Chip-Firing and Rotor-Routing on $\mathbb{Z}^d$ and on Trees

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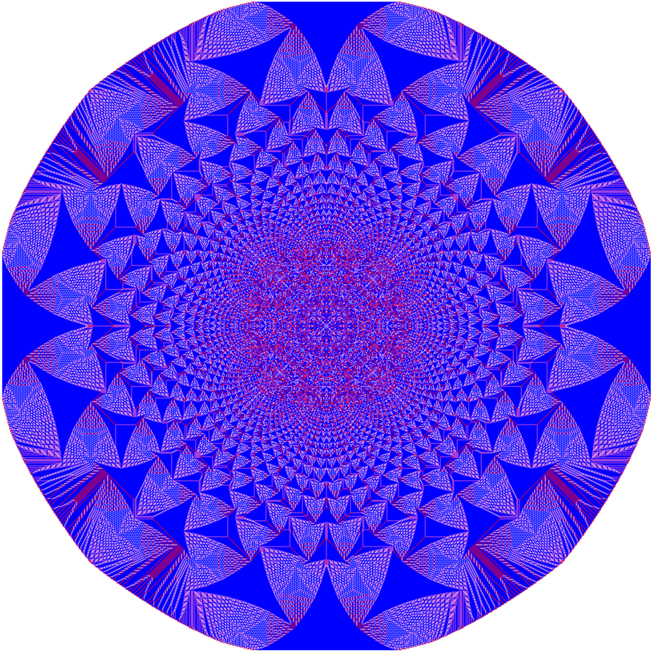
Joint work with Itamar Landau and Yuval Peres.

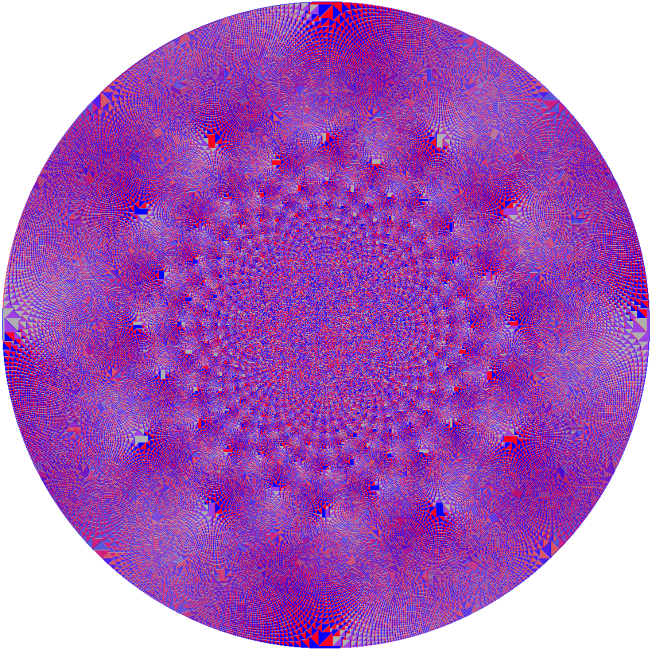
Chip-Firing on $\mathbb{Z}^d$

- Start with n chips at the origin.
- If a site has at least $2d$ chips, it can **fire** by sending one chip to each of the $2d$ neighboring sites.
- **Abelian property**: The final stable configuration does not depend on the order of the firings.
- Bak-Tang-Wiesenfeld '87, Björner-Lovász-Shor '91



Breaking Symmetry: Rotor-Routing

- Each site $x \in \mathbb{Z}^2$ has a **rotor** pointing North, South, East or West.
  (Start all rotors pointing North, say.)
- A site with more than one chip can **fire** by doing two things:
  1. Turning its rotor clockwise by 90 degrees;
  2. Sending one chip to the neighboring site pointed to by the rotor.
- The configuration is **stable** if every site has at most one chip.
- The final stable configuration does not depend on the order of the firings.
- For a general directed graph, fix a cyclic ordering of the outgoing neighbors of each vertex.



Spherical Asymptotics

- **Theorem** (L.-Peres) Let A_n be the region of n sites formed by rotor-router aggregation in $\mathbb{Z}^d$. Then

$$B_{r-c\log r} \subset A_n \subset B_{r(1+c'r^{-1/d}\log r)},$$

  where

  - B_ρ is the ball of radius ρ centered at the origin.
  - $n = \omega_d r^d$, where ω_d is the volume of the unit ball in $\mathbb{R}^d$.
  - c, c' depend only on d.

- **Corollary**: Inradius/Outradius $\to 1$ as $n \to \infty$.

Perfect Circularity on the Tree

- Let A_m be the region formed by rotor-router aggregation on the infinite d-regular tree, starting from m chips at the origin.
- **Theorem** (Landau-L.) If the initial rotor configuration is acyclic, then

$$A_{b_n} = B_n$$

where B_n is the ball of radius n centered at the origin, and $b_n = \#B_n$.

- In particular, if $b_n < m < b_{n+1}$, then

$$B_n \subset A_m \subset B_{n+1}.$$

Chip-Firing on General Graphs

- Finite connected graph G with a distinguished vertex s called the **sink**.
- **Chip configuration**: Each site $v \neq s$ has $\sigma(v) \geq 0$ chips.
- If $\sigma(v) \geq \deg(v)$, the vertex v can **fire**, sending one chip to each neighbor.
- The sink never fires.
- Order of topplings does not affect the final state σ°.

Recurrent Configurations

- A chip configuration σ on G is **stable** if

$$\sigma(v) \leq \deg(v) - 1$$

  for all vertices v.
- A stable configuration σ is **recurrent** if

$$\sigma = (\sigma + \tau)^\circ$$

  for some configuration $\tau \neq 0$.
  - Recurrent configurations are in bijection with G-**parking functions**.
- The recurrent configurations form a **group** SP(G) under the operation

$$(\sigma, \tau) \mapsto (\sigma + \tau)^\circ.$$

The Sandpile Group of a Graph

- $SP(G) \simeq \mathbb{Z}^{n-1}/\Delta\mathbb{Z}^{n-1}$, where

$$\Delta = D - A$$

  is the **reduced Laplacian** of G.
- Dhar '90, Lorenzini '91, Biggs '99 ("critical group"), Baker-Norine '07 ("Jacobian").
- **Matrix-tree theorem**:

$$\#SP(G) = \det \Delta = \#\{\text{spanning trees of } G\}.$$

The Action on Spanning Trees

- A spanning tree T of G defines a **rotor configuration** on G (the rotor at x points along the path from x to the sink).
- Let $e_x T$ be the final rotor configuration if we start a single chip at x and let it perform rotor-router walk until it reaches the sink.
- Since rotors point in the direction of last exit, $e_x T$ is also a spanning tree.
- Relations: $e_x^{\deg(x)} T = \left(\prod_{y \sim x} e_y\right) T$.
- Thus the sandpile group acts on spanning trees by

$$\sigma T = \left(\prod_x e_x^{\sigma(x)}\right) T.$$

- This is a **free**, **transitive** action.

Using the Action

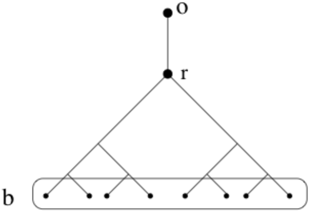
- In rotor-router aggregation on the d-regular tree, do the rotors in the first n levels of the tree ever return to their initial configuration?
  - Yes, if the intitial configuration is acyclic.
- How many chips do we need to add to get the initial configuration back?
  - Answer: The order of $\hat{r}$ in the sandpile group of the **wired regular tree** of height n, where

    $$\hat{r} = (\delta_r + e)^\circ$$

    where δ_r is a single chip at the root, and e is the identity element of the sandpile group.

The Sandpile Group of a Wired Tree

- Finite rooted tree T.
- Collapse the leaves to a single sink vertex.
- Add an edge from the root to the sink.



- What is the structure of the sandpile group?

Structure of the Sandpile Group

- **Theorem** (L.) Let T_n be the wired ternary tree of height n. Then

$$SP(T_n) \simeq \mathbb{Z}_{2^n-1} \oplus \mathbb{Z}_{2^{n-1}-1} \oplus \ldots \oplus (\mathbb{Z}_7)^{2^{n-4}} \oplus (\mathbb{Z}_3)^{2^{n-3}}.$$

- Similar decomposition for the d-regular tree:

$$SP(T_{n,d}) \simeq \mathbb{Z}_{m_n} \oplus (\mathbb{Z}_{m_{n-1}})^{a-1} \oplus \ldots \oplus (\mathbb{Z}_{m_3})^{a^{n-4}(a-1)} \oplus (\mathbb{Z}_{m_2})^{a^{n-3}(a-1)}.$$

  where $a = d-1$ and

$$m_k = a^{k-1} + \ldots + a + 1.$$

- Resolves a conjecture of Toumpakari (2004).

The Sandpile Group of a Tree, In Terms of its Branches

- **Lemma**: Let T be any finite wired tree, with principal branches $T_1, \dots, T_k$. Then

$$SP(T)/\langle \hat{r} \rangle \simeq \bigoplus_{i=1}^{k} SP(T_i) \Big/ \langle (\hat{r}_1, \dots, \hat{r}_k) \rangle$$

  where r, r_i are the roots of T, T_i respectively.

- **Proof sketch**: Map $\begin{pmatrix} a \\ \sigma_1, \dots, \sigma_k \end{pmatrix} \mapsto (\sigma_1, \dots, \sigma_k)$.
  - After modding out by $\hat{r}$, the branches become independent.
  - Since $(k+1)\hat{r} \mapsto (\hat{r}_1, \dots, \hat{r}_k)$ we have to mod out by this on the right.

Strengthening to a Direct Sum

- **Lemma**: Let T_n be the wired ternary tree of height n. Then

$$SP(T_n) = \mathbb{Z}_{2^n-1} \oplus SP(T_{n-1})^2 / \mathbb{Z}_{2^{n-1}-1}.$$

- **Proof sketch**: $\langle \hat{r} \rangle \simeq \mathbb{Z}_{2^n-1}$.
  - Need a projection map $p : SP(T_n) \longrightarrow \langle \hat{r} \rangle$.
  - Use the symmetrization map

$$p(\sigma)(x) = 2^{n+1-|x|} \sum_{|y|=|x|} \sigma(y).$$

Factoring Into Cyclic Subgroups

- $SP(T_2) = \mathbb{Z}_3$.
- $SP(T_3) = \mathbb{Z}_7 \oplus SP(T_2)^2/\mathbb{Z}_3 = \mathbb{Z}_7 \oplus \mathbb{Z}_3$.
- $SP(T_4) = \mathbb{Z}_{15} \oplus SP(T_3)^2/\mathbb{Z}_7 = \mathbb{Z}_{15} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_3^2$.
- $SP(T_5) = \mathbb{Z}_{31} \oplus SP(T_4)^2/\mathbb{Z}_{15} = \mathbb{Z}_{31} \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_7^2 \oplus \mathbb{Z}_3^4$.

$$\cdots$$

- $SP(T_n) = \mathbb{Z}_{2^n-1} \oplus \mathbb{Z}_{2^{n-1}-1} \oplus \ldots \oplus (\mathbb{Z}_7)^{2^{n-4}} \oplus (\mathbb{Z}_3)^{2^{n-3}}$.

Open Problems

- Find a bijective proof that

$$\#SP(T_n) = 3^{2^{n-3}} 7^{2^{n-4}} \cdots (2^{n-1}-1)(2^n-1)$$

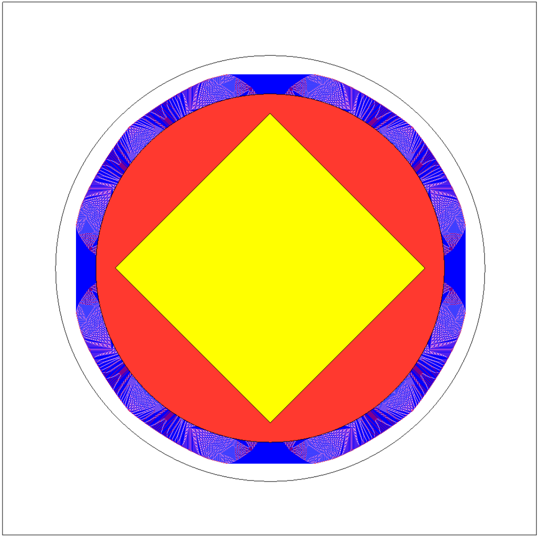
  for the wired ternary tree T_n.
- Aggregation on general trees: What takes the place of a ball?

Bounds for Chip-Firing in $\mathbb{Z}^d$

- **Theorem** (L.-Peres) Starting with n chips at the origin in $\mathbb{Z}^d$, let S_n be the set of sites that fire. Then

$$\left(\text{Ball of volume } \frac{n-o(n)}{2d-1}\right) \subset S_n \subset \left(\text{Ball of volume } \frac{n+o(n)}{d}\right).$$

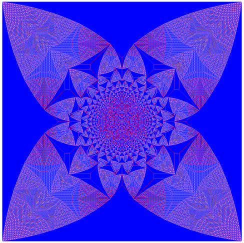
- Improves the bounds of Le Borgne and Rossin, Fey and Redig.



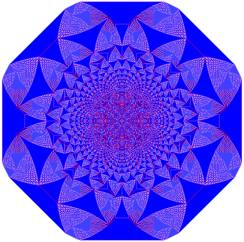
(Disk of area $n/3$) $\subset S_n \subset$ (Disk of area $n/2$)

An Open Problem in $\mathbb{Z}^2$

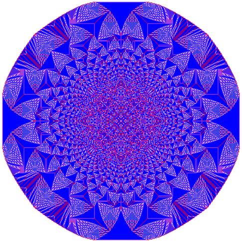
- Fix an integer $h \in (-\infty, 2]$.
- Start with n chips at the origin, and h chips at every other site in $\mathbb{Z}^2$. Let $S_{n,h}$ be the set of sites that fire.
- **Question**: As $n \to \infty$, is the limiting shape $S_{n,h}$ a regular $(12-4h)$-gon?
- Fey and Redig (2007) Case $h = 2$: The limiting shape of $S_{n,2}$ is a square.
- In all other cases, even the existence of a limiting shape is open.
- Even for $h = 2$, the rate of growth of the square is not known.



$h = 2$



$h = 1$



$h = 0$