REPRESENTATION THEORY OF THE 0-ARIKI-KOIKE-SHOJI ALGEBRAS

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Abstract. We investigate the representation theory of certain specializations of the Ariki-Koike algebras, obtained by setting $q = 0$ in a suitably normalized version of Shoji presentation. We classify the simple and projective modules, and describe restrictions, induction products, Cartan invariants and decomposition matrices. This allow us to identify the Grothendieck rings in terms of certain graded Hopf algebras known as the Mantaci-Reutenauer descent algebras, and Poirier Quasi-symmetric functions.

1. Introduction

Given an *inductive tower of algebras*, that is, a sequence of algebras

$$A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_n \hookrightarrow \ldots, \tag{1}$$

with embeddings $A_m \otimes A_n \hookrightarrow A_{m+n}$ satisfying an appropriate associativity condition, one can introduce two *Grothendieck rings*

$$\mathcal{G} := \bigoplus_{n\geq 0} G_0(A_n), \quad \mathcal{K} := \bigoplus_{n\geq 0} K_0(A_n), \tag{2}$$

where $G_0(A)$ and $K_0(A)$ are the (complexified) Grothendieck groups of the categories of finite-dimensional A-modules and projective A-modules respectively, with multiplication of the classes of an A_m-module M and an A_n-module N defined by

$$[M] \cdot [N] = [M \widehat{\otimes} N] = [M \otimes N \uparrow_{A_m \otimes A_n}^{A_{m+n}}]. \tag{3}$$

On each of these Grothendieck rings, one can define a coproduct by means of restriction of representations, turning these into mutually dual Hopf algebras.

The basic example of this situation is the character ring of symmetric groups (over $\mathbb{C}$), due to Frobenius. Here the $A_n = \mathbb{C}\,\mathfrak{S}_n$ are semi-simple algebras, so that

$$G_0(A_n) = K_0(A_n) = R(A_n), \tag{4}$$

where $R(A)$ denotes the vector space spanned by isomorphism classes of indecomposable modules which in this case are all simple and projective. The irreducible representations $[\lambda]$ of A_n are parametrized by partitions λ of n, and the Grothendieck ring is isomorphic to the algebra *Sym* of symmetric functions under the correspondence $[\lambda] \leftrightarrow s_\lambda$, where s_λ denotes the Schur function associated with λ.

Other known examples with towers of group algebras over the complex numbers, $A_n = \mathbb{C}\,G_n$, include the cases of wreath products $G_n = \Gamma \wr \mathfrak{S}_n$ (Specht), finite linear groups $G_n = GL(n, \mathbf{F}_q)$ (Green), *etc.*, all related to symmetric functions (see [11, 21]).

Examples involving non-semisimple specializations of Hecke algebras have also been worked out. Finite Hecke algebras of type A at roots of unity ($A_n = H_n(\zeta)$, $\zeta^r = 1$) yield quotients and subalgebras of Sym [10]:

$$\mathcal{G} = Sym/(p_{rm} = 0), \quad \mathcal{K} = \mathbb{C}[p_k | k \not\equiv 0 \mod r] \tag{5}$$

The Ariki-Koike algebras at roots of unity give rise to level r Fock spaces of affine Lie algebras of type A [2]. The 0-Hecke algebras $A_n = H_n(0)$ correspond to the pair Quasi-symmetric functions/ Noncommutative symmetric functions, $\mathcal{G} = QSym$, $\mathcal{K} = \mathbf{Sym}$ [8]. Affine Hecke algebras at roots of unity lead to $U(\widehat{sl}_r)$ and $U(\widehat{sl}_r)^*$ [1], and the cases of affine Hecke generic algebras can be reduced to a subcategory admitting as Grothendieck rings $U(\widehat{gl}_\infty)$ and $U(\widehat{gl}_\infty)^*$ [1].

A further interesting example is the tower of 0-Hecke-Clifford algebras [15, 4], giving rise to the peak algebras [9, 20].

Here, we shall show that appropriate versions at $q = 0$ of the Ariki-Koike algebras (presentation of Shoji [19, 18]) admit as Grothendieck rings two known combinatorial Hopf algebras, the Mantaci-Reutenauer descent algebras (associated with the corresponding wreath products) [12], and their duals, a generalization of quasi-symmetric functions, introduced by Poirier in [16] and more recently considered in [14, 3].

This article is structured as follows. We first define the 0-Ariki-Koike-Shoji algebras $\mathcal{H}_{n,r}(0)$, and introduce a special basis, well suited for analyzing representations. Next, we obtain the classification of simple $\mathcal{H}_{n,r}(0)$-modules, which turn out to be all one-dimensional, and labelled by $r(r+1)^{n-1}$ combinatorial objects called cyclotomic ribbons. We then describe induction products and restrictions of these simple modules, which allows us to identify the first Grothedieck ring $\mathcal{G}$ with a Hopf subalgebra of Poirier's Quasi-symmetric functions, dual to the Mantaci-Reutenauer Hopf algebra. This duality gives immediately the Grothendieck ring $\mathcal{K}$ associated with projective modules. An alternative labelling of the indecomposable projective modules leads then to a simple description of basic operations such as induction products, restriction to $H_n(0)$, or induction from a $H_n(0)$-projective module. Summarizing, we obtain an explicit description of the Cartan-Brauer triangle, in particular of the Cartan invariant and decomposition matrices.

In all the paper, we will make use of a set $C = \{1, \ldots, r\}$, called the *color set.*

2. The 0-Ariki-Koike-Shoji algberas

In [19], Shoji obtained a new presentation of the Ariki-Koike algebras defined in [2]. We shall first give a presentation very close to his, put $q = 0$ in the relations and then prove some simple results about another basis of the resulting algebra. To get our presentation from the one of Shoji, one has to replace qa_i by T_{i-1} and q^2 by q.

Let $u_1, \ldots, u_r$ be r distinct complex numbers. We shall denote by $P_k(X)$ the polynomial

$$P_k(X) := \prod_{1 \le l \le r,\ l \ne k} \frac{X - u_l}{u_k - u_l}. \tag{6}$$

Let $\mathcal{H}_{n,r}(q)$ be the associative algebra generated by the elements $T_1, \dots, T_{n-1}$ and $\xi_1, \dots, \xi_n$ subject to the following relations:

$$(T_i - q)(T_i + 1) = 0 \quad (1 \le i \le n-1), \tag{7}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \le i \le n-2), \tag{8}$$

$$T_i T_j = T_j T_i \qquad (|i-j| \ge 2), \tag{9}$$

$$(\xi_j - u_1) \cdots (\xi_j - u_r) = 0 \qquad (1 \le j \le n), \tag{10}$$

$$\xi_i \xi_j = \xi_j \xi_i \qquad (1 \le i, j \le n), \tag{11}$$

$$T_i \xi_i = \xi_{i+1} T_i - (q-1) \sum_{c_1 < c_2} (u_{c_2} - u_{c_1}) P_{c_1}(\xi_i) P_{c_2}(\xi_{i+1}) \quad (1 \le i \le n-1), \tag{12}$$

$$T_i(\xi_{i+1} + \xi_i) = (\xi_{i+1} + \xi_i) T_i \quad (1 \le i \le n-1) \tag{13}$$

$$T_i \xi_j = \xi_j T_i \qquad (j \ne i-1, i). \tag{14}$$

As noticed in [19], from this presentation, it is obvious that a generating set is given by the $\xi_1^{c_1} \cdots \xi_n^{c_n} \cdot T_\sigma$ with $\sigma \in \mathfrak{S}_n$ and c_i such that $0 \le c_i \le r-1$. Shoji proves that this is indeed a basis of $\mathcal{H}_{n,r}(q)$. Moreover, a simple adaptation of his proof enables us to conclude that this property still holds a $q = 0$. This can also be directly proved thanks to the multiplication relations between the Hecke generators and the Lagrange basis to be presented next. This algebra $\mathcal{H}_{n,r}(0)$, which we call the 0-Ariki-Koike-Shoki algebra, will be our main concern in the sequel.

If $\mathbf{c} = (c_1, \dots, c_n)$ is a word on C, we define

$$L_{\mathbf{c}} := P_{c_1}(\xi_1) \cdots P_{c_n}(\xi_n). \tag{15}$$

Since the Lagrange polynomials (Equation (6)) associated with r distincts complex numbers are a basis of $\mathbb{C}_{r-1}[X]$ (polynomials of degree at most $r-1$), the next proposition holds.

Proposition 2.1. *The set*

$$\{B_{\mathbf{c},\sigma} := L_{\mathbf{c}} T_\sigma\}, \tag{16}$$

where $\sigma \in \mathfrak{S}_n$ and $\mathbf{c} = (c_1, \dots, c_n)$ is a color word, is a basis of $\mathcal{H}_{n,r}(0)$.

Recall that a composition is any finite sequence of positive integers $I = (i_1, \dots, i_k)$. It can be pictured as a ribbon diagram, that is, a set of rows composed of square cells of respective lengths i_j, the first cell of each row being attached to the last cell of the previous one. I is called the *shape* of his ribbon diagram. Recall also that the descent composition $I = C(\sigma)$ of a permutation σ is the one whose diagram is obtained by writing the elements of σ one per cell so that the rows are weakly increasing and the columns are strictly decreasing (french notation).

We shall represent the basis element $B_{\mathbf{c},\sigma}$ by a filling of a composition diagram as follows: the composition is $C(\sigma)$ and its i-th cell is filled with c_i and σ_i.

Let us now describe the product by a generator on the left of a basis element: on $B_{\mathbf{c},\sigma}$, the generator ξ_i acts diagonally by multiplication by c_i, so that it only remains to explicit the product of T_i by $L_{\mathbf{c}}$. One finds

$$T_i L_{\mathbf{c}} = L_{\mathbf{c}\sigma_i} T_i + \begin{cases} -L_{\mathbf{c}} & \text{if} \quad c_i < c_{i+1}, \\ 0 & \text{if} \quad c_i = c_{i+1}, \\ L_{\mathbf{c}\sigma_i} & \text{if} \quad c_i > c_{i+1}. \end{cases} \tag{17}$$

where σ_i acts on the right of $\mathbf{c}$ by exchanging c_i and c_{i+1}.

This description enables us to analyze the left regular representation of $\mathcal{H}_{n,r}(0)$ in terms of our basis elements. As a first application, we shall obtain a classification of the simple $\mathcal{H}_{n,r}(0)$-modules.

3. Simple modules

3.1. **Definition.** Let I be a composition of n and $\mathbf{c} \in C^n$ be a color word of length n. The pair $[I, \mathbf{c}]$ will be called a *colored ribbon*, and depicted as the filling of I whose natural row reading is $\mathbf{c}$. We say that this filling is a *cyclotomic ribbon* (*cyclo-ribbon* for short) if it is weakly *increasing* in rows and weakly *decreasing* in columns. Notice that there are $r(r+1)^{n-1}$ cyclo-ribbons since when building the ribbon cell by cell, one has r possibilities for its first cell and then $r+1$ possibilities for the next ones: 1 possibility for the $r-1$ different choices than the previous one and 2 possibilities for the same choice (either go right or go down). Here are the five cyclo-ribbons of shape $(2, 1)$ with two colors.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2} \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2} \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2} \end{array} \tag{18}$$

3.2. **Eigenvalues.** Let I be a composition of n and $\mathbf{c} = (c_1, \ldots, c_n)$ be a color word. We say that their associated ribbon is a *anti-cyclotomic ribbon* (*anti-cyclo-ribbon* for short) if it is weakly *decreasing* in rows and weakly *increasing* in columns. There are as many anti-cyclo-ribbons as ribbons. The natural bijection ϕ from one set to the other is the restriction of an involution on all colored ribbons: read a ribbon R by rows from top to bottom (french notation), and build the corresponding ribbon cell by cell in the following way:

- if the $(i+1)$-th cell has the same content as the i-th cell, glue it in the same position as in R (right or down),
- if the $(i+1)$-th cell does not have the same content as the i-th cell, glue it in the other position (right or down).

For example, the colored ribbons below are exchanged by application of ϕ:

$$\begin{array}{|c|c|c|ccc}\cline{1-3} 1 & 1 & 3 & & & \\ \cline{1-3} \multicolumn{2}{c|}{} & 3 & \multicolumn{3}{c}{} \\ \cline{3-3} \multicolumn{2}{c|}{} & 3 & \multicolumn{3}{c}{} \\ \cline{3-3} \multicolumn{2}{c|}{} & 2 & \multicolumn{3}{c}{} \\ \cline{3-6} \multicolumn{2}{c|}{} & 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \cline{3-6} \end{array} \qquad\qquad \begin{array}{|c|c|c|c|c|}\cline{1-2} 1 & 1 & \multicolumn{3}{c}{} \\ \cline{1-2} \multicolumn{1}{c|}{} & 3 & \multicolumn{3}{c}{} \\ \cline{2-2} \multicolumn{1}{c|}{} & 3 & \multicolumn{3}{c}{} \\ \cline{2-5} \multicolumn{1}{c|}{} & 3 & 2 & 1 & 1 \\ \cline{2-5} \multicolumn{4}{c|}{} & 4 \\ \cline{5-5} \multicolumn{4}{c|}{} & 5 \\ \cline{5-5} \end{array} \tag{19}$$

FIGURE 1. A cyclo-ribbon R and its corresponding anti-cyclo-ribbon $\phi(R)$.

Let $[I, \mathbf{c}]$ be a cyclo-ribbon. Let $\eta_{[I,\mathbf{c}]}$ be the element of $\mathcal{H}_{n,r}(0)$ defined as

$$\eta_{[I,\mathbf{c}]} := L_{\mathbf{c}}\ \eta_I, \tag{20}$$

where η_I is the generator of the simple $H_n(0)$-module associated with I in the notation of [8] Proposition 5.3 (see also [13]). We will show later that this element generates a simple $\mathcal{H}_{n,r}(0)$-module.

Proposition 3.1. *Let $[I, \mathbf{c}]$ be a cyclo-ribbon. Then*

$$\eta_{[I,\mathbf{c}]} = \eta_{I'} L_{\mathbf{c}'}, \tag{21}$$

where $[I', \mathbf{c}']$ are the shape and the mirror image of the reading of the anti-cyclo-ribbon $\phi([I, \mathbf{c}])$.

Theorem 3.2. *Let $[I, \mathbf{c}]$ be a cyclo-ribbon. Then*

$$S_{[I,\mathbf{c}]} := \mathcal{H}_{n,r}(0)\eta_{[I,\mathbf{c}]} \tag{22}$$

is a simple module of $\mathcal{H}_{n,r}(0)$ realized as a minimal left ideal in its left regular representation. The eigenvalue of ξ_i is c_i, and that of T_i is -1 or 0 according to whether i is a descent of the shape of $\phi(R')$ or not. All these simple modules are pairwise non isomorphic and of dimension 1. Moreover, all simple $\mathcal{H}_{n,r}(0)$-modules are one-dimensional and isomorphic to some $S_{[I,\mathbf{c}]}$.

Proof. We only give a sketch of the proof.

The first part of the theorem follows directly from the previous propositions. Next, we prove that any simple module of dimension 1 is isomorphic to some $S_{[I,\mathbf{c}]}$. This comes easily from Equation (17).

It remains to show that all simple modules are of dimension 1. This is done by the same argument as in [4] in our context: compute the composition factors of the modules induced from simple modules of the 0-Hecke algebra. This is described in the next subparagraph. ∎

3.3. **Induction of the simple 0-Hecke modules.** To describe the induction process, we will need a partial order on the fillings of ribbons. Let I be a composition and $\mathbf{c} = (c_1, \ldots, c_n)$. The covering relation of the order $\leq_I$ corresponds to sort in increasing order any two adjacent elements in the rows of I or to sort in decreasing order any two adjacent elements in the columns of I. For example, the elements smaller than

$$T := \begin{array}{|c|c|c|c|} \hline 2 & 1 & & \\ \hline & 1 & 3 & 3 \\ \hline & & & 4 \\ \hline & & & 3 \\ \hline \end{array} \tag{23}$$

are

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & & \\ \hline & 1 & 3 & 3 \\ \hline & & & 4 \\ \hline & & & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|} \hline 2 & 1 & & \\ \hline & 1 & 3 & 4 \\ \hline & & & 3 \\ \hline & & & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|} \hline 1 & 2 & & \\ \hline & 1 & 3 & 4 \\ \hline & & & 3 \\ \hline & & & 3 \\ \hline \end{array} \tag{24}$$

If I is a composition of n, let $S_I := H_n(0)\eta_I$ be the corresponding simple module of $H_n(0)$ (notation as in [5]) and

$$M_I := S_I \uparrow_{H_n(0)}^{\mathcal{H}_{n,r}(0)} . \tag{25}$$

Clearly, M_I has dimension r^n and admits $L_{\mathbf{c}}\eta_I$ as linear basis, when $\mathbf{c}$ runs over color words. For $\mathbf{c} \in C^n$, let $M_{\mathbf{c},I}$ be the $\mathcal{H}_{n,r}(0)$-submodule of M_I generated by $L_{\mathbf{c}}\eta_I$.

Lemma 3.3.

$$M_{\mathbf{c},I} \subseteq M_{\mathbf{c}',I} \iff \mathbf{c} \leq_I \mathbf{c}'. \tag{26}$$

Proof. Let $i \in \{1, \dots, n-1\}$.

- If $c_i < c_{i+1}$, and T_i acts by 0 on η_I, we get $(1+T_i)L_{\mathbf{c}}\eta_I = L_{\mathbf{c}\sigma_i}T_i\eta_I = 0$.
- If $c_i < c_{i+1}$, and T_i acts by -1 on η_I, we get $(1+T_i)L_{\mathbf{c}}\eta_I = L_{\mathbf{c}\sigma_i}T_i\eta_I = -L_{\mathbf{c}\sigma_i}\eta_I$, and so $-(1+T_i)$ sorts in decreasing order in columns c_i and c_{i+1}.
- If $c_i = c_{i+1}$, then $T_iL_{\mathbf{c}}\eta_I = L_{\mathbf{c}}T_i\eta_I$ so the result is either 0 or -1 times $L_{\mathbf{c}}\eta_I$.
- If $c_i > c_{i+1}$, and T_i acts by -1 on η_I, we get $T_iL_{\mathbf{c}}\eta_I = L_{\mathbf{c}\sigma_i}(1+T_i)\eta_I = 0$.
- If $c_i > c_{i+1}$, and T_i acts by 0 on η_I, we get $T_iL_{\mathbf{c}}\eta_I = L_{\mathbf{c}\sigma_i}(1+T_i)\eta_I = L_{\mathbf{c}\sigma_i}\eta_I$, and so T_i sorts in increasing order in rows c_i and c_{i+1}.

∎

The $M_{\mathbf{c},I}$ such that the pair $\mathbf{c}, I$ is a cyclo-ribbon are simple. It follows easily from the previous lemma that these objects form the socle of M_I, and that M_I admits a composition series involving only one-dimensional modules. Since any simple $\mathcal{H}_{n,r}(0)$-module must appear as a composition factor of some M_I, this proves Theorem 3.2.

3.4. **First Grothendieck ring.** Let us first recall that the composition factors of the induction product of two simple 0-Hecke modules labelled by compositions I and J are easily described by means of permutations: let σ (resp. τ) be the permutation with the maximum inversion number in the descent class I (resp. J). Then the 0-Hecke graph associated with both simple modules is the same as the graph of the weak Bruhat order restrictede to the shifted shuffle product of the inverses of σ and τ. The composition factors are then the descent compositions of the elements of this shifted shuffle product.

In other words, the composition factors of the induction product of two simple 0-Hecke modules can be obtained by computing a product in **FQSym** (compute a shifted shuffle of permutations) and taking the image of each element by the morphism that sends a permutation to its descent composition (same morphism as taking the commutative image of the usual realization as noncommutative polynomials). Since this is an epimorphism onto $QSym$, this proves that the direct sum of all Grothendieck groups associated with the 0-Hecke algebra, is isomorphic to $QSym$.

The case at hand can be worked out in a similar way. Instead of permutations, one has to consider colored permutations as in [14]. Given a cyclo-ribbon $[I, \mathbf{c}]$, one associates with it the unique colored permutation (σ, u) where σ is the permutation with the maximum inversion number of the descent class I and $u = \mathbf{c}$. Recall that colored permutations index the Hopf algebra $\mathbf{FQSym}^{(r)}$ [14] and that the product of the **F** basis (dual to the **G** basis, considering the colors as a group) is given by the shifted shuffle of colored permutations.

Let $[I, \mathbf{c}]$ and $[J, \mathbf{c}']$ be two cyclo-ribbons. Then the graph associated with the induction product of the corresponding simple modules is the same as the graph of the shifted shuffle of the inverses (considering the colors as a group) of the colored permutations associated with both cyclo-ribbons. Both graphs have the same structure, the edges being given by T_i or $1+T_i$ depending on whether $c_i \geq c_{i+1}$ or not (see Equation (17)).

The simple module corresponding to a given colored permutation (σ, u) is the cyclo-ribbon associated with the colored descent composition of (σ, u). Recall that the colored descent composition of a colored permutation is the pair (I, v) where I is the unique composition which descents are either a descent of σ or a change of color in u, and v is the color of all elements of the corresponding block of the k-th row of I.

For example, Figure 2 presents the induction product of the two simple elements $[1,1],[2,1]$ and $[2],[1,2]$ belonging to $\mathcal{H}_{2,2}(0)$ to $\mathcal{H}_{4,2}(0)$.

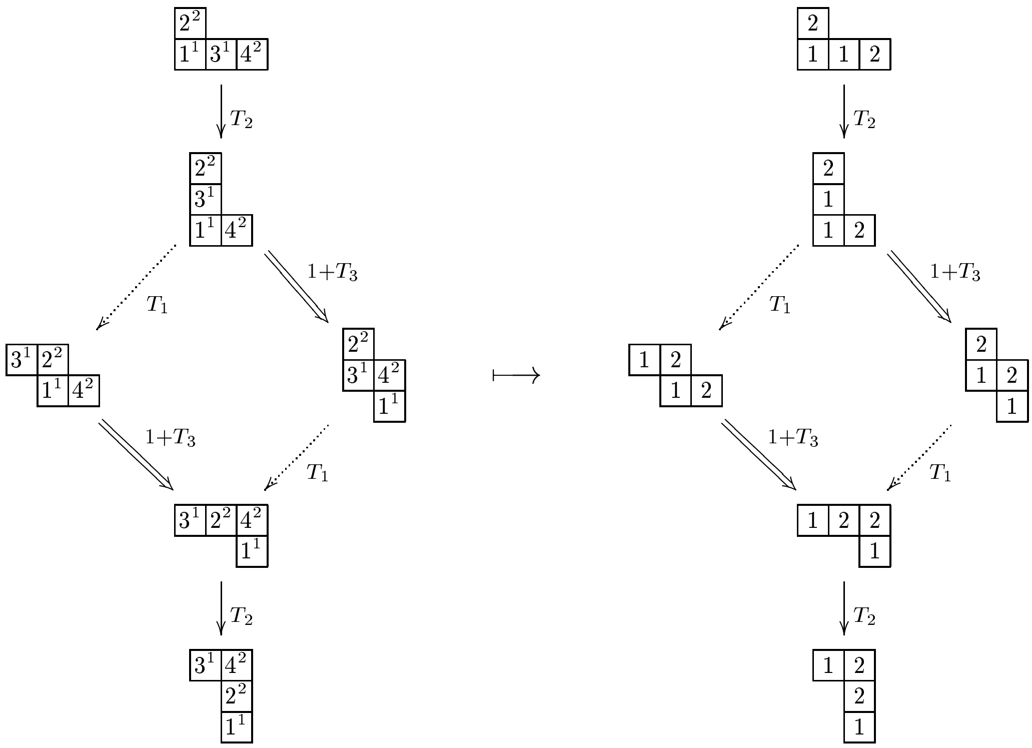

FIGURE 2. Induction of two simple modules of $\mathcal{H}_{2,2}(0)$ to $\mathcal{H}_{4,2}(0)$

3.5. **The Quasi-symmetric Mantaci-Reutenauer algebra.** Let $QSym^{(r)}$ be the algebra of level r quasi-symmetric functions, defined in [16, 14]. In [16], this algebra is indexed by *signed compositions*, which we will not define. Poirier also remarks (Lemma 11, p. 325) that the functions indexed by *descent signed compositions* (a subset of the previous set) can be computed through the operation "take the commutative image" of a noncommutative series, very similar to the definition of **FQSym** [5]. It happens that these functions (indexed by descent signed compositions) form a subalgebra $QMR^{(r)}$ of $QSym^{(r)}$ (see [3]). We obtained this algebra [14] as a quotient of $QSym^{(r)}$ by killing some monomials. We will prove that the Grothendieck ring of the tower of $\mathcal{H}_{n,r}(0)$ algebras is isomorphic to $QMR^{(r)}$.

There is a trivial bijection between descent signed compositions and cyclo-ribbons, so that we can speak of $F_{[I,\mathbf{c}]}$ without ambiguity.

Lemma 3.4 ([3, 14])**.** *The $F_{[I,\mathbf{c}]}$, where $[I,\mathbf{c}]$ runs over the cyclo-ribbons, span a Hopf subalgebra $QMR^{(r)}$ of $QSym^{(r)}$, isomorphic to the dual of the Mantaci-Reutenauer algebra $\mathbf{MR}^{(r)}$ defined in* [12].

Let

$$\mathcal{G}^{(r)} := \bigoplus_{n\geq 0} G_0(\mathcal{H}_{n,r}(0)) \tag{27}$$

be the Grothendieck ring of the tower of $\mathcal{H}_{n,r}(0)$ algebras. Define a characteristic map

$$\begin{array}{rrl} ch: & \mathcal{G}^{(r)} & \to QMR^{(r)} \\ & [S_{[I,\mathbf{c}]}] & \mapsto F_{[I,\mathbf{c}]}. \end{array} \tag{28}$$

Theorem 3.5. *ch is an isomorphism of Hopf algebras.*

Proof. Both $\mathcal{G}^{(r)}$ and $QMR^{(r)}$ are the image of $\mathbf{FQSym}^{(r)}$ over the morphism that sends any permutation to its colored descent composition: for $\mathcal{G}^{(r)}$, this follows from the previous paragraph, whereas for $\mathbf{FQSym}^{(r)}$, it is proved in [14]. ∎

4. Projective modules

The previous results immediately imply, by duality, a description of the Grothendieck ring of the category of projective $\mathcal{H}_{n,r}(0)$-modules. As a Hopf algebra,

$$\mathcal{K} = \bigoplus_{n\geq 0} K_0(\mathcal{H}_{n,r}(0)) \simeq \mathbf{MR}^{(r)} \tag{29}$$

is isomorphic to the Mantaci-Reutenauer algebra, and under this isomorphism, the class of indecomposable projective modules are mapped to the subfamily of the dual basis $F_{[I,\mathbf{c}]}$ of Poirier quasi-symmetric functions labelled by colored descent sets, or cyclo-ribbons.

This labelling is however not always convenient and it is useful to introduce another one, by *colored compositions* (I,u), that is, pairs formed by a composition $I = (i_1,\ldots,i_p)$, and a color word $u = (u_1,\ldots,u_p)$ of the same length.

The bijection between colored compositions and anti-cyclo-ribbons is easy to describe: starting with a colored composition, one rebuilds an anti-cycloribbon by separating adjacent blocks of the colored composition with different colors and gluing them back in the only possible way according to the criterion of being an anti-cyclo-ribbon. Conversely, one separates two adjacent blocks of different colors and glue them one below the other.

Recall that the $\mathbf{MR}^{(r)}$ Hopf algebra can be defined (see [14]) as the free associative algebra over the symbols $(S_j^{(i)})_{j\geq 1; 1\leq i\leq r}$, graded by $\deg S_j^{(i)} = j$, and with coproduct

$$\Delta S_n^{(k)} = \sum_{i=0}^{n} S_i^{(k)} \otimes S_j^{(k)}. \tag{30}$$

For a colored composition (I,u) as above, one defines

$$S^{(I,u)} := S_{i_1}^{(u_1)} \cdots S_{i_p}^{(u_p)}. \tag{31}$$

Clearly, the $S^{(I,u)}$ form a linear basis of $\mathbf{MR}^{(r)}$. There is a natural order on colored compositions, which generalizes the anti-refinement order on ordinary compositions: one says that $(J,v) \leq (I,u)$ if (J,v) can be obtained from (I,u) by adding up groups of consecutive parts of the same color.

For example, with two colors, the anti-refinements of $(2\overline{1}\overline{2}13)$ are

$$(2\overline{1}\overline{2}13), (2\overline{3}13), (2\overline{1}\overline{2}4), (2\overline{3}4). \tag{32}$$

The *colored ribbon basis* $R_{(I,u)}$ of $\mathbf{MR}^{(r)}$ can now be defined by the condition

$$S^{(I,u)} =: \sum_{(J,v)\leq (I,u)} R_{(J,v)}. \tag{33}$$

Let $(I,u) = (i_1,\ldots,i_p; u_1,\ldots,u_p)$ and $(J,v) = (j_1,\ldots,j_q; v_1,\ldots,v_q)$ be two colored compositions. We set

$$(I,u)\cdot(J,v) := (I\cdot J, u\cdot v), \tag{34}$$

where $a\cdot b$ denotes the concatenation.

Moreover, if $u_p = v_1$, we set

$$(I,u) \triangleright (J,v) := (i_1,\ldots,i_{p-1},(i_p+j_1),j_2,\ldots,j_q; u_1,\ldots,u_p,v_2,\ldots,v_q). \tag{35}$$

The colored ribbons satisfy the very simple multiplication rule:

$$R_{(I,u)}R_{(J,v)} = R_{(I,u)\cdot(J,v)} + \begin{cases} R_{(I,v)\triangleright(J,v)} & \text{if } u_p = v_1, \\ 0 & \text{if } u_p \neq v_1. \end{cases} \tag{36}$$

Let $[K,\mathbf{c}]$ be an anti-cyclo-ribbon. Let $P_{[K,\mathbf{c}]}$ be the indecomposable projective module whose unique simple quotient is the simple module labelled by $\phi(K,\mathbf{c})$ and let (I,u) be the corresponding colored composition.

Theorem 4.1. *The map*

$$\begin{array}{rcl} \mathbf{ch}: & \mathcal{K} & \to \mathbf{MR}^{(r)} \\ & [P_{[K,\mathbf{c}]}] & \mapsto R_{(I,u)} \end{array} \tag{37}$$

is an isomorphism of Hopf algebras.

The main interest of the labelling by colored compositions is that it allows immediate reading of some important information. For example, it follows from Theorem 4.1 that the products of complete functions $S^{(I,u)}$ are the characteristics of the projective $\mathcal{H}_{n,r}(0)$-modules obtained as induction products in which each factor $S_m^{(i)}$ is the characteristic of the one-dimensional projective $\mathcal{H}_{m,r}(0)$-module on which all the T_i act by 0 and all the ξ_j by the same eigenvalue u_i.

We see that, as in the case of $H_n(0)$, each indecomposable projective $\mathcal{H}_{n,r}(0)$-module occurs as a direct summand of such an induced module, and that the direct sum decomposition is given by the anti-refinement order. For example, the identity

$$S^{(2\overline{12}13)} = R_{(2\overline{12}13)} + R_{(2\overline{3}13)} + R_{(2\overline{12}4)} + R_{(2\overline{3}4)} \tag{38}$$

indicates what indecomposable projective direct summands compose the projective $\mathcal{H}_{9,2}(0)$-module defined as the outer tensor product

$$S_2 \widehat{\otimes} S_{\overline{1}} \widehat{\otimes} S_{\overline{2}} \widehat{\otimes} S_1 \widehat{\otimes} S_3. \tag{39}$$

Their restriction to $H_9(0)$ (and hence, their dimensions) can be computed by means of the following result:

Theorem 4.2. *The homomorphism of Hopf algebras*

$$\begin{array}{rll} \pi : & \mathbf{MR}^{(r)} & \to \mathbf{Sym} \\ & S_j^{(i)} & \mapsto S_j \end{array} \tag{40}$$

maps the class of a projective $\mathcal{H}_{n,r}(0)$-module to the class of its restriction to $H_n(0)$.

Continuing the previous example, we see that the restriction of $P_{2\overline{12}13}$ to $H_9(0)$ is given by

$$\pi(R_{2\overline{12}13}) = \pi(R_2 R_{\overline{12}} R_{13}) = R_2 R_{12} R_{13} = R_{21213} + R_{2133} + R_{3213} + R_{333}. \tag{41}$$

Dually, one can describe the induction product of projective $H_n(0)$-modules to $\mathcal{H}_{n,r}(0)$.

Theorem 4.3. *Let I be a composition of n and let M_I be the $\mathcal{H}_{n,r}(0)$-module induced by the indecomposable projective $H_n(0)$-module P_I. Then*

$$M_I \simeq \oplus P_{[I,\mathbf{c}]}, \tag{42}$$

where the sum runs over all the anti-cyclo-ribbons of shape I.

For example, let us complete the case $I = (2, 1)$ with two colors. The following five anti-cyclo-ribbons appear in the indution of P_I, with respective dimensions 3, 6, 3, 2, and 2

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline \end{array}\!\!\!\!\begin{array}{c} \\ \boxed{1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline \end{array}\!\!\!\!\begin{array}{c} \\ \boxed{2}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}\!\!\!\!\begin{array}{c} \\ \boxed{2}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}\!\!\!\!\begin{array}{c} \\ \boxed{2}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}\!\!\!\!\begin{array}{c} \\ \boxed{1}\end{array} \tag{43}$$

Finally, we can describe the Cartan invariants and the decomposition matrices by means of the maps

$$\begin{array}{rll} e : & \mathbf{MR}^{(r)} & \to Sym^{(r)} = (Sym)^{\otimes r} \simeq Sym(X_1, \ldots, X_r), \\ & S_j^{(i)} & \mapsto h_j(X_i) \end{array} \tag{44}$$

and

$$(45) \qquad \begin{array}{rrcl} d: & Sym^{(r)} & \hookrightarrow & QMR^{(r)} \\ & h_j(X_i) & \mapsto & F_{[(j),i^j]}. \end{array}$$

Then, the Cartan map is $c = d \circ e$, and the entry $C_{[I,\mathbf{c}],[J,\mathbf{d}]}$ of the Cartan matrix (giving the multiplicity of the simple module $S_{[J,\mathbf{d}]}$ as a composition factor of the indecomposable projective module $P_{[I,\mathbf{c}]}$) is equal to the coefficent of $F_{[J,\mathbf{d}]}$ in $c(R_{[I,\mathbf{c}]})$. The decomposition map is given by d and the decomposition matrix expresses the tensor product of Schur functions $S_\lambda = s_{\lambda^{(1)}}(X_1) \cdots s_{\lambda^{(r)}}(X_r)$ on the Poirier basis $F_{[I,\mathbf{c}]}$.

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