns p and p', we can treat p' as a word and check whether it encounters p. If it is the case, we denote this by $p|p'$ (which can also be read as "*p divides p'*"). The relation on E^* defined in this way is clearly reflexive and transitive, so it is a preorder on E^*. When $p|p'$ and $p'|p$ hold together, the patterns p and p' are said to be *equivalent*, and this occurs if and only if they differ by a permutation of E.

A pattern p is *avoidable on A* if there are infinitely many words in A^* that avoid p, i.e. if $p(A^+)$ is an avoidable set of words in A^*. This is equivalent, by König's lemma (Lemma 1.2.3) to the existence of one infinite word in A^ω avoiding p. If on the contrary every long enough word in A^* encounters p, then p is *unavoidable on A*.

If the cardinality of A is k and p is avoidable on A, then p is said to be *k-avoidable*. Obviously, changing the name of the letters has no influence on the patterns that can be avoided (the situation would be different if we were considering more general patterns where constants are allowed), therefore a pattern is k-avoidable if and only if it is avoidable on any k-letter alphabet. A pattern which is not k-avoidable is *k-unavoidable*.

In the above example, an infinite word avoiding $p = \alpha\alpha\beta\beta\alpha$ can be constructed, as we shall see in Lemma 3.3.2. We then say that p is 2-avoidable. On the other hand, p is 1-unavoidable, as are all other patterns (every unary word longer than p trivially contains p).

Finally, a pattern which is avoidable on A for some A will simply be called *avoidable*, and a pattern which is unavoidable on A for every A will be called *unavoidable*.

On several occasions we will need to delete certain variables from a pattern. If V is a subset of E, we will denote by δ_V the morphism from E^* to $(E \setminus V)^*$ that maps a variable in V to the empty word, and a variable in $E \setminus V$ to itself. In general, there is no link between the avoidability of p and that of $\delta_V(p)$.

3.1.2. Powers

The simplest class of patterns is certainly the class of powers of a single variable, α^n. The first two, $\alpha^0 = \varepsilon$, $\alpha^1 = \alpha$, are trivially unavoidable as they are encountered by any non-empty word. The situation changes radically for $n \geq 2$, with $\alpha^2 = \alpha\alpha$ being 3-avoidable, and α^n for $n \geq 3$ being 2-avoidable, as shown by the next proposition.

Recall that the Thue-Morse infinite word $t = abbabaab\ldots$ is the fixed point of the binary morphism $\theta\colon a \mapsto ab, b \mapsto ba$ (see Example 1.2.9 in Chapter 1). Moreover, let $u = abcacbabcbac\ldots$ be the fixed point of the ternary morphism $\mu\colon a \mapsto abc, b \mapsto ac, c \mapsto b$.

PROPOSITION 3.1.1.

(i) *The Thue-Morse infinite word t avoids the patterns $\alpha\alpha\alpha$ and $\alpha\beta\alpha\beta\alpha$.*
(ii) *The infinite word $u = abcacbabcbac\ldots$ avoids the pattern $\alpha\alpha$*

Proof. Property (ii) can be reduced to (i) using the following observation. Let $\pi\colon \{a,b,c\}^* \to \{a,b\}^*$ be the morphism defined by $\pi(a) = abb$, $\pi(b) = ab$, and $\pi(c) = a$. Then $\pi(u) = t$. Indeed, it is easy to check that $\pi \circ \mu = \theta \circ \pi$, hence $\theta(\pi(u)) = \pi(\mu(u)) = \pi(u)$. By Proposition 1.2.8, θ has a unique fixed point beginning with a, so that $\pi(u) = t$. Now, if a square is found in u, i.e. if vv occurs for some word $v \in \{a,b\}^+$, then $\pi(vv)$ occurs in t. Moreover, the first letter of $\pi(v)$ is a, as well as the letter following $\pi(vv)$ in t. But then we have found an occurrence of $\alpha\beta\alpha\beta\alpha$ or of $\alpha\alpha\alpha$ (if $\pi(v) = a$) in t.

Let us now prove (i). We proceed by contradiction, assuming that there is an occurrence of $\alpha\alpha\alpha$ or $\alpha\beta\alpha\beta\alpha$ in t. Consider the shortest such occurrence, $uvuvu$ with $u \in A^+$ and $v \in A^*$. It occurs for the first time in t at position n.

Since aaa and bbb are not factors of t, $|uvuvu| \geq 5$. All factors of length 5 of t contain either aa or bb as a factor, hence $uvuvu$ and therefore uvu contain aa or bb. These words occur only at odd positions in t, hence the position of all occurrences of uvu, among which n and $n + |uv|$, have the same parity. Thus $|uv|$ is even.

Let now u' be the word formed by letters of u that have an even position in t, and v' similarly from v. Then $u'v'u'v'u'$ occurs in t, and $|u'v'u'v'u'| < |uvuvu|$. If $u' \neq \varepsilon$, this contradicts the minimality of $uvuvu$. If $u' = \varepsilon$, letters at odd positions should be considered instead. ∎

Since the infinite word u avoids squares, it is said to be *square-free*. Similarly, since two overlapping occurrences of the same word contain one of the patterns $\alpha\alpha\alpha$ or $\alpha\beta\alpha\beta\alpha$, and the Thue-Morse infinite word t avoids them, it is said to be *overlap-free*.

Many other patterns are avoided by the Thue-Morse infinite word. This can usually be proved with arguments similar to those used in the proof of Proposition 3.1.1. One of these arguments is the presence of *synchronizing words*, here aa and bb. It is fairly general and is used in most avoidability proofs (see Section 3.3.2), as well as the general structure of the proof (consider an occurrence with minimal length, then construct a shorter one to reach a contradiction). The other argument, the use of parity, is specific to the Thue-Morse word, or at least of infinite words generated by uniform morphisms, and has to be replaced for other kinds of infinite words.

3.1.3. Sesquipowers

The only unavoidable patterns we have seen for the moment are the empty pattern ε and the pattern α. To construct other unavoidable patterns, we need new variables.

With two variables α and β, we can construct the pattern $\alpha\beta$ which is obviously unavoidable (any word of length at least 2, regardless of the alphabet, contains an occurrence of $\alpha\beta$). More interesting is the pattern $\alpha\beta\alpha$:

PROPOSITION 3.1.2. *The pattern $\alpha\beta\alpha$ is unavoidable. More precisely, if the cardinality of A is k, any word of length at least $2k+1$ contains an occurrence of $\alpha\beta\alpha$, and this bound is tight.*

Proof. Let $w \in A^*$ be a word of length at least $2k+1$. Then one of the letters in A, say a, occurs at least three times in w. Write $w = w_0aw_1aw_2aw_3$, and let $h(\alpha) = a$ and $h(\beta) = w_1aw_2$. Then $h\colon \{\alpha,\beta\}^* \to A^*$ is a non-erasing morphism and $h(\alpha\beta\alpha)$ is a factor of w. If $A = \{a_1, a_2, \ldots, a_k\}$, then $a_1a_1a_2a_2\ldots a_ka_k$ is a word of length $2k$ avoiding $\alpha\beta\alpha$. ∎

The above construction applies in fact to any pattern (although it is not so easy to get a tight bound then).

PROPOSITION 3.1.3. *Let p be a pattern unavoidable on A, and ζ a variable that does not occur in p. Then the pattern $p\zeta p$ is unavoidable on A.*

Proof. Let $k = \operatorname{Card} A$. Since p is unavoidable on A, there is an integer l such that any word in A^l contains p. This is a finite set with k^l elements. Let now $N = k^l(l+1)+l$, and consider any word $w \in A^N$. The word w can be viewed as the concatenation of k^l+1 words of length l, separated by individual letters. Among these k^l+1 factors of length l, at least two are equal, say $w = w_0vw_1vw_2$ with $|v| = l$ and $|w_1| \geq 1$. There is a non-erasing morphism $h\colon (\operatorname{alph} p)^* \to A^*$ such that $v = v_0h(p)v_1$, and letting $h(\zeta) = v_1w_1v_0$, we find that $h(p\zeta p)$ is a factor of w. ∎

If we apply recursively Proposition 3.1.3 starting with the empty word, we can construct an infinite family of unavoidable patterns. Let α_n, for $n \in \mathbb{N}$, be different variables in E. Let $Z_0 = \varepsilon$, and for all $n \in \mathbb{N}$, $Z_{n+1} = Z_n \alpha_n Z_n$. The patterns Z_n are called *Zimin words*, or *sesquipowers*.

PROPOSITION 3.1.4. *The Zimin patterns Z_n are all unavoidable.*

Proof. Let A be a finite alphabet. We have seen that $Z_0 = \varepsilon$, $Z_1 = \alpha_0$, and $Z_2 = \alpha_0\alpha_1\alpha_0$ are unavoidable on A. If Z_n is unavoidable on A, then by Proposition 3.1.3, $Z_{n+1} = Z_n\alpha_n Z_n$ is also unavoidable on A (note that we are allowed to apply Proposition 3.1.3 since α_n does not occur in Z_n). Since all sesquipowers Z_n are unavoidable on any A, they are unavoidable. ∎

3.2. Deciding avoidability: the Zimin algorithm

3.2.1. Reduction of patterns

To show that avoidability is decidable, we shall prove that it is equivalent to a property of *irreducibility*, defined below, which can itself be checked through a recursive algorithm.

THEOREM 3.2.1. *A pattern is avoidable if and only if it is irreducible*

The proof will be carried out in the following two subsections. Let us first define the reduction process.

Let $p \in E^*$ be a pattern. The *adjacency graph* of p is the bipartite graph $AG(p)$ with two copies of E as vertices, denoted by E^L and E^R, and with an edge between ξ^L and η^R if and only if $\xi\eta$ is a factor of p. For instance, the adjacency graph of $\alpha\beta\alpha\gamma\beta\alpha$, shown on Figure 3.1, has 6 vertices and 4 edges.

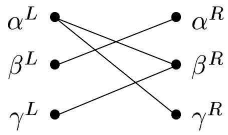

Figure 3.1. The adjacency graph of $\alpha\beta\alpha\gamma\beta\alpha$

A non-empty subset F of $\operatorname{alph} p$ is called a *free set* (for p) if there exists no path in $AG(p)$ linking a left-side vertex ξ^L to a right-side vertex η^R with ξ and η in F. To find all the free sets, one should first determine the connected components of $AG(p)$. In the example above, the adjacency graph has two connected components, and two free sets, $\{\alpha\}$ and $\{\beta\}$.

Given a pattern p and a free set F for p, we say that p reduces in one step to q by the deletion of F if $q = \delta_F(p)$ is the pattern obtained by deleting from p all occurrences of letters in F. We shall denote this fact as $p \xrightarrow{F} q$. We say that p reduces to q if there is a sequence of one-step reductions leading from p to q, and denote this as $p \xrightarrow{*} q$. Finally, a pattern p is *reducible* if it reduces to the empty pattern, $p \xrightarrow{*} \varepsilon$, and p is *irreducible* otherwise.

In the above example, $p = \alpha\beta\alpha\gamma\beta\alpha$ reduces to $\beta\gamma\beta$ by deletion of the free set $\{\alpha\}$, and $\beta\gamma\beta$ itself reduces to γ, which in turns reduces to ε. The pattern p is therefore reducible. However, if we had begun with the deletion of the free set $\{\beta\}$, we would have obtained the irreducible pattern $\alpha\alpha\gamma\alpha$. To prove that a pattern is irreducible, it is therefore necessary to recursively explore all possible sequences of one-step reductions and make sure that none of them leads to the empty pattern. (Since the exploration of a potentially large tree is needed, one should not expect this algorithm to be very efficient in practice.) This recursive algorithm will be called the *Zimin algorithm*.

It should be noted that it is sometimes necessary to delete free sets having more than one element, for instance for the pattern $p = \alpha\beta\alpha\gamma\alpha'\beta\alpha\delta\alpha\beta\alpha'\gamma\alpha'\beta\alpha'$ where all deletions of a free singleton lead to an irreducible pattern, while p can be reduced to the empty pattern starting with the deletion of the free set $\{\alpha, \alpha'\}$.

We shall need some additional notation concerning adjacency graphs. Given a pattern p and a set X of vertices of $AG(p)$, we denote by $C(X,p)$ the set of vertices of $AG(p)$ that are in the same connected component as an element of X, by $C_L(X,p)$ the set of variables $\xi \in E$ such that $\xi^L \in C(X,p)$, and by $C_R(X,p)$ the set of variables $\xi \in E$ such that $\xi^R \in C(X,p)$. If F is a free set and we apply these definitions to the set F^L of left-side vertices associated to F, then we obtain the two sets $C_L(F^L,p)$ and $C_R(F^L,p)$. The fact that F is a free set can then be expressed as $F \subset C_L(F^L,p) \setminus C_R(F^L,p)$.

3.2.2. Reducible patterns are unavoidable

Let us first prove the easier direction of Theorem 3.2.1: if a pattern is reducible, then it is unavoidable. This is a particular case of the following lemma, taking $q = \varepsilon$, the empty pattern being obviously unavoidable.

LEMMA 3.2.2. *If $p \xrightarrow{*} q$ and q is unavoidable, then p is also unavoidable.*

Proof. It suffices to prove Lemma 3.2.2 in the case where p reduces to q in one step by deletion of a free set F, the general case being then deduced by induction on the number of steps.

We shall prove that p is unavoidable on any alphabet A by induction on the size of A. For $\operatorname{Card} A = 1$ the assertion is obviously true. Assume now that $A = A' \cup \{a\}$ and that p is unavoidable on A'. Let $L = A'^+ \setminus A'^* p(A'^+) A'^*$ be the set of non-empty words on A' avoiding p, which is finite by assumption, and $M = aA^* \setminus A^* p(A^+) A^*$ be the set of words on A avoiding p and starting with a. Each word of M which is not a power of a can be represented as a non-empty product of words in $N = \{a^i w a^j \mid w \in L, 0 < i < |p|, 0 \leq j < |p|\}$, which is obviously finite. In other terms, if we view N as a new alphabet, $M \subset i(N^+) \cup a^+$, where i is the morphism from the free monoid N^* to A^* mapping an element of N (viewed as a letter) to itself (viewed as a word).

Let ζ be a variable that does not occur in p, and $E' = E \cup \{\zeta\}$. Then $q\zeta$ is an unavoidable pattern and every sufficiently long word on N contains an occurrence of this pattern. Consequently, for any $w \in N^*$ long enough, there is a non-erasing morphism f from E'^* to N^* such that $f(q\zeta)$ is a factor of w. For a variable ξ, $f(\xi) \in N^+$ hence $i(f(\xi)) \in aA^+$. We now define a new morphism g from E^* to A^* as follows:

1. $g(\xi) = i(f(\xi))$ if $\xi \in E \setminus (C_L(F^L, p) \cup C_R(F^L, p))$,
2. $g(\xi) = a^{-1}i(f(\xi))$ if $\xi \in C_R(F^L, p) \setminus C_L(F^L, p)$,
3. $g(\xi) = i(f(\xi))a$ if $\xi \in C_L(F^L, p) \setminus (C_R(F^L, p) \cup F)$,
4. $g(\xi) = a^{-1}i(f(\xi))a$ if $\xi \in C_L(F^L, p) \cap C_R(F^L, p)$,
5. $g(\xi) = a$ if $\xi \in F$.

Note that the five cases are indeed exclusive of each other, and that this defines a non-erasing morphism. Moreover, as we shall see below, $g(p)$ is a factor of $i(f(q\zeta))$. Consequently, $i(w)$ encounters p and cannot be in M, which means that M is finite and that p is unavoidable on A.

It remains to show that $g(p)$ is a factor of $i(f(q\zeta))$. We shall prove by induction on k, $1 \le k \le |p|$, that if p_k is the prefix of length k of p and $p_k \xrightarrow{F} q_k$, where q_k is a prefix of q, then $rg(p_k)$ is equal to $i(f(q_k))s_k$, where r is a or ε (depending on whether the first letter of p is in $C_R(F^L, p)$ or not) and s_k is a or ε (depending on whether the last letter of p_k is in $C_L(F^L, p)$ or not). For $k = 1$, this is obvious from the definition of g. Assume that $rg(p_k) = i(f(q_k))s_k$, and let $p_{k+1} = p_k\eta$. The last letter of p_k is denoted by ξ, so that there is an edge from ξ^L to η^R in $AG(p)$. We have to show that $rg(p_{k+1}) = i(f(q_{k+1}))s_{k+1}$, with $s_{k+1} = a$ if $\eta \in C_L(F^L, p)$, $s_{k+1} = \varepsilon$ otherwise. Writing $rg(p_{k+1}) = rg(p_k)g(\eta) = i(f(q_k))s_k g(\eta)$, it reduces to $s_k g(\eta) = i(f(\eta))s_{k+1}$ if $\eta \notin F$, or to $s_k g(\eta) = s_{k+1}$ if $\eta \in F$. This is again obvious from the definition of g, observing that $s_k = a$ occurs if and only if $\eta \in C_R(F^L, p)$, since this is equivalent to $\xi \in C_L(F^L, p)$. ∎

3.2.3. Irreducible patterns are avoidable

We now turn to the other direction of Theorem 3.2.1: if a pattern is unavoidable, then it is reducible. This part of the proof relies on several lemmas.

LEMMA 3.2.3. *Suppose that $f(q)$ is a factor of p for some non-erasing morphism f of E^* (so that q divides p), and that F is a free set for p. Let F' be the set of variables ξ such that $f(\xi) \in F^+$. Then F' is a free set for q. Moreover, if $p \xrightarrow{F} p'$ and $q \xrightarrow{F'} q'$, then $f'(q')$ is a factor of p', where $f' = \delta_F \circ f\big|_{E\setminus F'}$ is the non-erasing morphism from $(E \setminus F')^*$ to $(E \setminus F)^*$ mapping a variable ξ to the pattern obtained by deleting the elements of F from $f(\xi)$.*

Proof. Given a variable $\xi \in E$, we denote by $f_1(\xi)$ the first letter of $f(\xi)$ and by $f_2(\xi)$ the last letter of $f(\xi)$. If $\xi\eta$ occurs in q, then $f_2(\xi)f_1(\eta)$ occurs in p. Mapping ξ^L to $f_2(\xi)^L$ and ξ^R to $f_1(\xi)^R$, we see that $AG(q)$ is mapped to a subgraph of $AG(p)$. If there were a path from ξ^L to η^R in $AG(q)$, with ξ and η in F', then there would be a path from $f_2(\xi)^L$ to $f_1(\eta)^R$ in $AG(p)$, which is not the case since $f_2(\xi)$ and $f_1(\eta)$ are elements of the free set F. Therefore F' is a free set for q, and the rest of the lemma is obvious. ∎

Observe that if $\xi \in F'$ occurs in q, then $f(\xi)$ must have length 1, since two variables of F cannot occur consecutively in p.

LEMMA 3.2.4. *Suppose that p, p', q are patterns such that $p \xrightarrow{*} p'$ and $f(q)$ is a factor of p for some non-erasing morphism f. Then there exists a pattern q' and a non-erasing morphism f' such that $q \xrightarrow{*} q'$ and $f'(q')$ is a factor of p', with the additional condition that $f(\operatorname{alph} q \setminus \operatorname{alph} q') \subset (\operatorname{alph} p \setminus \operatorname{alph} p')^*$ (if a variable ξ is deleted from q, then $f(\xi)$ contains only variables deleted from p).*

Proof. This is just the iteration of Lemma 3.2.3, and can be proved by induction on the number of reduction steps from p to p'. If $p = p'$, then the result is obvious, with $q' = q$ and $f' = f$. Assume now that $p \xrightarrow{*} p'' \xrightarrow{F} p'$ and that we have constructed q'' and f'' such that $q \xrightarrow{*} q''$, $f''(q'')$ is a factor of p'', and $f(\operatorname{alph} q \setminus \operatorname{alph} q'') \subset (\operatorname{alph} p \setminus \operatorname{alph} p'')^*$. Then apply Lemma 3.2.3 to p'' and q'', to construct a free set F' for q'', a pattern q' such that $q'' \xrightarrow{F'} q'$ and a non-erasing morphism f' such that $f'(q')$ is a factor of p' and $f''(F') \subset F^+$.

$$\begin{array}{ccccc} p & \xrightarrow{*} & p'' & \xrightarrow{F} & p' \\ f\uparrow & & f''\uparrow & & f'\uparrow \\ q & \xrightarrow{*} & q'' & \xrightarrow{F'} & q' \end{array}$$

The additional condition also holds, since $\operatorname{alph} q \setminus \operatorname{alph} q' = (\operatorname{alph} q \setminus \operatorname{alph} q'') \cup F'$, with $f(\operatorname{alph} q \setminus \operatorname{alph} q'') \subset (\operatorname{alph} p \setminus \operatorname{alph} p'')^* \subset (\operatorname{alph} p \setminus \operatorname{alph} p')^*$ and $\delta_F(f''(F')) \subset \delta_F(F^+) \subset \{\varepsilon\}$, hence $f(F') \subset (\operatorname{alph} p \setminus \operatorname{alph} p')^*$. ∎

LEMMA 3.2.5. *Let $q = \delta_V(p)$ be a pattern obtained from another pattern p by deleting the variables in a set V (not necessarily a free set), Suppose that there is a pattern r and a non-erasing morphism f such that $r \xrightarrow{*} q$ and $f(p)$ is a factor of r, and that $\xi \in V$ if and only if $f(\xi) \in (\operatorname{alph} r \setminus \operatorname{alph} q)^*$. Then $p \xrightarrow{*} q$.*

Proof. Apply Lemma 3.2.4, with r and q playing respectively the roles of p and p'. There exist a pattern q' and a non-erasing morphism f' such that $p \xrightarrow{*} q'$ and $f'(q')$ is a factor of q, with $f(\operatorname{alph} p \setminus \operatorname{alph} q') \subset (\operatorname{alph} r \setminus \operatorname{alph} q)^*$. Then $\operatorname{alph} p \setminus \operatorname{alph} q' \subset V$, and $\delta_V(q') = \delta_V(p) = q$, hence $|q'| \geq |q|$. On the other hand, since $f'(q')$ is a factor of q, $|q'| \leq |q|$. We therefore have $|q| = |q'|$, and this is only possible if $q = q'$. We thus have $p \xrightarrow{*} q$. ∎

Let us define, for any positive integer k, a morphism φ_k on the $4k$-letter alphabet $A_k = \{a_0, a_1, \ldots, a_{2k-1}, b_0, b_1, \ldots, b_{2k-1}\}$. For $0 \le i \le 2k-1$, set $\varphi_k(a_i) = a_0 b_i a_1 b_{i+1} \ldots a_{k-1} b_{i+k-1}$ and $\varphi_k(b_i) = a_k b_i a_{k+1} b_{i+1} \ldots a_{2k-1} b_{i+k-1}$, where indices are taken modulo $2k$. The morphism φ_k is uniform with length $2k$. Using the morphism φ_k, we can construct an infinite word $w^{(k)} = \varphi_k^\omega(a_0)$, the fixed point of φ_k. We shall now prove that any irreducible pattern is avoided by $w^{(k)}$ for some k.

LEMMA 3.2.6. *Let v be a factor of length at least 2 of $w^{(k)}$. Then there is an integer i, $0 \le i \le 4k-1$, and a letter $x \in A_k$ such that, whenever v occurs at position $n \ge 0$ in $w^{(k)}$, one has*

(i) $n = i \pmod{4k}$,

(ii) *the letter at position* $n' = \lfloor \frac{n}{2k} \rfloor$ *in* $w^{(k)}$ *is* $w^{(k)}_{n'} = x$.

Proof. Let us assume that $|v| = 2$, the general case follows trivially.

Observe that the a's and b's alternate in the images under φ_k, hence in $w^{(k)}$. Consequently, the letter following an occurrence of $\varphi_k(a_i)$ is a_k, and the letter following an occurrence of $\varphi_k(b_i)$ is a_0, so that the letters at even positions in $w^{(k)}$ cycle periodically in the set $\{a_0, a_1, \ldots, a_{2k-1}\}$, i.e. $w^{(k)}_{2i} = a_i$, where the index in a_i is taken modulo $2k$. In other terms, the letters a_i are *synchronizing letters* that indicate the position in $w^{(k)}$ modulo $4k$. Since any factor of length 2 contains at least one letter a_i, the position of an occurrence of such a factor is unique modulo $4k$.

If v occurs at position n, then it starts within the image of $x = w^{(k)}_{n'}$. It is either a factor of $\varphi_k(x)$, or is formed with the last letter of $\varphi_k(x)$ followed by the first letter of the image of the next letter, that is a_k or a_0. There are exactly $8k^2$ such words of length 2, all different. Therefore x is uniquely defined by v, and can be computed using the following rules, where all indices are modulo $2k$:

- if $v = a_i b_j$ with $0 \le i \le k-1$, then $x = a_{j-i}$,
- if $v = b_j a_{i+1}$ with $0 \le i \le k-1$, then $x = a_{j-i}$,
- if $v = a_i b_j$ with $k \le i \le 2k-1$, then $x = b_{j+k-i}$,
- if $v = b_j a_{i+1}$ with $k \le i \le 2k-1$, then $x = b_{j+k-i}$.

In other terms, the letters b_j are *recognizing letters* that, together with a neighboring a_i, allow to reconstruct the word one particular factor of $w^{(k)}$ comes from under the action of φ_k. ∎

LEMMA 3.2.7. *Let p be a pattern, k an integer such that $2k > \operatorname{Card} \operatorname{alph} p$, and v a factor of $w^{(k)}$ such that $\varphi_k(v)$ encounters p. Then there is a pattern q such that $p \xrightarrow{*} q$ and v encounters q.*

Proof. Let h be a non-erasing morphism such that $h(p)$ is a factor of $\varphi_k(v)$. According to Lemma 3.2.6, to each letter $x \in A$ are associated $2k$ words of length 2 that recognize $\varphi_k(x)$. Since $2k > \operatorname{Card} \operatorname{alph} p$ and these $2k$ words end with different letters, at least one of these words ends with a letter which is not the first letter of any $h(\xi)$ with ξ occurring in p. Let us choose one such word d_x and call it the *decisive word* for x. By construction, whenever d_x occurs in $h(p)$ it occurs within some $h(\xi)$.

Let V be the set of variables ξ such that $h(\xi)$ contains no decisive word, and $q = \delta_V(p)$. We define a non-erasing morphism h' from $(E \setminus V)^*$ to A^* by $h'(\xi) = x_1 x_2 \dots x_m$, where d_{x_1}, d_{x_2}, ..., d_{x_m} are the decisive words occurring in $h(\xi)$, in the order in which they occur. Since each decisive word corresponds to exactly one letter in v, and consecutive decisive words correspond to consecutive letters, the word $h'(q)$ is a factor of v, i.e. v encounters q.

Note that V is not necessarily a free set, so we have not yet proved that $p \stackrel{*}{\to} q$. For this we shall define a morphism f such that $f(p) \stackrel{*}{\to} q$ and apply Lemma 3.2.5. The morphism f, from E^* to $(E \cup A)^*$ (elements of A being treated as additional variables), is defined as follows:

(i) if $\xi \in V$, then we set $f(\xi) = h(\xi)$,

(ii) if $h(\xi)$ does not contain a_0 or a_k, but contains a decisive word d_x (only one decisive word may occur in this case), then $\varphi_k(x) = a_i v_1 h(\xi) v_2$, with $i \in \{0, k\}$ and $v_1, v_2 \in A^*$, and we set $f(\xi) = h(\xi) v_2 \xi v_1 h(\xi)$,

(iii) if $\xi \notin V$ and $h(\xi) = v_1 \varphi_k(w) a_j v_2$ with $j \in \{0, k\}$ and $v_1, v_2, w \in A^*$, with w of maximal length, then let a_i be the first letter of $\varphi_k(w) a_j$, x_1 be the first letter of $h'(\xi)$, and x_2 be the last letter of $h'(\xi)$. We set $f(\xi) = v_1 v_1' \xi v_2' v_2$, where $v_1' = \varepsilon$ if the first decisive word d_{x_1} of $h(\xi)$ occurs in $v_1 a_i$, $v_1' = \varphi_k(x_1)$ otherwise, and similarly $v_2' = \varepsilon$ if the last decisive word d_{x_2} of $h(\xi)$ occurs in $a_j v_2$, $v_2' = (a_{j+k})^{-1} \varphi_k(x_2) a_j$ otherwise.

Deleting the elements of A from the pattern $f(p)$, one obtains exactly q, since $f(\xi) \in A^*$ when $\xi \in V$ and $f(\xi) \in A^* \xi A^*$ otherwise: $\delta_A(f(p)) = \delta_V(p) = q$. Moreover, two variables ξ_1 and ξ_2 consecutive in q are separated in $f(p)$ by the word $v_1 a_i v_2 \in A^*$, where $a_{i+k} v_1$ is the image by φ_k of the last letter in $h'(\xi_1)$ and $a_i v_2$ is the image by φ_k of the first letter in $h'(\xi_2)$. This allows to reduce $f(p)$ as follows. First, let $F_0 = \{b_0, b_1, \dots, b_{2k-1}\}$. This is a free set for $f(p)$, since an element of F_0 can only be followed by an element of $E \cup \{a_0, a_1, \dots, a_{2k-1}\}$, which in turn can only be preceded by an element of F_0. Thus $f(p) \stackrel{F_0}{\to} p_0$, where the pattern p_0 contains only variables in E and letters a_i, occurring in sequences $a_1 a_2 \dots a_{k-1} a_k a_{k+1} \dots a_{2k-1}$ or $a_{k+1} a_{k+2} \dots a_{2k-1} a_0 a_1 \dots a_{k-1}$ between two variables in E. Then $F_1 = \{a_1, a_{k+1}\}$ is a free set for p_0, allowing the reduction $p_0 \stackrel{F_1}{\to} p_1$. We continue deleting $F_i = \{a_i, a_{k+i}\}$ for i from 2 to $k-1$, until we get a pattern $p_{k-1} \in (E \cup \{a_0, a_k\})^*$, in which the elements of E and $F_k = \{a_0, a_k\}$ alternate, so that F_k is again a free set and $p_{k-1} \stackrel{F_k}{\to} q$.

Consequently, $f(p) \stackrel{*}{\to} q$ and the pattern $r = f(p)$ satisfies the hypotheses of Lemma 3.2.5, so that $p \stackrel{*}{\to} q$. ∎

LEMMA 3.2.8. *If the infinite word $w^{(k)}$ encounters a pattern p containing less than $2k$ distinct variables, then p is reducible.*

Proof. There is a positive integer m such that $\varphi_k^m(a_0)$ encounters p. Applying Lemma 3.2.7, we obtain a pattern p_1 such that $p \stackrel{*}{\to} p_1$ and $\varphi_k^{m-1}(a_0)$ encounters

p_1. This process can be repeated since the condition $2k > \operatorname{Card}\operatorname{alph} p_1$ still holds (the number of variables involved cannot increase), yielding patterns p_i for $1 \le i \le m$ such that $p \xrightarrow{*} p_i$ and $\varphi_k^{m-i}(a_0)$ encounters p_i. But then a_0 encounters p_m, which means that p_m is either a single variable or the empty pattern, hence p is reducible. ∎

Proof of Theorem 3.2.1. By Lemma 3.2.2, if p is reducible, then p is unavoidable. Conversely, assume that p is unavoidable. Then for all k, in particular for $k = \left\lceil \frac{\operatorname{Card}\operatorname{alph} p+1}{2} \right\rceil$, $w^{(k)}$ encounters p. By Lemma 3.2.8, p is reducible. ∎

Corollary 3.2.9. *The infinite word $w^{(k)}$ avoids all avoidable patterns with at most $2k-1$ variables.*

Proof. If $w^{(k)}$ encounters a pattern p with at most $2k-1$ variables, then by Lemma 3.2.8 p is reducible, hence by Lemma 3.2.2 p is unavoidable. ∎

Corollary 3.2.10. *If all variables that occur in a pattern p occur at least twice, then p is avoidable.*

Proof. If p is unavoidable, then it reduces to a single variable α. But then α is a variable that occurs only once in p. ∎

Corollary 3.2.11. *Let p be a pattern with n variables. If $|p| \ge 2^n$, then p is avoidable.*

Proof. We prove by induction on n that if p is unavoidable, then $|p| < 2^n$, For $n = 1$, the result holds according to Proposition 3.1.1. Assume that it holds for n, and consider an unavoidable pattern p with $n+1$ variables. According to Corollary 3.2.10, there is a variable α that occurs only once in p. Then p can be written as $p = p_1\alpha p_2$, where p_1 and p_2 are patterns with at most n variables, which are both unavoidable since they divide p. By the induction hypothesis, p_1 and p_2 have length at most $2^n - 1$, hence p has length at most $2^{n+1} - 1$. ∎

3.3. Avoidability on a fixed alphabet

3.3.1. The avoidability index

As shown by the case of the square $\alpha\alpha$, an avoidable pattern need not be avoidable on a 2-letter alphabet, in other terms the same pattern can be 2-unavoidable but k-avoidable for some larger k. This leads to define the *avoidability index* of a pattern $p \in E^*$, $\mu(p)$. It is the smallest integer k such that p is k-avoidable, or ∞ if p is unavoidable. Clearly, $2 \le \mu(p) \le \infty$ since no pattern is 1-avoidable. In some sense, the avoidability index of a pattern measures how easy it is to avoid this pattern. With the preorder on E^* defined by divisibility, the function $\mu\colon E^* \to \mathbb{N} \cup \{\infty\}$ is non-increasing: if $p|q$, then $\mu(p) \ge \mu(q)$.

Contrary to the situation of Section 3.2 where the size of the alphabet did not matter, there is no known algorithm to determine the status of a given pattern, that is to compute its avoidability index. Even for very short patterns the value of $\mu(p)$ may be unknown. For instance, it is not known at the time of writing whether $\mu(\alpha\alpha\beta\beta\gamma\gamma)$ is equal to 2 or 3, although there is some experimental evidence that the index is 2.

Therefore, the results that we shall present here are only partial. They are of two kinds: computations in a simple case where patterns can be completely classified according to their avoidability index, and bounds in the general case.

3.3.2. The binary case

We will now restrict to a particular class of patterns, namely binary patterns, i.e. patterns with at most two different variables. In this section, we set $E = \{\alpha, \beta\}$.

Let us first review the unary case, studied in Section 3.1.2. The empty pattern and the pattern reduced to one variable are unavoidable: $\mu(\varepsilon) = \mu(\alpha) = \infty$. Squares are 3-avoidable but 2-unavoidable, hence $\mu(\alpha\alpha) = 3$. Finally, cubes and larger powers are 2-avoidable, hence $\mu(\alpha^k) = 2$ for $k \geq 3$.

The fact that $\alpha\alpha$ is a 3-avoidable 2-unavoidable pattern gives us readily some information about binary patterns. Only a finite number of binary patterns are 3-unavoidable. Indeed, a pattern which is divisible by $\alpha\alpha$ must be 3-avoidable, and since $\alpha\alpha$ is 2-unavoidable, there are only finitely many binary patterns which are not divisible by $\alpha\alpha$, namely ε, α, β, $\alpha\beta$, $\beta\alpha$, $\alpha\beta\alpha$, and $\beta\alpha\beta$. All of these are in fact unavoidable, which implies that the avoidability index of a binary pattern can only be 2, 3, or ∞.

The remaining question is to distinguish patterns that have avoidability index 3 from patterns that have avoidability index 2.

LEMMA 3.3.1. *The binary patterns $\alpha\alpha$, $\alpha\alpha\beta$, $\alpha\alpha\beta\alpha$, $\alpha\beta\beta\alpha$, $\alpha\alpha\beta\beta$, $\alpha\beta\alpha\beta$, $\alpha\alpha\beta\alpha\alpha$, and $\alpha\alpha\beta\alpha\beta$ have avoidability index* 3.

Proof. Since all these patterns are divisible by $\alpha\alpha$, they are 3-avoidable. A simple backtracking algorithm is sufficient to check that they are 2-unavoidable. The results are summarized in the following table, where w is an example of a binary word avoiding p of maximal length, and N is the total number of binary words avoiding p, including the empty word and unary words.

p	w	$\|w\|$	N
$\alpha\alpha$	aba	3	7
$\alpha\alpha\beta$	$abab$	4	13
$\alpha\alpha\beta\alpha$	$ababab aaa$	9	91
$\alpha\beta\beta\alpha$	$aabbbaaabb$	10	93
$\alpha\alpha\beta\beta$	$abaaabaaaba$	11	147
$\alpha\beta\alpha\beta$	$abaabbaaabbbaabbab$	18	477
$\alpha\alpha\beta\alpha\alpha$	$abaaaabbbbabababab$	18	1699
$\alpha\alpha\beta\alpha\beta$	$abababab aabbaaabbbaaaabbbbaaabbbaabbab$	38	26241

■

LEMMA 3.3.2. *The binary patterns $\alpha\alpha\alpha$, $\alpha\beta\alpha\beta\alpha$, $\alpha\beta\alpha\beta\beta\alpha$, $\alpha\alpha\beta\alpha\beta\beta$, $\alpha\beta\alpha\alpha\beta$, and $\alpha\alpha\beta\beta\alpha$ have avoidability index 2.*

Proof. According to Proposition 3.1.1, $\alpha\alpha\alpha$ and $\alpha\beta\alpha\beta\alpha$ are avoided by the Thue-Morse infinite word, and therefore 2-avoidable.

The two patterns $\alpha\beta\alpha\beta\beta\alpha$ and $\alpha\alpha\beta\alpha\beta\beta$ are avoided by the infinite word $u = \nu^\omega(a)$, where ν is the uniform morphism that maps a to aab and b to bba. The proof is similar to that of Proposition 3.1.1, so we will only summarize it. Assume that the pattern p is not avoided by the infinite word u, and consider an element $h(p)$ of $\mathrm{Fact}(u) \cap p(A^+)$ of minimal length and the position n of its first occurrence in u. Then discuss on the value of n, $|h(\alpha)|$, and $|h(\beta)|$ modulo 3. In each of the 27 cases, either a contradiction is immediately reached, or an earlier or shorter occurrence of p in u can be constructed. Some preliminary observations on the structure of u, such as the fact that all squares that occur in u have a length multiple of 3, except aa and bb, can greatly reduce the number of cases that actually need to be considered.

The pattern $\alpha\beta\alpha\alpha\beta$ is avoided by the infinite word $v = \psi(\mu^\omega(a))$, where μ is the ternary morphism defined in Proposition 3.1.1 ($\mu(a) = abc$, $\mu(b) = ac$, $\mu(c) = b$) and ψ maps a to aaa, b to bbb, and c to $ababab$. We know by Proposition 3.1.1 that $\mu^\omega(a)$ is square free. The first step is to prove a *synchronization lemma* for ψ applied on $\mu^\omega(a)$: if x is a factor of v of length 7 or more, then there exist a unique triple (s, y, p) and two letters (not necessarily unique) d and e such that dye is a factor of $\mu^\omega(a)$, s is a proper suffix of $\psi(d)$, p is a proper prefix of $\psi(e)$, and $x = s\psi(y)p$. Then this synchronization lemma can be used to prove that the only squares that occur in v are a^2, b^2, $(aa)^2$, $(ab)^2$, $(ba)^2$, $(bb)^2$, and $(baba)^2$. Now, assume that $h(\alpha\beta\alpha\alpha\beta)$ occurs in v (there is no need to consider a minimal occurrence here). Then $h(\alpha)$ is one of a, b, aa, ab, ba, bb, or $baba$. If $h(\alpha\beta)$ has length 7 or more, then the synchronization lemma can be applied to it and it can be written as $s\psi(y)p$, leaving finitely many possibilities for s, p and $h(\alpha)$. Otherwise, there are finitely many possibilities for $h(\alpha)$ and $h(\beta)$. In each of the cases, either the word $h(\alpha\beta\alpha\alpha\beta)$ contains a small factor that obviously cannot occur in v, or unapplying ψ yields a square in $\mu^\omega(a)$.

The pattern $\alpha\alpha\beta\beta\alpha$ is avoided by the infinite word $v = \chi(\mu^\omega(a))$, where μ is as above and χ maps a to aa, b to aba, and c to $abbb$. The proof is similar. Here the synchronization lemma applies to words of length 5 or more, and the length of squares is not bounded. ■

THEOREM 3.3.3. *Binary patterns fall in three categories:*

- *The 7 binary patterns ε, α, β, $\alpha\beta$, $\beta\alpha$, $\alpha\beta\alpha$, and $\beta\alpha\beta$ are unavoidable (their avoidability index is ∞).*

- *The 22 binary patterns* $\alpha\alpha$, $\beta\beta$, $\alpha\alpha\beta$, $\alpha\beta\beta$, $\beta\alpha\alpha$, $\beta\beta\alpha$, $\alpha\alpha\beta\alpha$, $\alpha\alpha\beta\beta$, $\alpha\beta\alpha\alpha$, $\alpha\beta\alpha\beta$, $\alpha\beta\beta\alpha$, $\beta\alpha\alpha\beta$, $\beta\alpha\beta\alpha$, $\beta\alpha\beta\beta$, $\beta\beta\alpha\alpha$, $\beta\beta\alpha\beta$, $\alpha\alpha\beta\alpha\alpha$, $\alpha\alpha\beta\alpha\beta$, $\alpha\beta\alpha\beta\beta$, $\beta\alpha\beta\alpha\alpha$, $\beta\beta\alpha\beta\alpha$, *and* $\beta\beta\alpha\beta\beta$ *have avoidability index* 3.

- *All other binary patterns, and in particular all binary patterns of length* 6 *or more, have avoidability index* 2.

Proof. All binary patterns of length 6 are divisible by a pattern mentioned in Lemma 3.3.2, or by its mirror image: they are therefore 2-avoidable, and have avoidability index 2. Consequently, all larger patterns also have avoidability index 2. There remains a list of 29 patterns, 7 of which are unavoidable. The other 22 patterns are the patterns mentioned in Lemma 3.3.1, or their mirror images, up to a renaming of the variables. They all have avoidability index 3. ∎

3.3.3. A bound on the avoidability index

In our proof of Theorem 3.2.1, we constructed an infinite word that avoids a given irreducible pattern p. The construction uses a number of different letters that depend only of the number of variables in p. Thus, we have a bound (probably far from optimal) on $\mu(p)$ as a function of $\operatorname{Card}(\operatorname{alph} p)$.

THEOREM 3.3.4. *If p is an avoidable pattern, then* $\mu(p) \leq 4\left\lceil \frac{\operatorname{Card}\operatorname{alph} p+1}{2}\right\rceil \leq 2\operatorname{Card}\operatorname{alph} p + 4$.

Proof. Let $k = \left\lceil \frac{\operatorname{Card}\operatorname{alph} p+1}{2}\right\rceil$, so that $2k > \operatorname{Card}\operatorname{alph} p$. By Corollary 3.2.9, the infinite word $w^{(k)}$ avoids p. Since $w^{(k)}$ is defined over an alphabet of $4k$ letters, p is $4k$-avoidable. ∎

The proof is so short because all of the work has already been done in Section 3.2.3.

3.3.4. A bound on the length of 2-unavoidable patterns

We have seen in Theorem 3.3.3 that there are only finitely many 2-unavoidable binary patterns, namely that all binary patterns of length 6 or more are 2-avoidable. We shall now try to generalize this to a bound for patterns with more variables.

For $n, k \geq 1$, let us denote by ℓ_{nk} the smallest integer l (or ∞ if no such integer exists) such that every pattern of length l with n variables is k-avoidable. We extend the notation to $k = \infty$ with the convention that "∞-avoidable" means "avoidable". Clearly, $\ell_{nk} \leq \ell_{n'k'}$ if $n \leq n'$ and $k \geq k'$.

We know already that $\ell_{n1} = \infty$ since no pattern is 1-avoidable. Theorem 3.3.3 implies that $\ell_{22} = 6$ and $\ell_{2k} = 4$ for $k \geq 3$, and Proposition 3.1.1 implies that $\ell_{12} = 3$ and $\ell_{1k} = 2$ for $k \geq 3$. The value of $\ell_{n\infty}$ is given by Corollary 3.2.11: $\ell_{n\infty} = 2^n$.

If we are able to prove that all avoidable patterns on n variables are N_n-avoidable, then we can deduce that $\ell_{nk} = 2^n$ for all $k \geq N_n$. According to Theorem 3.3.4, this holds for $N_n = 4\left\lceil \frac{n+1}{2} \right\rceil$.

We shall now prove that ℓ_{nk} is finite for all $k \geq 2$. Obviously, it is sufficient for this to bound ℓ_{n2} for all n.

THEOREM 3.3.5. *For all $n \geq 1$, $\ell_{n2} < 200.5^n$.*

To prove Theorem 3.3.5, we have to construct a binary infinite word that avoids long enough patterns. We shall do so by starting with an infinite word $w_n \in A^\omega$ on a N_n-letter alphabet $A = \{a_0, a_1, \ldots, a_{N_n-1}\}$ avoiding all avoidable patterns on n variables, then recoding it on the binary alphabet $A' = \{a, b\}$ with a morphism $g: A^* \to A'^*$.

Such words w_n have been constructed in Section 3.2.3, where we called them $w^{(k)}$, with $k = \frac{N_n}{4} = \left\lceil \frac{n+1}{2} \right\rceil$. However, the construction of g is independent of the choice of w_n, so any other definition could be used instead for w_n.

Let $n \geq 1$, and $E = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Assume that $B_1, B_2, \ldots, B_{n-1}$ are $n-1$ fixed integers greater than 1. Let

$$M = T_n + 2B_{n-1} + \sum_{i=2}^{n} iB_{n+1-i} \ ,$$

where $T_n = \frac{1}{2}(N_n - 1)(n+1)(n+2) + n^2 + 2n + 4$, and let P be the set of patterns $p \in E^*$ of length at least M such that, for all i between 1 and $n-1$, every factor q of p of length B_i contains at least $i+1$ distinct variables.

Consider the morphism

$$\begin{aligned} g : A^* &\longrightarrow A'^* \\ a_i &\longmapsto a^{B_{n-1}+1} b^{n+i} z_1 b^i z_2 b^i \ldots z_{n-2} b^i z_{n-1} b^i a b^{n+nN_n+i} a \end{aligned}$$

where $z_i = (ab^i)^{B_{n-i}-1} ab^{n+iN_n}$.

Assume that p is a pattern in P that is not avoided by the infinite word $g(w_n)$. Then there is a prefix y of w_n, a non-erasing morphism $\varphi: E^* \to A'^*$ and words $t_1, t_2 \in A'^*$ such that $g(y) = t_1 \varphi(p) t_2$.

Let E' be the set of variables $\xi \in E$ such that $\varphi(\xi)$ contains aa. The value of M has been chosen so that $M = \max(|g(a_i)|) + B_{n-1} + 1$. Since $|\varphi(p)| \geq |p| \geq M$, the word $\varphi(p)$ contains at least one occurrence of $a^{B_{n-1}+2}$ at the boundary between two images of letters under g. Then either the set E' is non-empty, or there are B_{n-1} consecutive variables in p that are mapped to a by φ. In the latter case, since any variable in E occurs at least once in any factor of length B_{n-1} of p, the morphism φ maps all variables to a and $\varphi(p) = a^{|p|}$, which is obviously impossible since a^M is not a factor of $g(w_n)$.

Let us now define a new function $h: E \to \{0, 1, \ldots, n + (n+1)N_n - 1\}^*$ in the following way. For any $\xi \in E$, $\varphi(\xi)$ can be written in the form $\varphi(\xi) = b^{j_0} ab^{j_1} a \ldots ab^{j_m}$, with $0 \leq j_i < n + (n+1)N_n$, and we set $h(\xi) = j_1 j_2 \ldots j_{m-1}$, leaving out j_0 and j_m. Note that $h(\xi)$ can be the empty word if $m \leq 1$, but that $h(\xi)$ is never the empty word when $\xi \in E'$.

A *cut* of g is a pair of words (u_1, u_2) in A'^* such that $u_1u_2 = g(a_i)$ for some i and there exist $p_1, p_2 \in E^*$ and $s_1, s_2 \in A^*$ with $p_1p_2 = p$, $t_1\varphi(p_1) = g(s_1)u_1$ and $\varphi(p_2)t_2 = u_2g(s_2)$. A cut is called an *initial cut* when $h(\xi) = \varepsilon$ for any variable ξ in p_1.

If u is a factor occurring only once in all $g(a_i)$, we say that u is *cut* if there exists a non-initial cut (v_1u_1, u_2v_2) where u_1 and u_2 are non-empty words such that $u_1u_2 = u$.

LEMMA 3.3.6. *There exists an integer r with $0 \le r \le n$ such that for all $0 \le i < N_n$, $ab^{n+rN_n+i}a$ is not cut.*

Proof. Let r be the smallest non-negative integer such that the word $h(\xi)$ does not end in r for any variable $\xi \in E$. Since E has n elements, $r \le n$. We shall prove that $ab^{n+rN_n+i}a$ is never cut when $0 \le i < N_n$.

Note first that the word $ab^{n+rN_n+i}a$ occurs exactly once in $g(a_i)$ and not in any $g(a_j)$ with $j \ne i$. Indeed, it occurs as a factor of $a^{B_{n-1}+1}b^{n+i}z_1$ if $r = 0$, of $z_rb^iz_{r+1}$ if $1 \le r \le n-1$, or as a suffix if $r = n$.

Suppose that for some i, $ab^{n+rN_n+i}a$ is cut. Then there is a non-initial cut (v_1u_1, u_2v_2) with $u_1 = ab^j$, $u_2 = b^{n+rN_n+i-j}a$, $v_1u_1u_2v_2 = g(a_i)$, $p_1p_2 = p$, $t_1\varphi(p_1) = g(s_1)v_1u_1$ and $\varphi(p_2)t_2 = u_2v_2g(s_2)$. Since this cut is non-initial, there exist variables $\xi_0, \xi_1, \ldots, \xi_m$ such that p_1 ends in $\xi_0\xi_1\ldots\xi_m$, $h(\xi_0) \ne \varepsilon$ and $h(\xi_l) = \varepsilon$ for $1 \le l \le m$. In particular, for $1 \le l \le m$, $\varphi(\xi_l)$ contains at most one occurrence of a.

Assume first that $r = 0$. Then $\varphi(\xi_0\xi_1\ldots\xi_m)$ is a suffix of $t_1\varphi(p_1) = g(s_1)a^{B_{n-1}+1}b^j$. If $m \ge B_{n-1}$, then the factor $\xi_1\ldots\xi_m$ of p must contain n different variables, which contradicts the fact that E' is non-empty (and of course, $h(\xi) \ne \varepsilon$ when $\xi \in E'$). Therefore $m < B_{n-1}$, and $\varphi(\xi_1\ldots\xi_m)$ can take at most $B_{n-1} - 1$ of the a's in $a^{B_{n-1}+1}$. Consequently $\varphi(\xi_0)$ ends in aa or $aab^{j'}$, and $h(\xi_0)$ ends in 0, a contradiction with the definition of r.

Assume now that $r = n$. Then for all ξ in E, $h(\xi)$ is non-empty and ends in an integer between 0 and $n-1$. Necessarily $m = 0$ and $t_1\varphi(p_1)$ ends in $ab^{n+(n-1)N_n+i}ab^j$, hence $h(\xi_0)$ ends in $n + (n-1)N_n + i$, which is not an integer between 0 and $n-1$.

Finally, assume that $1 \le r \le n-1$. Then $t_1\varphi(p_1) = g(s_1)v_1u_1$ ends in the word $ab^{n+(r-1)N_n}(ab^r)^{B_{n-r}-1}ab^j$. Let F be the set of variables ξ such that $h(\xi)$ is either empty or ends in a value larger than or equal to n. The cardinal of F is at most $n-r$, hence p does not contain more than $B_{n-r-1} - 1$ consecutive variables in F. In particular, $m < B_{n-r-1}$. If $m < B_{n-r-1} - 1$, one finds that $\varphi(\xi_0)$ ends in $ab^rab^{j'}$, hence $h(\xi_0)$ ends in r, a contradiction with the definition of r. If $m = B_{n-r-1} - 1$, then either $\varphi(\xi_0)$ ends in $ab^rab^{j'}$ as above, or it ends in $ab^{n+(r-1)N_n}ab^{j'}$ and $\xi_0 \in F$, so that we have B_{n-r-1} consecutive variables in F, which is again excluded. ∎

LEMMA 3.3.7. *The infinite word $g(w_n)$ avoids all patterns $p \in P$.*

Proof. Assume that p is a pattern in P that is not avoided by $g(w_n)$. Let r be as given by Lemma 3.3.6, and ψ be the morphism from E^* to A^* defined by $\psi = \pi \circ h$, where $\pi(n + rN_n + i) = a_i$ and $\pi(j) = \varepsilon$ otherwise, and h is naturally extended to a morphism. Then $\psi(p)$ is a factor of y (recall that $\varphi(p)$ is a factor of $g(y)$). Indeed, let first p' be the suffix of p starting with the first occurrence of a variable ξ such that $h(\xi) \neq \varepsilon$. Obviously $\psi(p') = \psi(p)$. Any occurrence of a letter a_i in y such that $g(a_i)$ falls inside $\varphi(p')$ is marked by the word $ab^{n+rN_n+i}a$, which is not cut (not even by an initial cut, thanks to the definition of p'). This word corresponds therefore to exactly one occurrence of a_i in $\psi(p')$. Conversely, each letter in $\psi(p')$ corresponds to one occurrence of a marker $ab^{n+rN_n+i}a$ in $\varphi(p')$, hence to a letter a_i in y.

Let $q = \delta_V(p)$, where V is the set of variables that are mapped to ε by ψ. The pattern q is not empty. Indeed, cut p into three approximately equal parts, $p = p_1p_2p_3$. Since $|p| \geq M \geq 4B_{n-1}$, one can take $|p_j| \geq B_{n-1}$ for each $j = 1, 2, 3$, i.e. p_j must contain at least one occurrence of each letter in E. In particular, there is a letter in E' in p_1, which implies that $|p'| > |p_2p_3|$. There is also a letter in E' in p_3. Since $|\varphi(p)| \geq |p| > B_{n-1} + 2$, there is at least one variable ξ such that $\varphi(\xi)$ contains at least one b, and this variable occurs at least once in p_2. Consequently, p_1 and p_3 overlap two different blocks $a^{B_{n-1}+2}$, and between them there is some $g(a_i)$, which occurs completely within $\varphi(p')$, except maybe for some power of a at the beginning, and which contains one marker $ab^{n+rN_n+i}a$, so that $\psi(p') = \psi(q)$ is not empty.

Since $|p| \geq M \geq 4B_{n-1}$, p contains at least four occurrences of each variable in E, so that q contains at least four occurrences of each variable that occurs in it, and is therefore avoidable according to Corollary 3.2.10. The infinite word $g(w_n)$ contains $\psi(q)$ as a factor, in contradiction with the fact that $g(w_n)$ avoids all avoidable patterns on at most n variables. ∎

Proof of Theorem 3.3.5. Assume that we have proved that for $i < n$, every pattern of length S_i with i variables is 2-avoidable. Let $B_1 = S_1$, $B_2 = S_2$, $\ldots$, $B_{n-1} = S_{n-1}$ and apply Lemma 3.3.7. Then a pattern p of length M with n variables is either in P, in which case it is avoided by $g(w_n)$ according to Lemma 3.3.7, or for some $i < n$ it contains a factor q of length B_i with at most i variables, which is itself 2-avoidable by the induction hypothesis.

We just have proved by induction that every pattern of length S_n with n variables is 2-avoidable, where (S_n) is the numeric sequence starting with $S_1 = 3$ and satisfying the recurrence relation

$$S_n = T_n + 2S_{n-1} + \sum_{i=2}^{n} iS_{n+1-i} \ .$$

Let $\lambda > \left(1 - 2^{-\frac{1}{3}}\right)^{-1}$, so that $K = \left(1 - \frac{1}{\lambda}\right)^{-2} + \frac{2}{\lambda} - 1 < 1$, and

$$C = \max\left(\frac{3}{\lambda}, \frac{1}{1-K} \max_{n\geq 1}\left(\frac{T_n}{\lambda^n}\right)\right) \ .$$

Then $S_n \leq C\lambda^n$ for all $n \geq 1$, since $3 = S_1 \leq C\lambda$ and, if $S_i \leq C\lambda^i$ for $1 \leq i < n$, then

$$S_n \le C\lambda^n \left(\frac{T_n}{C\lambda^n} + \frac{2}{\lambda} + \sum_{i=2}^{n} i\lambda^{1-i} \right)$$

with $\frac{2}{\lambda} + \sum\limits_{i=2}^{n} i\lambda^{1-i} < K$ and $\frac{T_n}{C\lambda^n} \le 1 - K$.

Since $\left(1 - 2^{-\frac{1}{3}}\right)^{-1} \simeq 4.847$, we can take $\lambda = 5$. Then, with $N_n = 4\left\lceil \frac{n+1}{2} \right\rceil$, we find that $C < 200$. Note that the actual value of N_n influences only the constant C, as long as $N_n = o(n^{-2}\lambda^n)$. ■

Problems

Section 3.1

3.1.1 A set of patterns $P \subset E^*$ is said to be avoidable on A^* if there exists an infinite word in A^ω that avoids all elements of P. P is said to be k-avoidable if it is avoidable on a k-letter alphabet, and avoidable if it is k-avoidable for some k. Show that if P is finite, then P is avoidable if and only if all its elements are avoidable. Does this equivalence still hold if P is infinite? What if avoidability is replaced with k-avoidability?

3.1.2 A *simple formula* is a finite set of patterns, and is denoted

$$f = p_1 \cdot p_2 \cdot \ldots \cdot p_m \ ,$$

the order being unimportant. A word u is said to encounter the simple formula f if there exists a non-erasing morphism $h: E^* \to A^*$ such that all the words $h(p_1)$, $h(p_2)$, ..., $h(p_m)$ are factors of u. Avoidability and k-avoidability of simple formulas is then defined as for patterns. (Note that it is a different notion from the avoidability of sets in Problem 3.1.1.) Show that f is avoidable on A if and only if the pattern $p_1\zeta_1p_2\zeta_2\ldots p_{m-1}\zeta_{m-1}p_m$ is avoidable on A, where ζ_1, ..., ζ_{m-1} are distinct variables that do not occur in f. How can this fact be used for practical checking of unavoidability? (Compare the number of binary words avoiding $\alpha\alpha\beta\gamma\beta\alpha\beta\alpha$ to the number of binary words avoiding the equivalent simple formula $\alpha\alpha\beta \cdot \beta\alpha\beta\alpha$.)

*3.1.3 Definitions of Problems 3.1.1 and 3.1.2 can be combined to construct the *algebra of formulas* $\mathcal{F}(\mathcal{E})$: a *formula* is a set of simple formulas, denoted $f_1 + f_2 + \cdots + f_k$. $(\mathcal{F}(\mathcal{E}), +, \cdot)$ is a commutative algebra, the neutral elements being 0 (the formula containing no simple formula) and 1 (the simple formula containing no pattern). A word u is said to encounter the formula f if it encounters one of its elements f_i. All words encounter 1, and no word encounters 0. Design an equivalence relation $\sim$ on $\mathcal{F}(\mathcal{E})$ such that $f \sim f'$ implies that $\mu(f) = \mu(f')$, and that allows rewriting sequences such as

$$\begin{aligned}\alpha\alpha\beta\alpha\gamma\alpha\beta\delta\beta\alpha\alpha + \alpha\beta\beta\alpha\beta &\sim \alpha\alpha\beta\alpha \cdot \alpha\beta \cdot \beta\alpha\alpha + \alpha\beta\beta\alpha\beta \\ &\sim \alpha\alpha\beta\alpha \cdot \beta\alpha\alpha + \alpha\beta\beta\alpha\beta \\ &\sim \alpha\alpha\beta\alpha \cdot \beta\alpha\alpha + \beta\alpha\alpha\beta\alpha \\ &\sim \alpha\alpha\beta\alpha \cdot \beta\alpha\alpha \\ &\sim \alpha\alpha\beta\alpha \cdot \beta\alpha\alpha + \alpha\alpha\beta\alpha\alpha \\ &\sim \alpha\alpha\beta\alpha \cdot \beta\alpha\alpha + \alpha\alpha \cdot \alpha\alpha \\ &\sim \alpha\alpha\beta\alpha \cdot \beta\alpha\alpha + \alpha\alpha \\ &\sim \alpha\alpha \ .\end{aligned}$$

Two patterns equivalent in the sense of Section 3.1.1 should be also equivalent for this relation.

3.1.4 A pattern p is said to be *D0L-avoidable* on A if there exist a morphism $f: A^* \to A^*$ and a letter $a \in A$ such that the morphic infinite word $f^\omega(a)$ is properly defined (see Proposition 1.2.8) and avoids p. Let here $A = \{a, b\}$.

1. Show that the patterns $\alpha\beta\beta\gamma\alpha\beta\beta\gamma$ and $\alpha\beta\alpha\gamma\alpha\beta\alpha\gamma$ are avoidable on A but D0L-unavoidable on A. Hint: a binary morphic word contains arbitrarily long squares.
2. Show that the pattern $\alpha\beta\alpha\alpha\beta\alpha$ is D0L-unavoidable on A. Hint: a binary morphic word is either cube-free or contains arbitrarily long cubes.
3. More generally, show that for all $n \geq 2$, the pattern $Z_n Z_n$, i.e. the square of the sesquipower of order n, is avoidable on A but is D0L-unavoidable on A.

**3.1.5 Prove or disprove that, if p is avoidable on A, there exist an alphabet B (possibly larger than A), two morphisms $f: B^* \to B^*$ and $g: B^* \to A^*$, and a letter $a \in B$ such that the infinite word $g(f^\omega(a))$ is well-defined and avoids p.

Section 3.2

3.2.1 Show that a pattern is unavoidable if and only if it divides some sesquipower Z_n. Hint: if p reduces to q in one step, and q divides Z_n, then p divides Z_{n+1}.

*3.2.2 A pattern p such that $AG(p)$ is a connected graph is called a *locked pattern*. Show that all locked patterns are 4-avoidable. Hint: modify Lemma 3.2.7 to prove that $w^{(1)}$ avoids locked patterns, using the fact that a variable ξ of p such that $|h(\xi)|$ is odd would constitute a free set, and that consequently only two different letters can occur at the beginning of all $h(\xi)$, so that decisive words can be found.

3.2.3 Given a pattern $p = \xi_1\xi_2 \ldots \xi_m$ and $k \leq m$, the *k-chop* if p is the pattern

$$(\xi_1\xi_2 \ldots \xi_k)\zeta_1(\xi_2\xi_3 \ldots \xi_{k+1})\zeta_2 \ldots \zeta_{m-k}(\xi_{m-k+1}\xi_{m-k+2} \ldots \xi_m)$$

where ζ_1, ..., ζ_{m-k} are new variables, or equivalently the simple formula (see Problem 3.1.2)

$$\xi_1\xi_2 \ldots \xi_k \cdot \xi_2\xi_3 \ldots \xi_{k+1} \cdot \ldots \cdot \xi_{m-k+1}\xi_{m-k+2} \ldots \xi_m$$

formed of all factors of length k of p. Show that the 2-chop of a pattern is either 4-avoidable or unavoidable. Hint: if the 2-chop of p is irreducible, then the reduction algorithm stops on a locked 2-chop (see Problem 3.2.2), which is 4-avoidable and divides the 2-chop of p. What prevents this proof from extending to k-chops with arbitrary k?

Section 3.3

3.3.1 Construct a family of 2-unavoidable patterns R_n on n variables, similar to sesquipowers, such that $|R_n| = 3.2^{n-1} - 1$. Conclude that $\forall n \geq 1, \ell_{n2} \geq 3.2^{n-1}$.

**3.3.2 Show that $\forall n \geq 1, \ell_{n2} = 3.2^{n-1}$ and $\forall n \geq 1, \forall k \geq 3, \ell_{nk} = 2^n$.

3.3.3 Given a positive integer l, we say that a word u contains an *l-occurrence* of p if it contains a factor $h(p)$ where $|h(\xi)| \geq l$ for all $\xi \in E$. The pattern p is said to be *weakly k-avoidable* if there exist an integer l and an infinite word on k letters without l-occurrences of p. Show that every avoidable pattern is weakly 2-avoidable.

3.3.4 Show that the pattern $\alpha\beta\zeta_1\beta\gamma\zeta_2\gamma\alpha\zeta_3\beta\alpha\zeta_4\alpha\gamma$ has avoidability index 4. Hint: this pattern is locked (see Problems 3.2.2 and 3.2.3) and the equivalent simple formula $\alpha\beta \cdot \beta\gamma \cdot \gamma\alpha \cdot \beta\alpha \cdot \alpha\gamma$ (see Problem 3.1.2) is 3-unavoidable.

**3.3.5 Prove or disprove that all avoidable patterns have avoidability index at most 4.

Notes

Finding infinite words that avoid repetitions (mainly squares or cubes) is an old problem, that can be traced back to Thue (1906, 1912), and was rediscovered or studied by many authors, including Adian (1979), Aršon (1937), Berstel (1979a, 1984), Dean (1965), Dekking (1976), Entringer, Jackson, and Schatz (1974), Evdokimov (1968), Hawkins and Mientka (1956), Istrail (1977), Leech (1957), Morse and Hedlund (1944), Pleasants (1970), Salomaa (1981), Shyr (1977), Zech (1958).

The present notion of pattern was introduced independently by Bean, Ehrenfeucht, and McNulty (1979) and Zimin (1979, 1982). We adopt here the vocabulary of Bean et al. Zimin calls *blocking term* what we call *unavoidable pattern*, and *σ-deletion of variables* what we call *deletion of a free set*. Theorem 3.2.1

and most lemmas in its proof are taken from Zimin (Lemmas 3.2.2, 3.2.3, 3.2.4, and 3.2.5 are Zimin's Lemmas 4, 6, 7 and 8). Zimin also introduced sesquipowers as a family of unavoidable patterns.

Baker, McNulty, and Taylor (1989) introduced the adjacency graph and free sets, that allow a nice presentation of pattern reduction. They defined locked patterns (see Problem 3.2.2) and gave the first example of an avoidable pattern which is not avoidable on a ternary alphabet (see Problem 3.3.4). They also gave a first linear bound on the avoidability index of patterns with a given number of variables (non-linear bounds can be derived from the proofs of Zimin and Bean et al., but they did not make them explicit). The bound we give here in Theorem 3.3.4 was found by Mel'ničuk (an unpublished paper communicated by P. Goralčik). We adapted Zimin's proof to make use of Mel'ničuk's construction, slightly modified, with the help of notes from lectures of Goralčik given at LITP, Paris in 1992 and Volkov in a talk given at Marquette University, Milwaukee in 1991.

The classification of binary patterns presented in Section 3.3.2 was started by Schmidt (1986, 1989), who proved that binary patterns of length 13 are 2-avoidable. The bound was reduced to the optimal value 6 by Roth (1992) and the classification completed by Cassaigne (1993b, 1994b). Vaniček (1989) independently established the classification (see also Goralčik and Vaniček 1991). A similar classification for ternary patterns was started by Nilgens (1991) and continued (with semi-automatized proofs of avoidability) by Cassaigne (1994b), but is not yet complete.

The bound in Theorem 3.3.5 was found by Cassaigne and Roth (see Cassaigne 1994b).

Formulas (see Problems 3.1.2 and 3.1.3) were defined by Cassaigne (1993b). Their study has not been carried very far. For instance, there is no classification of binary formulas. Avoidability by D0L and HD0L words (see Problems 3.1.4 and 3.1.5) was also studied by Cassaigne (1993b, 1994a). The notion of l-occurrences of patterns (see Problem 3.3.3) was studied by Roth (1991).

A list of open problems on patterns was published by Currie (1993), who offers prizes for the resolution of some of them. None of them seems to have been solved yet. Problems 3.1.5 and 3.3.5 are in that list.

One question we did not include here is the study of the set of words (finite or infinite) avoiding a given pattern : its growth, its topological structure, etc. For references, see e.g. Cassaigne (1993a) and Currie (1993).