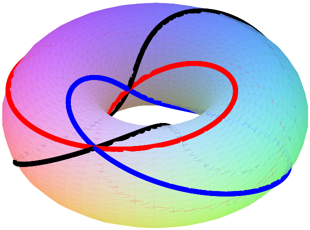

Affine descents and the Steinberg torus

T. Kyle Petersen
University of Michigan

(joint with Kevin Dilks and John Stembridge, arXiv:0709.4291)

FPSAC 08

Affine descents and the Steinberg torus

Eulerian polynomials

Coxeter complexes

Affine Eulerian polynomials

The Steinberg torus

Eulerian polynomials

The Eulerian polynomials, $A_n(t) = \sum_{k=0}^{n} a_{n,k} t^k$ (classical):

$$
\begin{aligned}
A_1(t) &= 1 + t \\
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- symmetric, unimodal coefficients
- real-rooted (Harper '67)

Combinatorial interpretation

Let $d(w) := \#\{i : w_i > w_{i+1}\}$. Then,

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$$A_2(t) = 1 + 4t + t^2$$

A generalization

The notion of descent makes sense in any Coxeter system (W, S) (and simple roots Δ):

$$\begin{aligned} d(w) &:= \#\{s \in S : \ell(ws) < \ell(w)\} \\ &= \#\{\alpha \in \Delta : w(\alpha) < 0\} \end{aligned}$$

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- Brenti has conjectured real-rootedness as well (D_n remains unproved)

Affine descents and the Steinberg torus

The f- and h-polynomials

Let Σ be a finite set of simplices, $f_k(\Sigma) =$ number of faces of dimension $k-1$

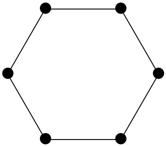
$$f(\Sigma; t) := \sum_{k=0}^{n} f_k(\Sigma) t^k$$

$(f_0, f_1, \ldots, f_n)$ is the *f-vector*

$$h(\Sigma; t) := (1-t)^n f(\Sigma; t/(1-t)) = \sum_{k=0}^{n} h_k(\Sigma) t^k$$

$(h_0, h_1, \ldots, h_n)$ is the *h-vector*

The f- and h-polynomials

Σ : 

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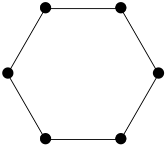
Σ : 

- $f_0 = 1$

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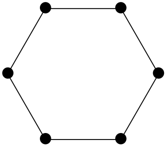
Σ : 

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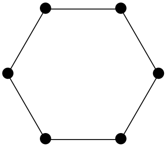

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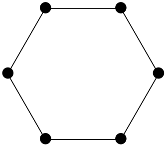
Σ : 

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$f(\Sigma; t) = 1 + 6t + 6t^2$

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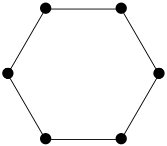
Σ : 

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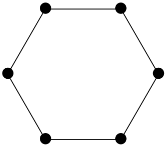
Σ : 

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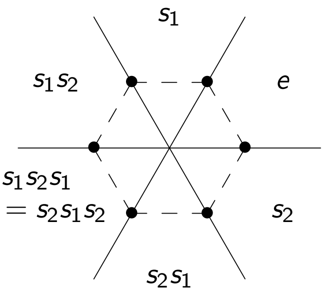
$f(\Sigma; t) = 1 + 6t + 6t^2$
$h(\Sigma; t) = 1 + 4t + t^2 = A_2(t)$ (hmm. . .)

The Coxeter complex

$\Sigma(A_2)$:



For a Coxeter system (W, S), the reflecting hyperplanes partition the ambient vector space

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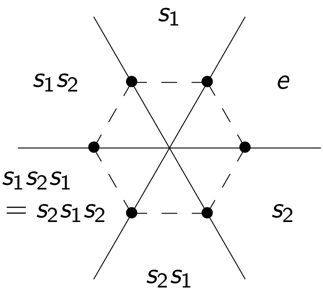

For a Coxeter system (W, S), the reflecting hyperplanes partition the ambient vector space

By intersecting the hyperplanes with the unit sphere we achieve a topological realization of the *Coxeter complex*, $\Sigma(W)$

The W-Eulerian polynomial

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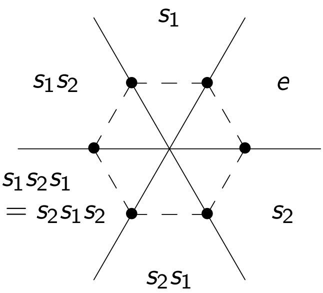

Theorem (Björner '84, Brenti '94)

For any finite Coxeter group W,

$$h(\Sigma(W); t) = \sum_{w \in W} t^{d(w)} = W(t)$$

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Affine descents

If W is crystallographic, it has a unique lowest root $\alpha_0 = -\widetilde{\alpha}$

Let s_0 be the corresponding reflection, $\Delta_0 = \Delta \cup \{\alpha_0\}$, and

$$\begin{aligned} \widetilde{d}(w) &:= d(w) + \chi(\ell(ws_0) > \ell(w)) \\ &= \#\{\alpha \in \Delta_0 : w(\alpha) < 0\} \end{aligned}$$

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Definition (Dilks-Petersen-Stembridge '07)

The affine W-Eulerian polynomial is

$$\widetilde{W}(t) := \sum_{w \in W} t^{\widetilde{d}(w)}$$

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Results of D-P-S, $\widetilde{W}(t)$ is:

- γ-nonnegative ($\Rightarrow$ symmetric, unimodal)
- conjecturally real-rooted ($\widetilde{A}_n$, $\widetilde{C}_n$, exceptional groups are proved; $\widetilde{B}_n$ and $\widetilde{D}_n$ are verified for $n \leq 100$)

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$$\widetilde{A}_2(t) = 3t + 3t^2$$

Jeopardy!

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It is the topological construction whose h-vector is encoded by the affine Eulerian polynomial.

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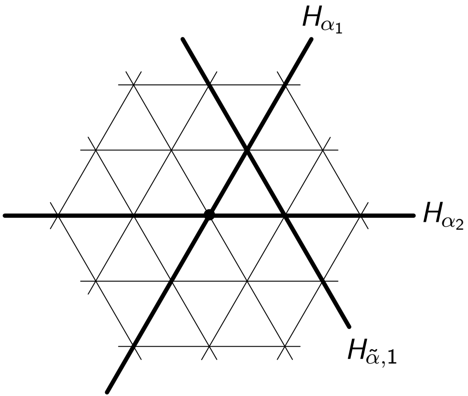
It is the topological construction whose h-vector is encoded by the affine Eulerian polynomial.

What is the Steinberg torus? (Correct!)

Affine Coxeter complexes

$\Sigma(\widetilde{A}_2)$:



The affine Weyl group $\widetilde{W}$ is generated by S along with the reflection through $H_{\widetilde{\alpha},1} := \{\lambda : \langle\widetilde{\alpha}, \lambda\rangle = 1\}$, drawing all hyperplanes gives $\Sigma(\widetilde{W})$ (. . . if W is irreducible. . .)

The Steinberg torus

Standard fact: the coroot lattice is a translation subgroup;

$$\widetilde{W} \cong W \ltimes \mathbb{Z}\Phi^\vee$$

Thus $\widetilde{W}$-action on V restricts to a W-action on the torus $V/\mathbb{Z}\Phi^\vee$ (Steinberg '68 - Bott's formula for Poincaré series of $\widetilde{W}$)

Definition (D-P-S)

The **Steinberg torus** *of* $\widetilde{W}$ *is*

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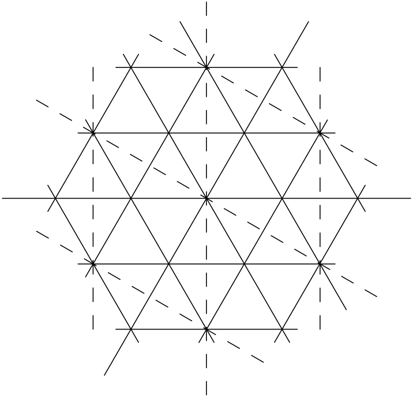
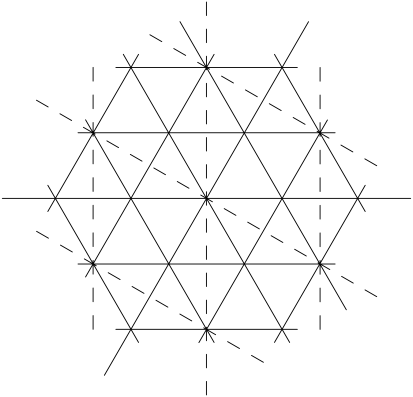
- maximal cells of in bijection with elements of W

The Steinberg torus



$$\Sigma_T(\widetilde{A}_2) := \Sigma(\widetilde{A}_2)/\mathbb{Z}\{\alpha_1^\vee, \alpha_2^\vee\}$$

The Steinberg torus



Equivalently, observe that exactly one vertex of every alcove is in $\mathbb{Z}\Phi^\vee$, so we translate can translate to the origin

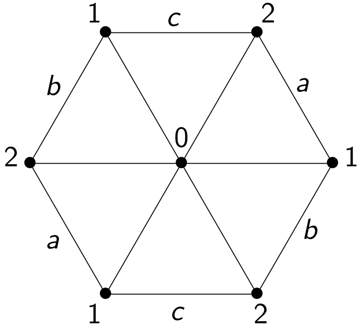
The Steinberg torus

The union of (closures of) the alcoves neighboring the origin is a convex, W-invariant simplicial polytope:

$$P_\Phi := \{\lambda \in V : -1 \leq \langle \lambda, \beta \rangle \leq 1 \text{ for all } \beta \in \Phi\}$$

We obtain the Steinberg torus by identifying opposite faces of P_Φ

$\Sigma_T(\widetilde{A}_2)$:



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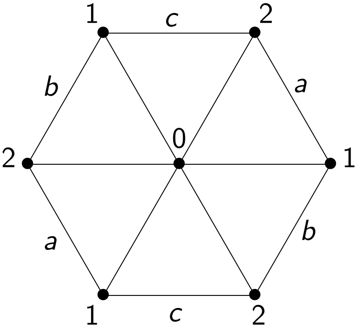

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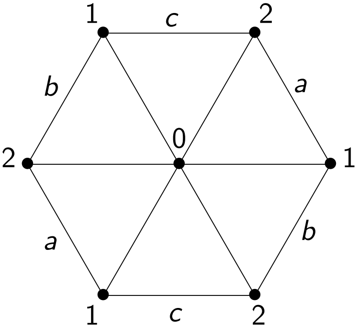

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The Steinberg torus

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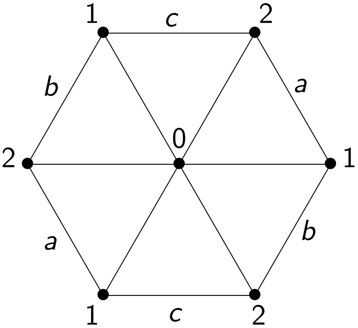

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The Steinberg torus

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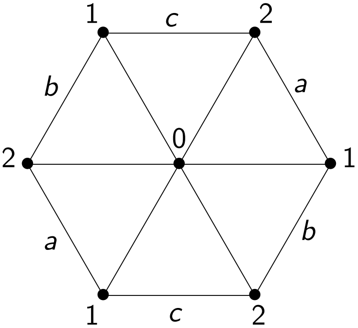

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- $f_1 = 3$
- $f_2 = 9$

The Steinberg torus

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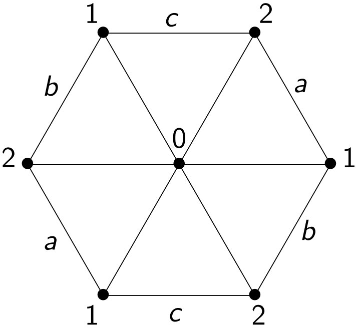

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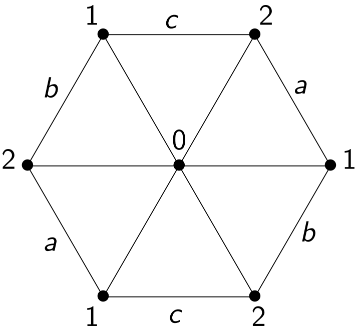

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The Steinberg torus

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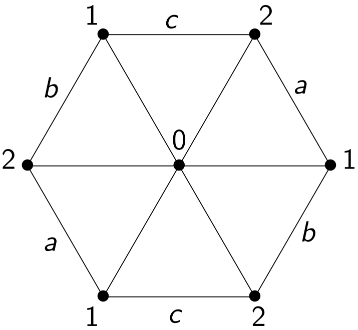

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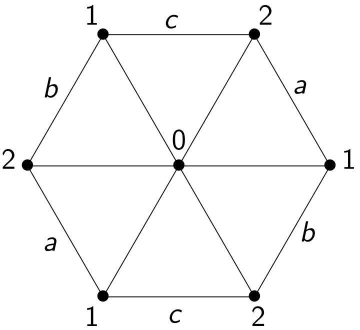

$$f(\Sigma_T; t) = 3t + 9t^2 + 6t^3 \qquad h(\Sigma_T; t) = 3t + 3t^2 = \widetilde{A}_2(t)$$

The Steinberg torus

Theorem (D-P-S)

For any irreducible affine Weyl group $\widetilde{W} = W \ltimes \mathbb{Z}\Phi^\vee$,

$$h(\Sigma_T(\widetilde{W}); t) = \sum_{w \in W} t^{\tilde{d}(w)} = \widetilde{W}(t)$$

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Dehn-Sommerville relations for the torus now imply symmetry of $\widetilde{W}(t)$

Next

- Real-rootedness: The remaining cases, $D_n, \widetilde{B}_n, \widetilde{D}_n$ are all γ-nonnegative (a necessary condition for real roots in this situation)

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- What does γ-nonnegativity mean topologically? (related to Gal's conjecture in spherical case?)

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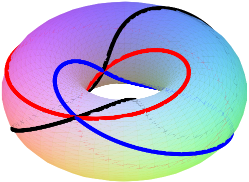
  we ignore the empty face (compare with reduced/unreduced homology)

Questions?

Art gallery:



http://www.math.lsa.umich.edu/~tkpeters/steinberg