DESCENT REPRESENTATIONS AND MULTIVARIATE STATISTICS (EXTENDED ABSTRACT)

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ABSTRACT. Combinatorial identities on Weyl groups of types A and B are derived from special bases of the corresponding coinvariant algebras. Using the Garsia-Stanton descent basis of the coinvariant algebra of type A we give a new construction of the Solomon descent representations. An extension of the descent basis to type B, using new multivariate statistics on the group, yields a refinement of the descent representations. These constructions are then applied to refine well-known decomposition rules of the coinvariant algebra and to generalize various identities.

RÉSUMÉ. Nous démontrons certaines identités combinatoires sur les groupes de Weyl de type A et B à partir de bases spéciales des algèbres coinvariantes associées. En utilisant la base de Garsia-Stanton de l'algèbre coinvariante de type A, nous donnons une nouvelle construction des représentations de descentes de Solomon. Une extension de la base des descentes de type B, utilisant de nouvelles statistiques multivariées sur le groupe, mène à un raffinement des représentations des descentes. Nous appliquons ensuite ces constructions pour raffiner des règles de décomposition bien connues de l'algèbre coinvariante, et pour généraliser diverses identités.

1. Introduction

This paper studies the interplay between representations of classical Weyl groups of types A and B and combinatorial identities on these groups. New combinatorial statistics on these groups are introduced, which lead to a new construction of representations. The Hilbert series which emerge give rise to multivariate identities generalizing known ones.

The set of elements in a Coxeter group having a fixed descent set carries a natural representation of the group, called a descent representation. Descent representations of Weyl groups were first introduced by Solomon [23] as alternating sums of permutation representations. This concept was extended to arbitrary Coxeter groups, using a different construction, by Kazhdan and Lusztig [20] [19, §7.15].

For Weyl groups of type A, these representations also appear in the top homology of certain (Cohen-Macaulay) rank-selected posets [25]. Another description (for type A) is by means of zig-zag diagrams [17]. In [2] we give a new construction of descent representations for Weyl groups of type A, using the coinvariant algebra as a representation space. This viewpoint gives rise to a new extension for type B, which refines the one by Solomon.

The construction of a basis for the coinvariant algebra is important for many applications, and has been approached from different viewpoints. A geometric approach identifies the coinvariant algebra with the cohomology ring $H^*(G/B)$ of the flag variety. This leads to the Schubert basis, and applies to any Weyl group. This identification also appears in Springer's construction of irreducible representations. Barcelo [7] found bases for the resulting quotients. An algebraic approach, applying Young symmetrizers, was used by

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Ariki, Terasoma and Yamada [6] to produce a basis compatible with the decomposition into irreducible representations.

A combinatorial approach, which produces a basis of monomials, was presented by Garsia and Stanton in [15] [13]. They actually presented a basis for a finite dimensional quotient of the Stanley-Reisner ring arising from a finite Weyl group. For type A, unlike other types, this quotient is isomorphic to the coinvariant algebra. The Garsia-Stanton descent basis for type A may be constructed from the coinvariant algebra via a straightening algorithm [5]. Using a reformulation of this algorithm we give a natural construction of Solomon's descent representations as factors of the coinvariant algebra of type A.

An analogue of the descent basis for type B is now given. This analogue (again consisting of monomials) involves extended descent sets and new combinatorial statistics. An extension of the construction of descent representations, using the new basis for type B, gives rise to a family of descent representations, refining Solomon's. A decomposition of these descent representations into irreducibles, refining theorems of Kraskiewicz-Weyman (for type A) [22, Theorem 8.8] and Stembridge (for type B) [30], is carried out using a multivariate version of Stanley's formula for the principal specialization of Schur functions.

This algebraic setting is then applied to obtain new multivariate combinatorial identities. Suitable Hilbert series are computed and compared to each other and to generating functions of multivariate statistics. The resulting identities present a far reaching generalization of bivariate identities from [16], [14], and [1].

2. **Preliminaries**

2.1. **Notations.** Let $\boldsymbol{P} := \{1, 2, 3, \ldots\}$, $\boldsymbol{N} := \boldsymbol{P} \cup \{0\}$, $\boldsymbol{Z}$ be the ring of integers, and $\mathbf{Q}$ be the field of rational numbers; for $a \in \boldsymbol{N}$ let $[a] := \{1, 2, \ldots, a\}$ (where $[0] := \emptyset$). Given $n, m \in \boldsymbol{Z}$, $n \leq m$, let $[n, m] := \{n, n+1, \ldots, m\}$. The cardinality of a set A will be denoted by $|A|$.

Given a variable q and a commutative ring R, denote by $R[q]$ (respectively, $R[[q]]$) the ring of polynomials (respectively, formal power series) in q with coefficients in R. For $i \in \boldsymbol{N}$ let, as customary, $[i]_q := 1 + q + q^2 + \ldots + q^{i-1}$ (so $[0]_q = 0$).

2.2. **Sequences and Permutations.** Let Σ be a linearly ordered alphabet. Given a sequence $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma^n$ we say that a pair $(i, j) \in [n] \times [n]$ is an *inversion* of σ if $i < j$ and $\sigma_i > \sigma_j$. We say that $i \in [n-1]$ is a *descent* of σ if $\sigma_i > \sigma_{i+1}$;

$$Des(\sigma) := \{1 \leq i \leq n-1 \mid \sigma_i > \sigma_{i+1}\}$$

is the *descent set* of σ. Denote by $inv(\sigma)$ (respectively, $des(\sigma)$) the number of inversions (respectively, descents) of σ. We also let

$$maj(\sigma) := \sum_{i \in Des(\sigma)} i$$

and call it the *major index* of σ.

Given a set T let $S(T)$ be the set of all bijections $\pi : T \to T$, and $S_n := S([n])$. For $\pi \in S_n$ write $\pi = \pi_1 \ldots \pi_n$ to mean that $\pi(i) = \pi_i$, for $i = 1, \ldots, n$. Any $\pi \in S_n$ may also be written in *disjoint cycle form* (see, e.g., [26, p.17]), usually omitting the 1-cycles of π. For example, $\pi = 365492187$ may also be written as $\pi = (9, 7, 1, 3, 5)(2, 6)$. Given $\pi, \tau \in S_n$ let $\pi\tau := \pi \circ \tau$ (composition of functions) so that, for example, $(1, 2)(2, 3) = (1, 2, 3)$.

Denote by B_n the group of all bijections σ of the set $[-n, n] \setminus \{0\}$ onto itself such that

$$\sigma(-a) = -\sigma(a)$$

for all $a \in [-n,n] \setminus \{0\}$, with composition as the group operation. This group is usually known as the group of "signed permutations" on $[n]$, or as the *hyperoctahedral group* of rank n. We identify S_n as a subgroup of B_n, and B_n as a subgroup of S_{2n}, in the natural ways.

For $\sigma \in B_n$ write $\sigma = [a_1, \ldots, a_n]$ to mean that $\sigma(i) = a_i$ for $i = 1, \ldots, n$, and (using the natural linear order on $[-n,n] \setminus \{0\}$) let

$$\begin{aligned} inv(\sigma) &:= inv(a_1,\ldots,a_n), & des(\sigma) &:= des(a_1,\ldots,a_n),\\ maj(\sigma) &:= maj(a_1,\ldots,a_n), & Neg(\sigma) &:= \{i \in [n] :\ a_i < 0\},\\ neg(\sigma) &:= |Neg(\sigma)|, & fmaj(\sigma) &:= 2\cdot maj(\sigma) + neg(\sigma). \end{aligned}$$

The statistic *fmaj* was introduced in [3] [4] and further studied in [1].

2.3. **Partitions and Tableaux.** Let n be a nonnegative integer. A *partition* of n is an infinite sequence of nonnegative integers with finitely many nonzero terms $\lambda = (\lambda_1, \lambda_2, \ldots)$, where $\lambda_1 \geq \lambda_2 \geq \ldots$ and $\sum_{i=1}^{\infty} \lambda_i = n$. The sum $\sum \lambda_i = n$ is called the *size* of λ, denoted $|\lambda|$; write also $\lambda \vdash n$. The number of parts of λ, $\ell(\lambda)$, is the maximal j for which $\lambda_j > 0$.

The *dominance* partial order on partitions is defined as follows : For any two partitions μ and λ of the same integer, μ *dominates* λ (denoted $\mu \trianglerighteq \lambda$) if and only if $\sum_{j=1}^{i} \mu_j \geq \sum_{j=1}^{i} \lambda_j$ for all i (and, by assumption, $\sum_{j=1}^{\infty} \mu_j = \sum_{j=1}^{\infty} \lambda_j$).

The subset $\{(i,j) \mid i,j \in \mathbf{P},\ j \leq \lambda_i\}$ of $\mathbf{P}^2$ is called the *Young diagram* of *shape* λ. (i,j) is the *cell* in row i and column j. A *Young tableau* of shape λ is obtained by inserting the integers $1, 2, \ldots, n$ (where $n = |\lambda|$) as *entries* in the cells of the Young diagram of shape λ, allowing no repetitions. A *standard Young tableau* of shape λ is a Young tableau whose entries increase along rows and columns.

A *descent* in a standard Young tableau T is an entry i such that $i+1$ is strictly south (and hence weakly west) of i. Denote the set of all descents in T by $Des(T)$. The *descent number* and the *major index* (for tableaux) are defined as follows :

$$des(T) := \sum_{i \in Des(T)} 1 \ ; \qquad maj(T) := \sum_{i \in Des(T)} i.$$

A *bipartition* of n is a pair (λ^1, λ^2) of partitions of total size $|\lambda^1| + |\lambda^2| = n$. A skew diagram of shape (λ^1, λ^2) is a disjoint union of a diagram of shape λ^1 and a diagram of shape λ^2, where the second diagram lies southwest of the first. A standard Young tableau $T = (T^1, T^2)$ of shape (λ^1, λ^2) is obtained by inserting the integers $1, 2, \ldots, n$ as entries in the cells, such that the entries increase along rows and columns. The descent set $Des(T)$, the descent number $des(T)$, and the major index $maj(T)$ of T are defined as above. The *negative set*, $Neg(T)$, of such a tableau T is the set of entries in the cells of λ^2. Define $neg(T) := |Neg(T)|$ and

$$fmaj(T) := 2 \cdot maj(T) + neg(T).$$

Example 1. Let T be

$$\begin{array}{ccccc} & & 2 & 5 & 6 \\ & & 3 & & \\ 1 & 7 & & & \\ 4 & 9 & & & \\ 8 & & & & \end{array}$$

T is a standard Young tableau of shape $((3,1),(2,2,1))$, $Des(T) = \{2,3,6,7\}$, $des(T) = 4$, $maj(T) = 18$, $Neg(T) = \{1,4,7,8,9\}$, $neg(T) = 5$ and $fmaj(T) = 41$.

Denote by $SYT(\lambda)$ the set of all standard Young tableaux of shape λ, and by $SYT(\lambda^1, \lambda^2)$ the set of all standard Young tableaux of shape (λ^1, λ^2).

2.4. **The Coinvariant Algebra.** The groups S_n and B_n have natural actions on the ring of polynomials P_n (cf. [19, §3.1]). S_n acts by permuting the variables, and B_n acts by permuting the variables and multiplying by ± 1. The ring of S_n-invariant polynomials is Λ_n, the ring of symmetric functions in $x_1, \ldots, x_n$. Similarly, the ring of B_n-invariant polynomials is Λ_n^B, the ring of symmetric functions in $x_1^2, \ldots, x_n^2$. Let I_n, I_n^B be the ideals of P_n generated by the elements of Λ_n, Λ_n^B (respectively) without constant term. The quotient P_n/I_n (P_n/I_n^B) is called the *coinvariant algebra* of S_n (B_n). Each group acts naturally on its coinvariant algebra. The resulting representation is isomorphic to the regular representation. See, e.g., [19, §3.6] and [18, §II.3].

Let R_k $(0 \le k \le \binom{n}{2})$ be the k-th homogeneous component of the coinvariant algebra of S_n: $P_n/I_n = \oplus_k R_k$. Each R_k is an S_n-module. The following theorem is attributed by Reutenauer to Kraskiewicz and Weyman [22, p. 215].

Kraskiewicz-Weyman Theorem. [22, Theorem 8.8] *For any $0 \le k \le \binom{n}{2}$ and $\mu \vdash n$, the multiplicity in R_k of the irreducible S_n-representation corresponding to μ is*

$$m_{k,\mu} = |\ \{\ T \in SYT(\mu) \mid maj(T) = k\ \}\ |.$$

The following B-analogue (in different terminology) was proved in [30]. Here R_k^B is the k-th homogeneous component of the coinvariant algebra of B_n.

Stembridge's Theorem. *For any $0 \le k \le n^2$ and bipartition (μ^1, μ^2) of n, the multiplicity in R_k^B of the irreducible B_n-representation corresponding to (μ^1, μ^2) is*

$$m_{k,\mu^1,\mu^2} = |\ \{\ T \in SYT(\mu^1, \mu^2) \mid fmaj(T) = k\ \}\ |.$$

3. Main Results

For any partition λ with (at most) n parts, let $P_\lambda^{\trianglelefteq}$ be the subspace of the polynomial ring $P_n = \mathbf{Q}[x_1, \ldots, x_n]$ spanned by all monomials whose exponent partition is dominated by λ, and let R_λ be a distinguished quotient of the image of $P_\lambda^{\trianglelefteq}$ under the projection of P_n onto the coinvariant algebra. We will show that the homogeneous components of the coinvariant algebra decompose as direct sums of certain R_λ-s. This will be done using an explicit construction of a basis for R_λ. The construction of this basis involves combinatorial statistics.

3.1. **New Statistics.** Let Σ be a linearly ordered alphabet. For any finite sequence $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n$ of letters in Σ define

$$Des(\sigma) := \{i \mid \sigma_i > \sigma_{i+1}\},$$

the descent set of σ, and

$$d_i(\sigma) := |\{j \in Des(\sigma) :\ j \ge i\}|,$$

the number of descents in σ from position i on.

If Σ consists of integers, let

$$Neg(\sigma) := \{i \mid \sigma_i < 0\};$$

$$n_i(\sigma) := |\{j \in Neg(\sigma) :\ j \ge i\}|; \qquad \varepsilon_i(\sigma) := \begin{cases} 1, & \text{if } \sigma_i < 0, \\ 0, & \text{otherwise;} \end{cases}$$

and

$$f_i(\sigma) := 2d_i(\sigma) + \varepsilon_i(\sigma).$$

The statistics $f_i(\sigma)$ refine the flag-major index $fmaj(\sigma)$, which was introduced and studied in [3] [4] [1].

3.2. **The Garsia-Stanton Descent Basis and its Extension.** To any $\pi \in S_n$ Garsia and Stanton [15] associated the monomial

$$a_\pi := \prod_{i \in Des(\pi)} (x_{\pi(1)} \cdots x_{\pi(i)}).$$

It should be noted that in our notation $a_\pi = \prod_{i=1}^{n} x_{\pi(i)}^{d_i(\pi)}$. Using Stanley-Reisner rings Garsia and Stanton showed that the set $\{a_\pi + I_n \mid \pi \in S_n\}$ forms a basis for the coinvariant algebra of type A [15]. This basis will be called the *descent basis*. The Garsia-Stanton approach is not applicable to the coinvariant algebras of other Weyl groups. In [2] we extend the descent basis to the Weyl groups of type B.

To any $\sigma \in B_n$ we associate the monomial

$$b_\sigma := \prod_{i=1}^{n} x_{|\sigma(i)|}^{f_i(\sigma)}.$$

Theorem 3.1. *The set*

$$\{b_\sigma + I_n^B \mid \sigma \in B_n\}$$

forms a basis for the coinvariant algebra of type B.

3.3. **Descent Representations.** For a monomial m in the polynomial ring $P_n = \mathbf{Q}[x_1, \ldots, x_n]$, let the exponent partition $\lambda(m)$ be the partition obtained by rearranging the exponents in a weakly decreasing order. For any partition λ with at most n parts, let $P_\lambda^{\trianglelefteq}$ be the subspace of P_n spanned by all monomials whose exponent partition is dominated by λ :

$$P_\lambda^{\trianglelefteq} := span_{\mathbf{Q}}\{m \mid \lambda(m) \trianglelefteq \lambda\}.$$

Similarly, define $P_\lambda^{\triangleleft}$ by strict dominance :

$$P_\lambda^{\triangleleft} := span_{\mathbf{Q}}\{m \mid \lambda(m) \triangleleft \lambda\}.$$

Consider now the canonical projection of P_n onto the coinvariant algebra

$$\psi : P_n \longrightarrow P_n/I_n.$$

Define R_λ to be a quotient of images under this map :

$$R_\lambda := \psi(P_\lambda^{\trianglelefteq})/\psi(P_\lambda^{\triangleleft}).$$

Then R_λ is an S_n-module.

Lemma 3.2. $R_\lambda \neq 0$ *if and only if* $\lambda = \lambda(a_\pi)$ *for some* $\pi \in S_n$.

For any subset $S \subseteq \{1, \ldots, n\}$ define a partition

$$\lambda_S := (\lambda_1, \ldots, \lambda_n)$$

by

$$\lambda_i := |\, S \cap \{i, \ldots, n\}\, |.$$

Using a straightening algorithm for the descent basis it is shown that $R_\lambda \neq 0$ if and only if $\lambda = \lambda_S$ for some $S \subseteq [n-1]$, and that a basis for R_{λ_S} may be indexed by the permutations with descent set equal to S. Let R_k be the k-th homogeneous component of the coinvariant algebra P_n/I_n.

Theorem 3.3. *For every* $0 \le k \le \binom{n}{2}$,

$$R_k \cong \bigoplus_S R_{\lambda_S}$$

as S_n*-modules, where the sum is over all subsets* $S \subseteq [n-1]$ *such that* $\sum\limits_{i \in S} i = k$.

Let

$$R^B_\lambda := \psi^B(P^{\trianglelefteq}_\lambda) \;/\; \psi^B(P^{\triangleleft}_\lambda),$$

where $\psi^B : P_n \longrightarrow P_n/I^B_n$ is the canonical map from P_n onto the coinvariant algebra of type B.

Lemma 3.4. $R^B_\lambda \neq 0$ *if and only if* $\lambda = \lambda(b_\sigma)$ *for some* $\sigma \in B_n$.

For subsets $S_1 \subseteq [n-1]$ and $S_2 \subseteq [n]$, let λ_{S_1,S_2} be the vector

$$\lambda_{S_1,S_2} := 2\lambda_{S_1} + \mathbf{1}_{S_2},$$

where λ_{S_1} is as above and $\mathbf{1}_{S_2} \in \{0,1\}^n$ is the characteristic vector of S_2. This is not always a partition. Indeed, for a partition λ, $R^B_\lambda \neq 0$ if and only if $\lambda = \lambda_{S_1,S_2}$ for some $S_1 \subseteq [n-1], S_2 \subseteq [n]$. In this case, a basis for $R_{\lambda_{S_1,S_2}}$ may be indexed by the signed permutations $\sigma \in B_n$ with $Des(\sigma) = S_1$ and $Neg(\sigma) = S_2$.

Let R^B_k for the k-th homogeneous component of the coinvariant algebra of type B. The following theorem is a B-analogue of Theorem 3.3.

Theorem 3.5. *For every* $0 \le k \le n^2$,

$$R^B_k \cong \bigoplus_{S_1,S_2} R_{\lambda_{S_1,S_2}}$$

as B_n*-modules, where the sum is over all subsets* $S_1 \subseteq [n-1]$ *and* $S_2 \subseteq [n]$ *such that* λ_{S_1,S_2} *is a partition and*

$$2 \cdot \sum_{i \in S_1} i + |S_2| = k.$$

3.4. Decomposition into Irreducibles.

Theorem 3.6. *For any subset* $S \subseteq [n-1]$ *and partition* $\mu \vdash n$*, the multiplicity in* R_{λ_S} *of the irreducible* S_n*-representation corresponding to* μ *is*

$$m_{S,\mu} := |\{ T \in SYT(\mu) \mid Des(T) = S \}|,$$

the number of standard Young tableaux of shape μ *and descent set* S.

This theorem implies the well known Kraskiewicz-Weyman theorem for decomposing the homogeneous components of the coinvariant algebra into irreducibles. See Subsection 2.4.

For type B we have

Theorem 3.7. *For any pair of subsets* $S_1 \subseteq [n-1]$, $S_2 \subseteq [n]$*, and a bipartition* (μ^1, μ^2) *of* n*, the multiplicity of the irreducible* B_n*-representation corresponding to* (μ^1, μ^2) *in* $R_{\lambda_{S_1,S_2}}$ *is*

$$m_{S_1,S_2,\mu^1,\mu^2} := |\{ T \in SYT(\mu^1, \mu^2) \mid Des(T) = S_1 \text{ and } Neg(T) = S_2 \}|,$$

the number of pairs of standard Young tableaux of shapes μ^1 *and* μ^2 *with descent set* S_1 *and sets of entries* $[n] \setminus S_2$ *and* S_2*, respectively.*

The proofs apply multivariate extensions of Stanley's formula for the principal specialization of a Schur function [27, Prop. 7.19.11] to obtain the graded character.

3.5. **Combinatorial Identities.** For any partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with at most n parts define

$$m_j(\lambda) := |\ \{1 \le i \le n \mid \lambda_i = j\}\ | \qquad (\forall j \ge 0).$$

By considering Hilbert series of the polynomial ring with respect to rearranged multi-degree and applying the Straightening Lemma for the coinvariant algebra of type A we obtain

Theorem 3.8. *For any positive integer n*

$$\sum_{\ell(\lambda)\le n} \binom{n}{m_0(\lambda), m_1(\lambda), \ldots} \prod_{i=1}^{n} q_i^{\lambda_i} = \frac{\sum_{\pi\in S_n} \prod_{i=1}^{n} q_i^{d_i(\pi)}}{\prod_{i=1}^{n}(1 - q_1 \cdots q_i)}$$

in $\boldsymbol{Z}[[q_1, \ldots, q_n]]$, where the sum on the left-hand side is taken over all partitions with at most n parts.

Note that the multinomial coefficient in the theorem is the number of monomials with exponent partition λ. Taking $q_1 = qt$ and $q_2 = \ldots = q_n = q$ yields the following well known result (attributed by Garsia [14] to Gessel [16]; see also [10]).

Corollary 3.9. *Let $n \in \boldsymbol{P}$. Then*

$$\frac{\sum_{\pi\in S_n} t^{des(\pi)} q^{maj(\pi)}}{\prod_{i=0}^{n}(1 - tq^i)} = \sum_{r\ge 0} [r+1]_q^n t^r.$$

in $\boldsymbol{Z}[q][[t]]$.

The Hilbert series of P_n rearranged by multi-degree may be computed in a different way, by considering the signed descent basis for the coinvariant algebra of type B and applying the Straightening Lemma for this type.

Theorem 3.10. *With notations as in Theorem 3.8*

$$\sum_{\ell(\lambda)\le n} \binom{n}{m_0(\lambda), m_1(\lambda), \ldots} \prod_{i=1}^{n} q_i^{\lambda_i} = \frac{\sum_{\sigma\in B_n} \prod_{i=1}^{n} q_i^{f_i(\sigma)}}{\prod_{i=1}^{n}(1 - q_1^2 \cdots q_i^2)}$$

in $\boldsymbol{Z}[[q_1, \ldots, q_n]]$, where the sum on the left-hand side runs through all partitions with at most n parts.

The main combinatorial result for type B is a far reaching generalization of [1, Corollary 4.5].

Theorem 3.11. *For any positive integer n*

$$\sum_{\sigma\in B_n} \prod_{i=1}^{n} q_i^{d_i(\sigma)+n_i(\sigma^{-1})} = \sum_{\sigma\in B_n} \prod_{i=1}^{n} q_i^{2d_i(\sigma)+\varepsilon_i(\sigma)}.$$

For further identities see [2, Section 6].

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