# Algebraic constructions on set partitions

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Abstract. The main contribution of this article is to provide a combinatorial Hopf algebra on set partitions, which can be seen as a Hopf subalgebra of the free quasi-symmetric functions, and which is free, cofree and self-dual. This algebraic construction comes naturally from a combinatorial algorithm, an analogous of the Robinson-Schensted correspondence on set partitions. This correspondence also gives us a monoid structure analogous to the plactic monoid, whose product can be described combinatorically with an analogous of the jeu de taquin on set partitions. We also introduce a new partial order on set partitions. We explain how these objects are related to each others through our Hopf algebra.

Résumé. La principale contribution de cet article est de fournir une algèbre de Hopf combinatoire sur les partitions ensemblistes, qui peut être vue comme une sous-algèbre de Hopf des fonctions quasi-symétriques libres, et qui est libre, co-libre et auto-duale. Cette construction algébrique provient naturellement d'un algorithme combinatoire qui est un analogue de la correspondance de Robinson-Schensted sur les partitions ensemblistes. Cette correspondance nous fournit également une structure de monoïde analogue  celle du monoïde plaxique, dont le produit peut être décrit de manière combinatoire avec un analogue du jeu de taquin sur les partitions ensemblistes. Nous introduisons également un nouvel ordre partiel sur les partitions ensemblistes. Nous expliquons comment ces objets sont reliés les uns aux autres au travers de notre algèbre de Hopf.

## 1. Introduction

In the past years several authors mixed Hopf algebras, partial orders, and combinatorial algorithms. Such a construction has been made on Young tableaux by Poirier and Reutenauer [**9**] for the Hopf algebra part, thus Reiner and Taskin [**10**] for the connection with partial orders, and with the Robinson-Schensted correspondence as combinatorial algorithm. A similar construction has also been made on binary trees by Hivert, Novelli and Thibon [**6**] with the Hopf algebra of Loday and Ronco [**8**], involving the Tamari order (see [**12**]) and the binary search tree insertion as combinatorial algorithm. It also exists a jeu de taquin-like on binary trees that match the product of the Loday-Ronco algebra in some unpublished work of Viennot.

The main contribution of this article is to provide a combinatorial Hopf algebra on set partitions. Several authors [**11**], [**1**], have studied a Hopf algebra on set partitions but their algebra is cocommutative and our algebra is noncommutative, non cocommutative and self-dual. This algebraic construction comes naturally from a simple combinatorial algorithm, an analogous of the Robinson-Schensted correspondence on set partitions, also studied by Burnstein and Lankham in [**2**]. This correspondence gives us a monoid structure analogous to the plactic monoid (see [**7**]), whose product can be described combinatorically with an analogous of the jeu de taquin on set partitions. We also introduce the *Bell order* on set partitions. This partial order admits the Tamari order as suborder. All these tools allow us to build a Hopf algebra on set partitions which can be seen as a Hopf subalgebra of the free quasi-symmetric functions. The product of this algebra can be described with the jeu de taquin on set partitions, as well as specific intervals of the Bell order on set partitions. It can be proved that this Hopf algebra is free, cofree and self-dual, thanks to the bidendriform bialgebra structures defined by Foissy [**5**].

2000 *Mathematics Subject Classification.* Primary 05-06; Secondary 05E99.

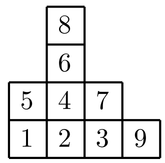
*Key words and phrases.* Algebraic combinatorics, Robinson-Schensted correspondence like, combinatorial Hopf algebra, partial order, jeu de taquin like, monoid, dendriform dialgebras.

This paper hence reinforce the belief of a more general theory involving such mathematical objects.

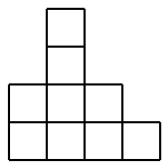
In Section 2 we provide a Robinson-Schensted-like defined on words and involving set partitions in the special case of permutations. Then, in Section 3 we introduce the Bell monoid and the jeu de taquin on set partition. Section 4 is mainly devoted to a new partial order on set partitions. At last Setion 5 provide the announced Hopf algebra on set partitions and state its properties.

## 2. A Robinson-Schensted-like correspondence

Throughout this article, the notation $[n]$ represent the set $\{1, 2, 3, \ldots, n\}$. We say then that $S \subseteq 2^{[n]} \backslash \{\emptyset\}$ with $S = \{S_1, S_2, \ldots, S_k\}$ is a set partition of $n$ (denoted $S \vdash [n]$) if $S_i \cap S_j = \emptyset$ for $i \neq j$ and $S_1 \cup S_2 \cup \cdots \cup S_k = [n]$. The set partition $S$ is a sub set partition of $T$ if all $S_i$ are included in some $T_j$. A set partition can be represented in a canonical way by ordering sets according to their minimal elements and ordering the sets themselves according to the natural order on integers. Then, $\{\{5,1\},\{8,6,4,2\},\{7,3\},\{9\}\} \vdash [9]$, sometimes also written $\{5,1\,|\,8,6,4,2\,|\,7,3\,|\,9\}$ is its canonical representation and can be drawn in the plane as a set of boxes like that


.

Let the *shape* of a set partition $\zeta$ of $n$, with $k$ columns or blocks, is its underlying composition, that is the vector $(|c_1|, |c_2|, \ldots, |c_k|) \models n$ where the $c_i$ are the columns of $\zeta$. For example, the shape of $\{\{5,1\},\{8,6,4,2\},\{7,3\},\{9\}\}$ is $(2,4,2,1)$ whose composition diagram is graphically represented as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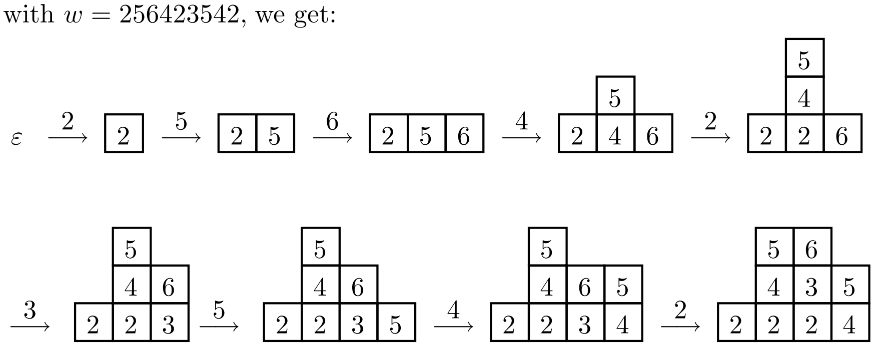
We now give an algorithm defined on words, related to the Patience Sorting algorithm of [**2**], that produces a set partition from a permutation and in all generality produces a lexicographic set partition. A *lexicographic set partition* or *l.s.p.* for short, that is a composition diagram labelled by integers weakly increasing on the bottom line from left to right and strictly increasing on columns from bottom to top. The set of l.s.p. will be denoted $LSP$.

ALGORITHM 2.1. The input of this algorithm is a word $w = w_1 w_2 \cdots w_n$. The output is a l.s.p..

(1) $r \leftarrow \varepsilon$; $l \leftarrow w_1$;
(2)
- If $l$ is greater than the smallest element of the right-most column of $r$, then create a new column with one box labelled by $l$, at the right-most position of $r$ and go to Step 3;
- Otherwise, insert in $r$ a box labelled by $l$ under the left-most column of $r$ whose smallest element is strictly greater than $l$ and go to Step 3.

(3) $l$ becomes the next letter of $w$ and go back to Step 2 or ends if $l$ is already the last letter of $w$.

For example, with $w = 256423542$, we get: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In the case of the permutation $\sigma = 586714923$, we have $P(\sigma) = \{\{5,1\},\{8,6,4,2\},\{7,3\},\{9\}\}$. For sake of clarity, we will indifferently write $P(\sigma) = 51 \cdot 8642 \cdot 73 \cdot 9$, and consider $P(\sigma)$ as a permutation. Moreover, by the way we construct $P(\sigma)$, it is clear that we can associate a second set partition $Q(\sigma)$ that records when each box was created. In our example,

$$P(\sigma) = \begin{array}{|c|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & 8 & \multicolumn{2}{c}{} \\ \cline{2-2} \multicolumn{1}{c|}{} & 6 & \multicolumn{2}{c}{} \\ \cline{1-3} 5 & 4 & 7 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 & 9 \\ \hline \end{array} \quad , \quad Q(\sigma) = \begin{array}{|c|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & 8 & \multicolumn{2}{c}{} \\ \cline{2-2} \multicolumn{1}{c|}{} & 6 & \multicolumn{2}{c}{} \\ \cline{1-3} 5 & 3 & 9 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 & 7 \\ \hline \end{array} \quad .$$

Remark 2.2. For every permutation $\sigma$, $P(\sigma) = Q(\sigma^{-1})$. Although this map shares some properties with the RSK algorithm, it is *not* bijective. In other words, pairs of set partitions of same shape are not in bijection with permutations, as a simple counting argument can show.

Definition 2.3. Let $u, v$ be two words. We write $u \equiv v$, if $P(u) = P(v)$. The equivalence classes induced by this equivalence relation are called the *l.s.p. classes*. The l.s.p. class of a permutation is also called a *Bell class*.

Then $P(\sigma)$ can be considered as the representative permutation of its Bell class. There is yet another way to describe the Bell classes, connected to Algorithm 2.1. Indeed, columns of set partitions can be geometrically interpreted on permutations as Viennot's shadow lines, as shown on Figure 1. Then, Bell classes can be defined as permutations whose shadow lines share the same $y$-coordinates. It results that a Bell class can be obtained by horizontally streching shadow lines of a given set partition belonging to this Bell class.

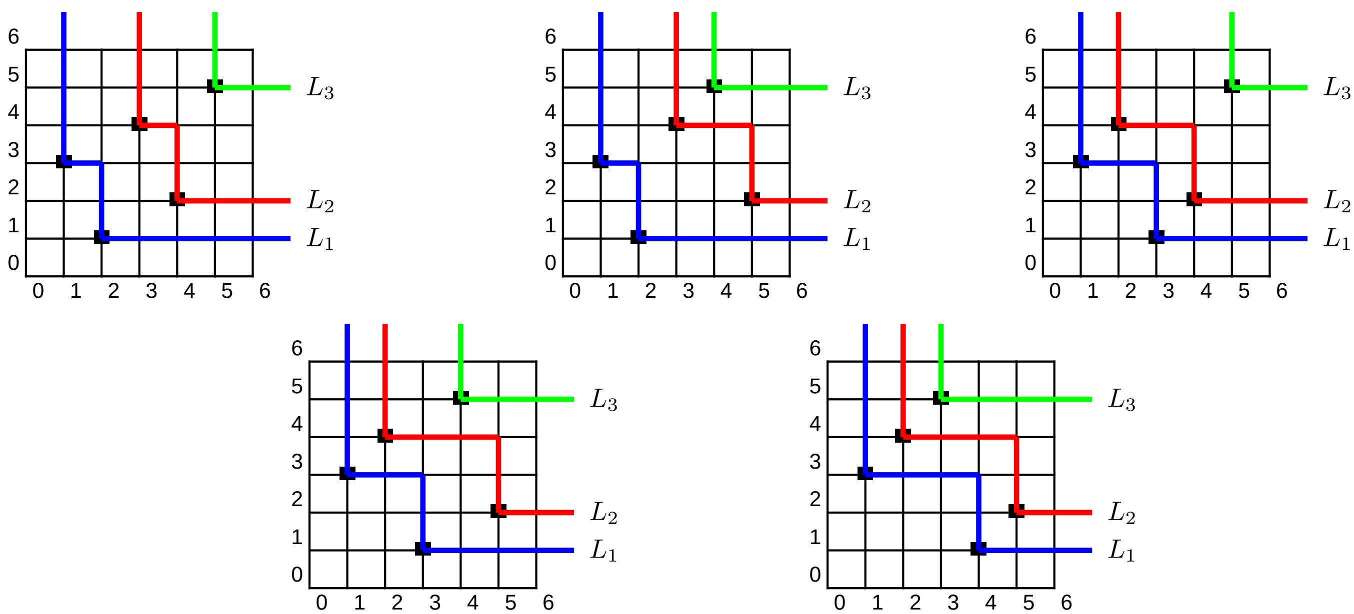



Figure 1. The five elements 31425, 31452, 34235, 34152 and 34512, of the Bell class of $\{\{3,1\},\{4,2\},\{5\}\}$, represented as Viennot's shadow lines.

## 3. An analogue of the jeu de taquin

**3.1. Monoid and compatibility properties.** We build a monoid similar to the plactic monoid [**7**]. This construction is possible because algorithm 2.1 satisfies some properties, also satisfied by the Robinson-Schensted correspondence. This first theorem states the compatibility of $P$ with the concatenation.

THEOREM 3.1. *Let $u, u', v, v'$ be words. If $u \equiv u'$ and $v \equiv v'$, then $u \cdot v \equiv u' \cdot v'$.*

For example, with $25642 \equiv 25462$ and $3542 \equiv 3254$, one can check that

$$\begin{aligned} P(25642 \cdot 3542) &= P(25462 \cdot 3254)\,, \\ P(25642 \cdot 3254) &= P(25462 \cdot 3542)\,, \\ P(25462 \cdot 3542) &= P(25642 \cdot 3254)\,, \\ P(3542 \cdot 25642) &= P(3254 \cdot 25462)\,, \\ P(3254 \cdot 3254) &= P(3542 \cdot 3542)\,. \end{aligned}$$

It is possible from Theorem 3.1 to build a monoid structure, namely the *Bell monoid*, whose product is defined by

$$\zeta \times \delta := P(\zeta \cdot \delta)\,, \tag{3.1}$$

with $\zeta, \delta$, two arbitrary l.s.p.. Another property of Algorithm 2.1, also shared with the Schensted algorithm, is stated in the following Theorem.

THEOREM 3.2. *Let $w$ be a word. Then,*

$$P(w \cap I) \equiv P(w) \cap I,$$

*for any $I$ interval of $\mathbb{N}$.*

For example,

$$\begin{aligned} &P(586614726 \cap [4,7]) \;=\; P(566476) \;=\; 54 \cdot 6 \cdot 6 \cdot 76\,. \\ &P(P(586614726) \cap [4,7]) \;=\; P(51 \cdot 8642 \cdot 6 \cdot 76 \cap [4,7]) \;=\; P(564676) \;=\; 54 \cdot 6 \cdot 6 \cdot 76\,. \end{aligned}$$

**3.2. Jeu de taquin-like: scrolling.** Let us present a combinatorial version of the product introduced in Section 3.1, similar to the jeu de taquin in the plactic monoid context. Indeed, the Schensted $P$-symbol of a word is obtained by applying the jeu de taquin to its diagonal strip. In our case, we apply the *scroll* to ribbon tableau of the word, as explained in the sequel.

ALGORITHM 3.3. Let $\gamma$ be a *ribbon tableau* that may already have been *scrolled* or not. We perform a *scroll* of $\gamma$ written $s(\gamma)$, as follows.

(1) Consider the box in the highest inner corner of $\gamma$, if it exists.
(2) According to the immediate neighborhood of the current box three cases arise:

(a) $\begin{array}{|c|c} \cline{1-1} b & \\ \hline \circ & \multicolumn{1}{c|}{a} \\ \hline \end{array}$ with $a < b$ or with $b$ the empty box, becomes $\begin{array}{|c|c} \cline{1-1} b & \\ \hline a & \multicolumn{1}{c|}{\circ} \\ \hline \end{array}$ and go back to Step 2.

(b) $\begin{array}{|c|c} \cline{1-1} \star & \\ \cline{1-1} a & \\ \hline \circ & \multicolumn{1}{c|}{b} \\ \hline \end{array}$ with $a \leq b$ or with $b$ the empty box, becomes $\begin{array}{|c|c} \cline{1-1} \star & \\ \hline a & \multicolumn{1}{c|}{b} \\ \hline \end{array}$ and exit.

(c) If $\begin{array}{|c|c} \cline{1-1} \emptyset & \\ \hline \circ & \multicolumn{1}{c|}{\emptyset} \\ \hline \end{array}$, then exit.

In other words, this algorithm moves the highest inner corner of a ribbon tableau out of the ribbon.

Applying Algorithm 3.3 to a l.s.p. $\gamma$ until there is no more inner corners, on gets the *scrolling* $S(\gamma)$. In order to define the final set of the scrolling, a first consideration which is immediate by analysis of rules of Algorithm 3.3, is that the shape of $S(\gamma)$ is a composition diagram.

Moreover a ribbon tableau, by definition, satisfies the weakly increasing condition on its labels in each of its rows, and the increasing condition on its columns. This consideration stands for the initial case of our induction. From Algorithm 3.3, a scroll is the application of a sequence of rules (a) eventually followed by rule (b). Then, since the scroll always starts from the higher inner corner, it yields that,

- labels of each column of a scrolled ribbon tableau are increasing,
- labels of the rows of the lower outline of a scrolled ribbon tableau satisfy the weakly increasing condition.

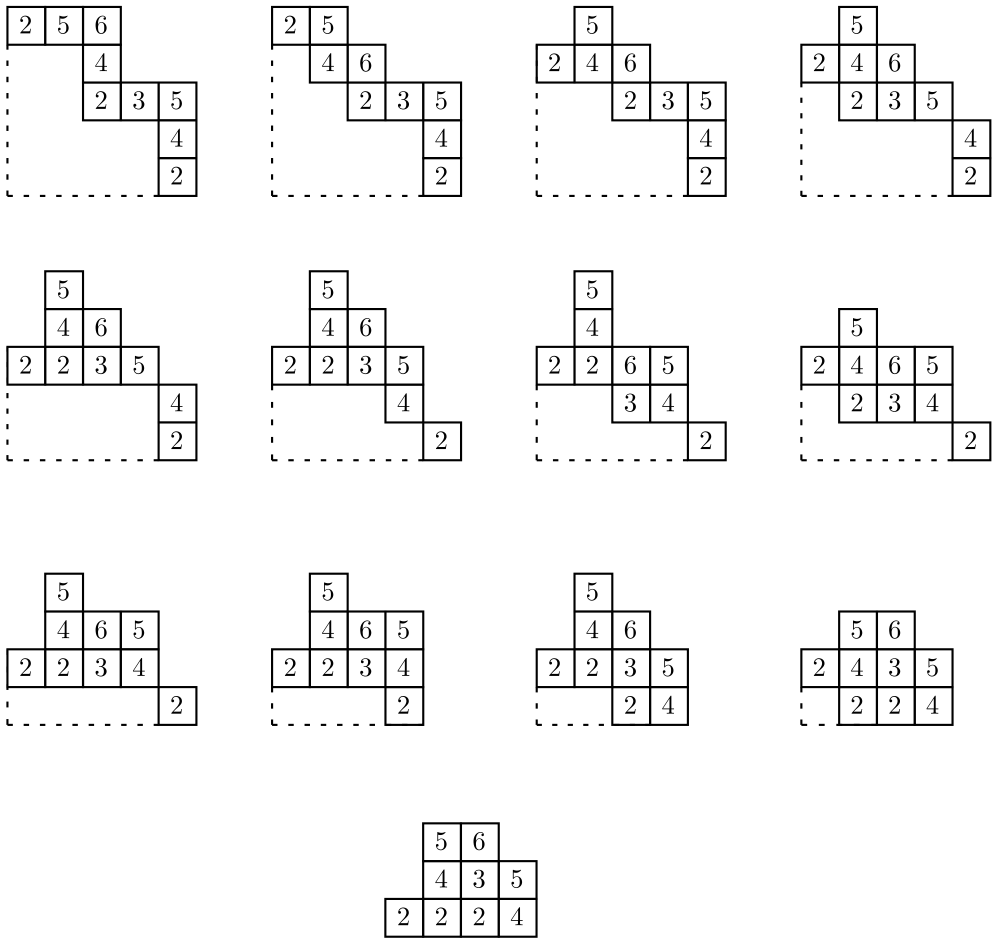
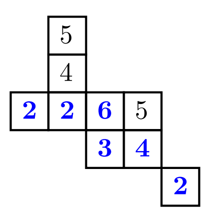


FIGURE 2. This is the scrolling of the word $w = 256423542$, identified to its ribbon tableau; scrolls are made from left to right and from top to bottom. The final result is the same that the computation of $P$ on $w$.

For example, rows of the **lower outline** in this scrolled ribbon tableau (from Figure 2) are weakly increasing but not the full row due to the presence of 5 after 6:

 .

From these considerations it results that for any ribbon tableau $\gamma$, its scrolling $S(\gamma)$ is a l.s.p.. Thus, a scrolled ribbon tableau will be called a *skew l.s.p.*. We give a full example with the scrolling of $w = 256423542$ in Figure 2. In the sequel, we identify a word with its ribbon tableau. Note that $S(256423542)$ is precisely $P(256423542)$. It is true in the general case:

THEOREM 3.4. *Let $w$ be a word. Then, $S(w) = P(w)$.*

If one defines the *reading* $r(\gamma)$ of a skew l.s.p. $\gamma$, as the reading of its columns from top to bottom and from left to right, the Theorem 3.4 comes from the fact that the reading of $\gamma$ is bell equivalent to $s(\gamma)$, and so is $S(\gamma)$.

In the plactic theory, the jeu de taquin gives a combinatorial way to describe the product of two Young tableaux. In our case the scrolling plays an equivalent role. Indeed, Theorem 3.4 provides a combinatorial description of the product of two l.s.p., given in Section 3. Let $\delta, \gamma$ be two l.s.p.. Then, we write $\delta\neg_\gamma$ the skew l.s.p. built with the l.s.p. $\delta$ and the ribbon $\gamma$ such that all boxes of $\delta$ are strictly above and strictly to the left to those of $\gamma$. For example, with $\delta = 2 \cdot 652 \cdot 43 \cdot 76$ and $\gamma = 541 \cdot 321 \cdot 3$, we have:

$$\delta\neg_\gamma := \begin{array}{ccccccc} & 6 & & & & & \\ & 5 & 4 & 7 & & & \\ 2 & 2 & 3 & 6 & & & \\ & & & & 5 & & \\ & & & & 4 & & \\ & & & & 1 & 3 & \\ & & & & & 2 & \\ & & & & & 1 & 3 \end{array} .$$

COROLLARY 3.5. *Let $\delta, \gamma$ be two l.s.p.. Then, $S(\delta\neg_\gamma) = \delta \times \gamma$.*

For example, with $\delta = 2 \cdot 652 \cdot 43 \cdot 76$ and $\gamma = 541 \cdot 321 \cdot 3$, one can check that $P(\delta \cdot \gamma) = 21 \cdot 6521 \cdot 432 \cdot 76543 \cdot 3$ and

$$S(\delta \cdot \gamma) = \begin{array}{ccccc} & & & 7 & \\ & 6 & & 6 & \\ & 5 & 4 & 5 & \\ 2 & 2 & 3 & 4 & \\ 1 & 1 & 2 & 3 & 3 \end{array} .$$

## 4. A partial order on set partitions

We consider the weak order on permutations that defines the so-called permutohedron. This section explains how the Bell classes play a special role on the permutohedron.

THEOREM 4.1. *Bell classes are intervals of the permutohedron.*

Figure 3 shows all Bell classes with $n = 4$. More precisely, if $\sigma$ is a permutation, then $P(\sigma)$ is the unique permutation of its Bell class having the smallest number of inversions. Moreover, in each Bell class, there is also a unique permutation with a maximal number of inversions. This corresponds to the way to strech shadow lines such that each point of shadow lines are at the leftmost possible position, with priority to the points of greater $y$-coordinate. The permutations of a Bell class are those which are in the interval of the permutohedron defined by these two extremal permutations. For example, in Figure 1, the minimal element is 31425 and the maximal element is 34512. It follows naturally from Theorem 4.1 the following Definition.

DEFINITION 4.2. Let $\delta$ and $\zeta$ be two set partitions. We say that $\delta$ is greater than $\zeta$, and write $\delta > \zeta$, if it exists some permutations $\sigma$ with $P(\sigma) = \delta$, and $\pi$ with $P(\pi) = \zeta$, such that $\sigma > \pi$. It is called the *Bell order*.

This order admits a useful combinatorial definition.

PROPOSITION 4.1. *Let $G$ be the graph whose vertices are set partitions of $[1, n]$ and whose edges are defined according to the following operator $O$.*

- *This operator moves an element $b$ of the $(i+1)$-th column of the set partition $\delta = \{\delta_1, \delta_2, ..., \delta_n\}$ to its $i$-th column $\delta_i = \{\delta_i^1, \delta_i^2, ..., \delta_i^k\}$, when $b$ is the greatest element of $\delta_{i+1}$ which belongs to one of the intervals $([\delta_i^j, \delta_i^{j+1}])_{j\in[1,k-1]}$ or $[\delta_i^k, +\infty[$.*

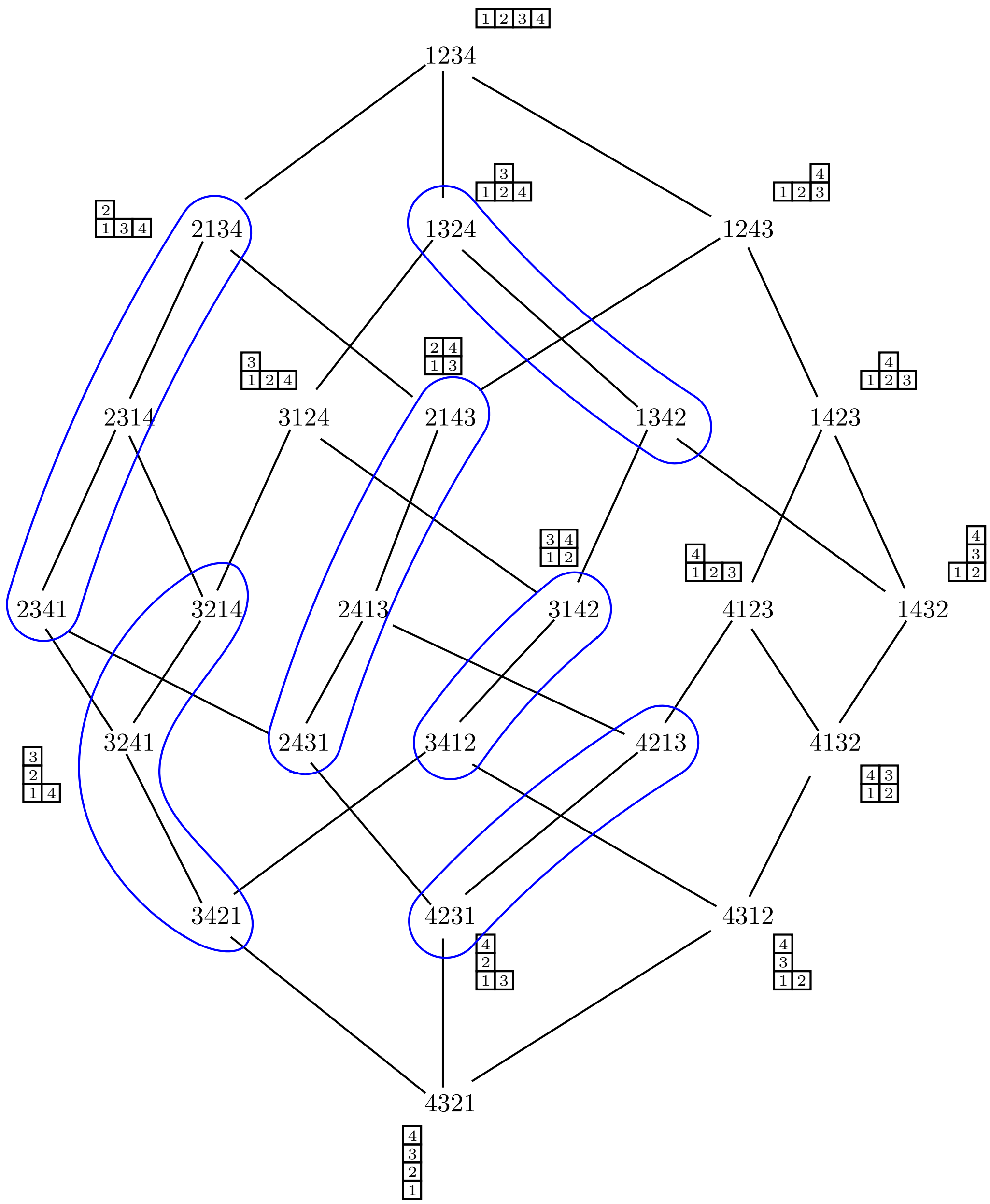
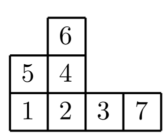



FIGURE 3. Bell classes drawn on the permutohedron of permutations of size 4.

*The order of Definition 4.2 coincide with the partial order induces by this graph, that is the transitive closure of operator $O$.*

For example, consider the following set partition $\delta$:

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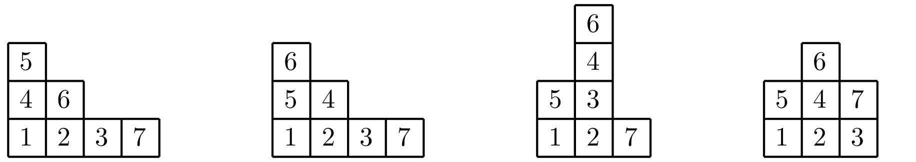
The integers satisfying the condition of Proposition 4.1 are $4, 6, 3$ and $7$. One can check that 2 does not satisfy this condition since $2 \in [1, 5]$ but 4 also belongs to this interval. Then, operator $O$ produces from it these following set partitions:



(A) (B) (C) (D)

This example allows us to see that operator $O$ does *not* compute exactly the immediate successors of order $>$ for set partitions. Indeed, set partition (C) can also be obtained from (D) and then it is not an immediate successor of $\delta$ but of (D). Nevertheless, it is easy to deduce the operator which computes exactly the immediate successors of a given set partition from operator $O$. Indeed, in our previous example, such an event happens because 3 is alone in its column and $7 \notin [2, 4]$. This holds in general.

This order coincides with the Tamari order for permutations of size 3. It is different to it from permutations of size 4, as shown on Figure 3. In this particular case, all Bell classes are the same as Tamari or Sylvester classes (see [**6**]) but for $\{3124, 3142, 3412\}$ that is split in the two parts $\{3124\}$ and $\{3142, 3412\}$. Note that the Bell order on set partitions of size 4 is a lattice. Maybe this property holds in general.

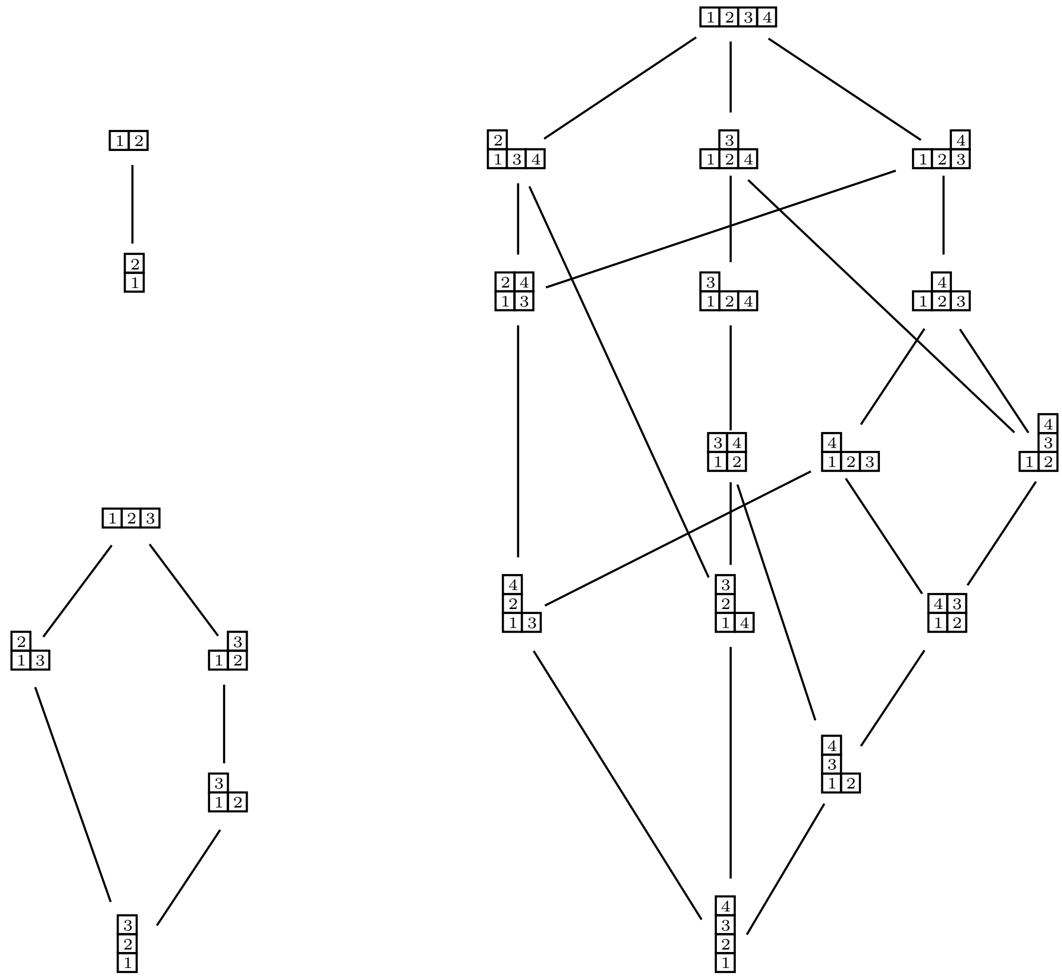


FIGURE 4. Bell order on set partitions of size 2, 3 and 4.

## 5. A self-dual Hopf algebra on set partitions

In this section we show how Bell classes naturally lead to a Hopf subalgebra of **FQSym** . Note that this construction is very similar to those of **PBT**, **FSym**, and **NCSF**. Indeed, in all cases it consists in defining an analogue of the Robinson-Schensted correspondence from which the construction of the Hopf algebra follows naturally.

**5.1. Definition.** We consider the following elements of **FQSym** :

$$\mathbf{P}_\delta := \sum_{P(\sigma)=\delta} \mathbf{F}_\sigma\,, \tag{5.1}$$

for every set partitions $\delta$. For example,

$$\mathbf{P}_{\{\{3,2,1\},\{4\},\{5\}\}} = \mathbf{F}_{32145} + \mathbf{F}_{32415} + \mathbf{F}_{34215} + \mathbf{F}_{32451} + \mathbf{F}_{34251} + \mathbf{F}_{34521}\,.$$

Let us define an associative operation $\mid$ on set partitions $\delta = \{\delta_1, \delta_2, ..., \delta_k\} \vdash [n]$ and $\varsigma = \{\varsigma_1, \varsigma_2, ..., \varsigma_l\} \vdash [m]$. Let $\delta \,|\, \varsigma \vdash [n+m]$ represent $\{\delta_1, \delta_2, ..., \delta_k, \varsigma_1 + n, \varsigma_2 + n, ..., \varsigma_l + n\}$ where $\varsigma + n$ stands for the set $\varsigma$ where each of its element has been increased by $n$. Let $\delta \sqcup \varsigma$ be $P(\varsigma[n] \cdot \delta)$.

THEOREM 5.1. *The elements $(\mathbf{P}_\delta)_\delta$ generate a Hopf subalgebra of* **FQSym**. *We write $\mathfrak{P}$ this Hopf algebra on set partitions.*

*More precisely, let $\delta = \{d_1, d_2, ..., d_m\} \vdash k, \varsigma = \{\varsigma_1, \varsigma_2, ..., \varsigma_n\} \vdash l$ be two set partitions. Then, the product $\mathbf{P}_\delta \mathbf{P}_\varsigma$ is equal to*

$$\sum_\gamma \mathbf{P}_\gamma\,,$$

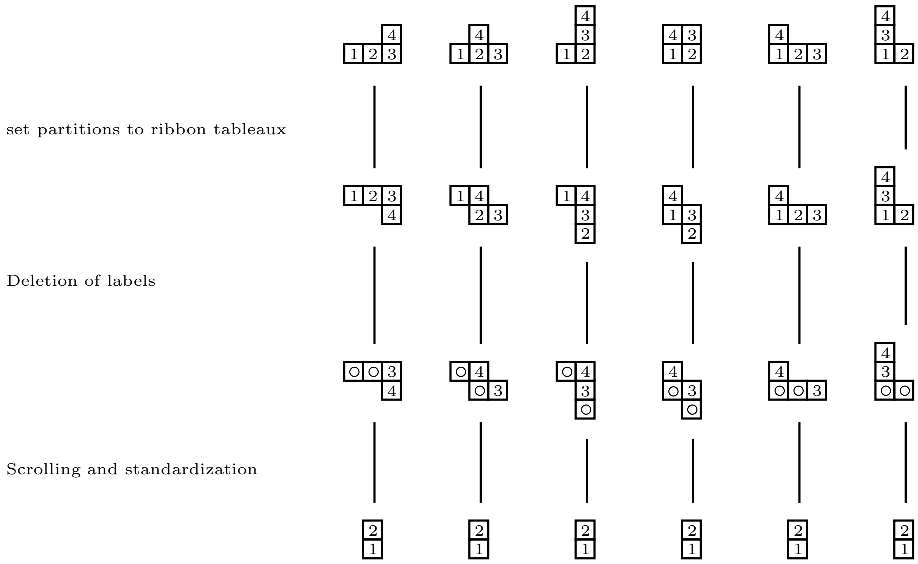
*where $\gamma$ runs over the equivalent set:*

- ***Shuffle:*** *set partitions $P(\sigma)$ that appear in the shuffle of the Bell classes of $\delta$ and $\varsigma$;*
- ***Interval:*** *set partitions greater than or equal to $\delta \,|\, \varsigma$ and smaller than or equal to $\delta \sqcup \varsigma$, according to the Bell order.*

For example,

$$\begin{aligned}
\mathbf{P}_{\{1,3|2\}}\mathbf{P}_{\{1\}} &= \mathbf{P}_{\{1,3,4|2\}} + \mathbf{P}_{\{1,3|2,4\}} + \mathbf{P}_{\{1,3|2|4\}}\,,\\
\mathbf{P}_{\{1|2\}}\mathbf{P}_{\{1|2\}} &= \mathbf{P}_{\{1|2|3|4\}} + \mathbf{P}_{\{1|2,3|4\}} + \mathbf{P}_{\{1,3|2|4\}} + \mathbf{P}_{\{1,3|2,4\}}\,,\\
\mathbf{P}_{\{1|2\}}\mathbf{P}_{\{1,2\}} &= \mathbf{P}_{\{1|2|3,4\}} + \mathbf{P}_{\{1|2,4|3\}} + \mathbf{P}_{\{1|2,3,4\}}\\
&\quad + \mathbf{P}_{\{1,4|2,3\}} + \mathbf{P}_{\{1,4|2|3\}} + \mathbf{P}_{\{1,3,4|2\}}\,,\\
\mathbf{P}_{\{4,1|3,2\}}\mathbf{P}_{\{2,1|3\}} &= \mathbf{P}_{\{1,4|2,3|5,6|7\}} + \mathbf{P}_{\{1,4|2,3,6|5|7\}} + \mathbf{P}_{\{1,4|2,3,6|5,7\}}\\
&\quad + \mathbf{P}_{\{1,4|2,3,5,6|7\}} + \mathbf{P}_{\{1,4,6|2,3|5|7\}} + \mathbf{P}_{\{1,4,6|2,3|5,7\}}\\
&\quad + \mathbf{P}_{\{1,4,6|2,3,7|5\}} + \mathbf{P}_{\{1,4,6|2,3,5|7\}} + \mathbf{P}_{\{1,4,6|2,3,5,7\}}\\
&\quad + \mathbf{P}_{\{1,4,5,6|2,3|7\}} + \mathbf{P}_{\{1,4,5,6|2,3,7\}}\,.
\end{aligned}$$

One can check on Figure 4 that, for example, the set partitions appearing as index in the product $\mathbf{P}_{\{\{1\},\{2\}\}}\mathbf{P}_{\{\{1,2\}\}}$ are exactly the set partitions greater than $\{\{1\},\{2\}\} \,|\, \{\{1,2\}\} = \{\{1\},\{2\},\{3,4\}\}$ and smaller than $\{\{1\},\{2\}\} \sqcup \{\{1,2\}\} = \{\{1,3,4\},\{2\}\}$. Moreover we can also describe the product of $\mathfrak{P}$ thanks to the scrolling. Indeed, considering again the product $\mathbf{P}_{\{\{1\},\{2\}\}}\mathbf{P}_{\{\{1,2\}\}}$ one can check that the set partitions appearing as index of $\mathbf{P}$ elements, are exactly those which have the set partition $\{\{1\},\{2\}\}$ as sub set partition such that if we delete the labels of $\{\{1\},\{2\}\}$ on their associated ribbon tableau and then apply the scrolling, we obtain $\{\{1,2\}\}$ once standardized, as shown below:




This holds in general.

PROPOSITION 5.1. *Let $\delta$ be a set partition. The coproduct of $\mathbf{P}_\delta$ is given by*

$$\Delta(\mathbf{P}_\delta) = \sum_{(\varsigma,\gamma)\in Dec(\delta)} \mathbf{P}_\varsigma \otimes \mathbf{P}_\gamma\,, \tag{5.2}$$

*with $Dec(\delta)$ the set of pairs of set partitions $(\varsigma,\gamma)$ such that $\varsigma$ and $\gamma$ respectively are the standardized words of $u$ and $v$, where $u\cdot v$ is in the Bell class of $\delta$.*

For example,

$$\begin{aligned}\Delta(\mathbf{P}_{\{1,2|3|4,5\}}) = \; & 1\otimes \mathbf{P}_{\{1,2|3|4,5\}} + \mathbf{P}_{\{1\}}\otimes \mathbf{P}_{\{1,2|3,4\}} + \mathbf{P}_{\{1\}}\otimes \mathbf{P}_{\{1|2|3,4\}}\\ & +\mathbf{P}_{\{1,2\}}\otimes \mathbf{P}_{\{1|2,3\}} + \mathbf{P}_{\{1|2\}}\otimes \mathbf{P}_{\{1|2,3\}} + \mathbf{P}_{\{1|2\}}\otimes \mathbf{P}_{\{1,3|2\}}\\ & +\mathbf{P}_{\{1|2\}}\otimes \mathbf{P}_{\{1,2,3\}} + \mathbf{P}_{\{1,2|3\}}\otimes \mathbf{P}_{\{1,2\}} + \mathbf{P}_{\{1|2|3\}}\otimes \mathbf{P}_{\{1,2\}}\\ & +\mathbf{P}_{\{1|2|3\}}\otimes \mathbf{P}_{\{1|2\}} + \mathbf{P}_{\{1,2|3|4\}}\otimes \mathbf{P}_{\{1\}} + \mathbf{P}_{\{1|2|3,4\}}\otimes \mathbf{P}_{\{1\}}\\ & +\mathbf{P}_{\{1,2|3|4,5\}}\otimes 1\,.\end{aligned}$$

**5.2. Properties of the Hopf algebra structure.** In this Section we show that this Hopf algebra is free, cofree and self-dual. We can prove directly the *freeness*, by providing first some multiplicative bases of this algebra, thanks to the Bell order. Let $\delta$ be a set partition. We define the following elements on $\mathfrak{P}$:

$$\mathbf{E}_\delta := \sum_{\gamma\geq\delta} \mathbf{P}_\gamma\,,$$

$$\mathbf{H}_\delta := \sum_{\gamma\leq\delta} \mathbf{P}_\gamma\,.$$

The families $(\mathbf{H}_\varsigma)_\varsigma$ and $(\mathbf{E}_\varsigma)_\varsigma$ are respectively called *complete* and *elementary* functions of $\mathfrak{P}$. Families $(\mathbf{H}_\varsigma)_\varsigma$ and $(\mathbf{E}_\varsigma)_\varsigma$ are bases of $\mathfrak{P}$.

PROPOSITION 5.2. *Let $\delta\vdash[n]$ and $\varsigma\vdash[m]$. Then,*

$$\mathbf{E}_\delta\,\mathbf{E}_\varsigma = \mathbf{E}_{\delta|\varsigma}\,, \tag{5.3}$$

$$\mathbf{H}_\delta\,\mathbf{H}_\varsigma = \mathbf{H}_{\varsigma\sqcup\delta}\,. \tag{5.4}$$

A set partition $\delta$ is *reducible* if there exists non-empty set partitions $\varsigma$ and $\gamma$ such that $\delta = \varsigma\,|\,\gamma$. An *irreducible* set partition is a non reducible set partition (see A074664). Thus, immediately Formula (5.3) implies:

Proposition 5.3. *$\mathfrak{P}$ is free, as associative algebra, on irreducible set partitions.*

In order to prove that this Hopf algebra is also *cofree* and *self-dual* we prove that its image is bidendriform (see [**5**]) through a map $\phi$ defined as follows:

$$\mathbf{F}_\sigma \xrightarrow{\phi} \mathbf{F}_{\sigma^r} ,$$
$$\mathbf{F}_\sigma \otimes \mathbf{F}_\pi \xrightarrow{\phi} \mathbf{F}_{\pi^r} \otimes \mathbf{F}_{\sigma^r} ,$$

where $\sigma^r$ stands for the mirror image of $\sigma$. Thus, $\phi$ is a isomorphism of associative algebra and an anti-isomorphism of coalgebra. First, it is clear by Algorithm 2.1 that every permutations belonging to a same Bell class have the same first letter. This consideration immediately implies that the associative algebra of $\phi(\mathfrak{P})$ is a sub dendriform algebra of the dendriform algebra of the free quasi-symmetric functions, since no elements $\mathbf{F}$ indexed by permutations of a same Bell class are split between the two products $\prec$ and $\succ$. Moreover, by Theorem 3.1, it follows that no elements $\mathbf{F}$ indexed by permutations of a same Bell class are split neither by coproducts $\Delta_\prec$ nor $\Delta_\succ$. Then, the compatibilities relations needed to be a bidendriform bialgebra immediately follow from the fact that **FQSym** is already such an algebraic structure. Hence $\phi(\mathfrak{P})$ is a bidendriform bialgebra and so is automatically free, cofree and self-dual. These properties are shared through the map $\Phi$.

Theorem 5.2. *$\mathfrak{P}$ is free, cofree and self-dual.*

It is possible to build some graded graphs in duality in the sense of [**4**], in a classical way, from the analogue of Robinson-Schensted on set partitions of Section 2. These graphs encode respectively the right multiplication by $\mathbf{P}_{\{1\}}$ and the right multiplication by $\mathbf{Q}_{\{1\}}$, where the $\mathbf{Q}$ elements are the elements of the dual basis $\mathbf{P}$.

At last, we have the following inclusion diagram on Hopf algebra:

$$\mathbf{PBT}^{op} \hookrightarrow \mathfrak{P} \hookrightarrow \mathbf{FQSym}\,,$$

where $\mathbf{PBT}^{op}$ is the image of $\mathbf{PBT}$ through $\phi$.

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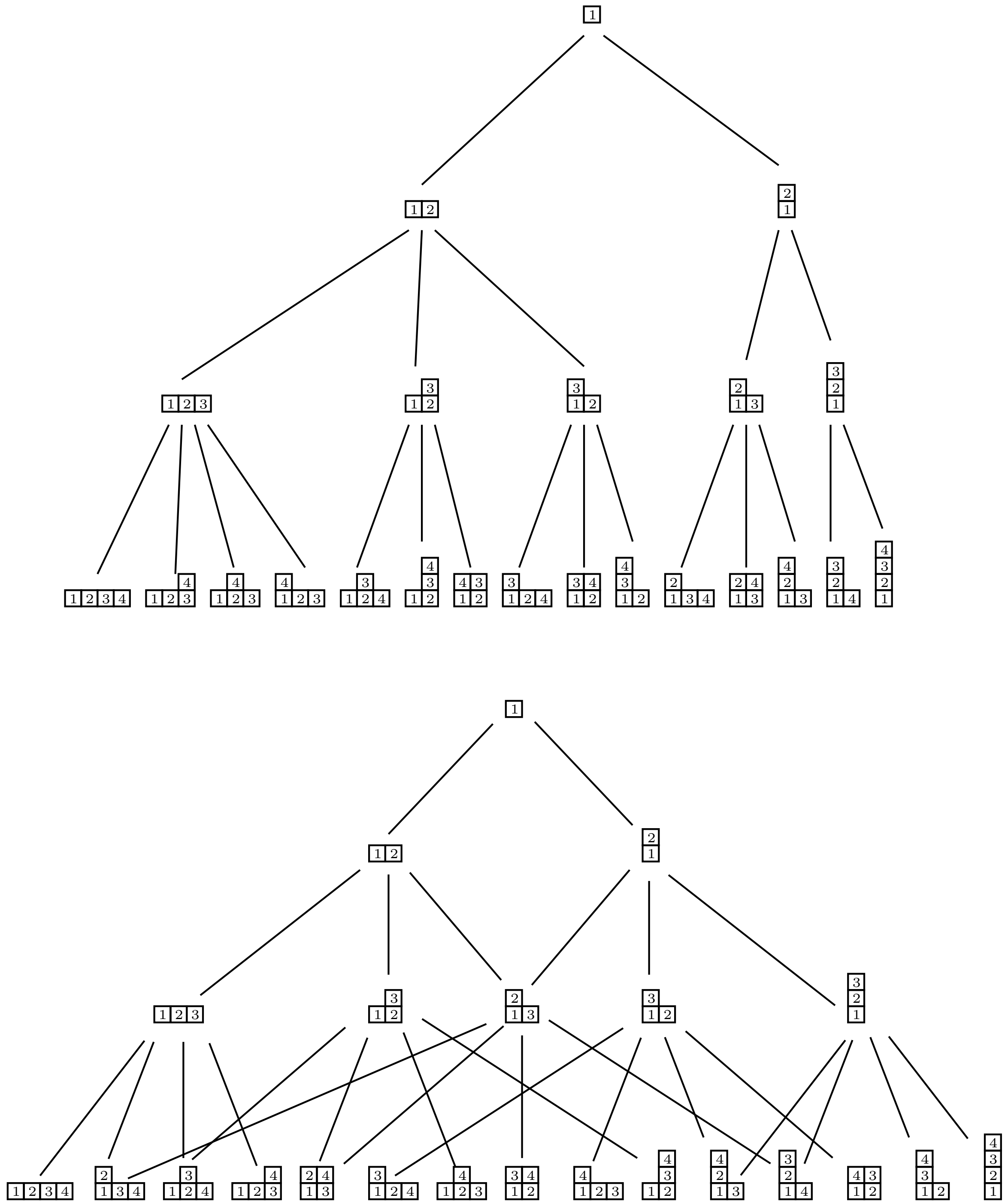

FIGURE 5. These graded graphs are in duality and respectively describes the right multiplication by $\mathbf{P}_{\{1\}}$ and $\mathbf{Q}_{\{1\}}$.