LOCAL ACTION OF THE SYMMETRIC GROUP AND GENERALIZATIONS OF QUASI-SYMMETRIC FUNCTIONS

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Dedicated to Adriano Garsia for the occasion of his 75-th birthday.

Abstract. This paper is the result of a tentative to better understand the result of Garsia and Wallach showing that the ring of quasi-symmetric functions is free as a module over the ring of symmetric functions. We conjecture that their result is a particular case of a more general situation whose construction is decribed here.

We consider a certain class of actions of the symmetric group $\mathfrak{S}_n$ on polynomials $\mathbb{Q}[X_n] = \mathbb{Q}[x_1, \ldots, x_n]$ called *local actions*, which are very similar to the one of [10, 11]. After classifying these actions, we study the sub-class of these actions whose set classifying these action, we study the class of these actions whose set of of fixed polynomial is a sub-algebra of $\mathbb{Q}[X_n]$. This gives rise to an infinite hierarchy of sub-Hopf-algebras of QSym, interpolating between QSym and Sym. We conjecture that these algebra are free modules over Sym as is suggested by an explicit formula giving the Hilbert series of the quotient.

During the time of preparation of the final version of this paper, Adriano Garsia and Nolan Wallack announced a proof the the main conjecture.

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Contents

1. Introduction

The symmetric polynomials Sym are by definition the polynomials of $\mathbb{Q}[X_n] := \mathbb{Q}[x_1, \dots, x_n]$ which are invariant under the action of the symmetric group $\mathfrak{S}_n$ on polynomials. They form a sub-algebra Sym $:= \mathbb{Q}[X_n]^{\mathfrak{S}_n}$ of $\mathbb{Q}[X_n]$, and consequently, polynomials can be considered as a module over Sym. It is well known that this module is free of rank $n!$ and this result is now widely interpreted for example in the cohomology or the Grothendieck ring of the flag manifold.

In his study of descents of permutations, Gessel defined a generalization of the notion of symmetric functions namely the quasi-symmetric functions QSym [8]. This algebra as been extensively studied: it can also be endowed with a natural Hopf algebra structure which is dual to the Hopf algebra **NCSF** of noncommutative symmetric functions [16, 7]. Quasi-symmetric functions are defined as the set of polynomials with a certain partial symmetry property. However, in [10, 11], the author described a new action of the symmetric group and its Hecke algebra on the space of polynomials whose invariant are exactly the quasi-symmetric polynomials. At this point, it is important to notice that the action is not compatible with the product, so that there is no way to deduce from it that quasi-symmetric polynomials form an algebra.

A further step in the study of quasi-symmetric polynomials is the result of Garsia and Wallach [6] showing that QSym is free as a module on Sym. This was conjectured by F. Bergeron and C. Reutenauer in 2000 and independently by J.-Y. Thibon in 1996 when he gave me my PhD subject. Unfortunately I was not able to solve it. However as we will see in the present paper, I was not completely wrong, at least I eventually got a PhD ! The action of the symmetric group I defined in [10, 11] tells something on the freeness.

As in the case of polynomials the module of QSym over Sym is of rank $n!$, which can be easily deduced from the generating series. However the proof of Garsia and Wallach relies on a very subtle analyze of the generating series and do not gives no good explanation of the link between the $n!$ and the symmetric group. The present paper is actually a tentative to understand their result. Though there is apparently no link between the rank $n!$ of the module and the existence of the action of the symmetric group, both of these properties seem to generalize. This give a new light, which allows us to give a combinatorial interpretation of the Hilbert series of the module, assuming its freeness:

$$\mathrm{Hilb}_t(\mathrm{QSym}^r(X_n)) = \sum_{\sigma\in\mathfrak{S}_n} \frac{t^{\mathrm{Maj}(\sigma)+r(n-\mathrm{Fix}(\sigma))}}{(1-t)(1-t^2)\cdots(1-t^n)}\,, \tag{1}$$

where Maj is the major index ant Fix is the number of fixed points. Such a result is quite unexpected even in the case where $r = 1$ that is for classical quasi-symmetric functions.

The paper is structured as follows. In a first part, we study a class of actions of the symmetric group on polynomials generalizing those of [10, 11]. In these actions, the elementary transposition σ_i acts only on the variables x_i and x_{i+1}, and therefore we call these action *local*. Then, we investigate the case where the set of fixed points under one of these action form a sub-algebra of the algebra of polynomials. This gives rise to an infinite hierarchy of sub-algebras of QSym, interpolating between QSym and Sym. We further show that, when the number of variables is infinite, these algebras are actually sub-Hopf-algebras of QSym and describe the dual Hopf algebras as quotient of **NCSF**. We show that, as in the case of QSym these algebras are free, and that, in the finite number of variable case, they seem to be Cohen-Macaulay as indicated by their generating series. Finally, we remark that most of this theory naturally extends to the case of non-commutative variables.

Acknowledgments. During the preparation of the final version of this paper, I receive a mail from Adriano Garsia together with, in attachment, a preprint of a paper whose goal is to show the main conjecture of this paper. As a consequence, they give a much simpler proof of the freeness of QSym, achieving the main goal of the present research.

2. Background

2.1. **Quasi-symmetric functions.** Let $X := \{x_1 < x_2 < \cdots < x_n\}$ denote a totally ordered set of commutative indeterminates. X is called the *alphabet*. By $\mathcal{P}(X)$ (resp. $\mathcal{P}_k(X)$), we mean the set of the subsets (resp. k-elements subsets) of the alphabet X.

Let m be the monomial $x_1^{m_1} \dots x_n^{m_n}$ where the m_i's are possibly zero. For readability, we identify m with the integer vector $[m_1, m_2, \dots, m_n]$. We define the *support* of m as the set $A \in \mathcal{P}(X)$ of the x_i's whose exponent are non-zero, as well as the composition I obtained by removing the zeros in the sequence $(m_1, m_2, \dots, m_n)$. In the sequel we write A^I in place of the monomial m. For example if $X = \{x_1 < x_2 < x_3 < x_4\}$, we write $x_1^2 x_3 = [2, 0, 1, 0] = \{x_1, x_3\}^{(2,1)}$ and $x_1^3 x_2^5 x_4 = [3, 5, 0, 1] = \{x_1, x_2, x_4\}^{(3,5,1)}$.

A polynomial $f \in \mathbb{Q}[X]$ is said to be *quasi-symmetric* if and only if for each composition $I = (i_1, \dots, i_r)$ the coefficient of the monomial

$$x_{j_1}^{i_1} x_{j_2}^{i_2} \dots x_{j_r}^{i_r} \tag{2}$$

is independent of the choice of $j_1 < j_2 < \cdots < j_r$. With our support and exponent notations it says that the coefficient of A^I is independent of the set of variables $A \in \mathcal{P}_r(X)$.

The quasi-symmetric polynomials form a subalgebra of $\mathbb{Q}[X]$ denoted by QSym_n. It is often convenient to let n tends toward ∞ and to take the inverse limit in the category of graded ring. Then, we get an algebra called the algebra of *quasi-symmetric functions* [8]. Such functions can be seen as formal sums of monomials on an infinite alphabet $X := \{x_1 < x_2 < \cdots < x_n < \cdots\}$.

It is clear that the family of so-called *quasi-monomial functions* defined by

$$M_I := \sum_{A \in \mathcal{P}_r(X)} A^I = \sum_{j_1 < \cdots < j_r} x_{j_1}^{i_1} \cdots x_{j_r}^{i_r} = \sum_{k \to I} x^k \tag{3}$$

labeled by compositions $I := (i_1 \dots, i_r)$ form a basis of QSym, and the last sum is extended to all integer vectors $k \to I$ of length n obtained by inserting zeros in the composition I. For example,

$$M_{(2,1)} = \{x_1, x_2\}^{(2,1)} + \{x_1, x_3\}^{(2,1)} + \{x_1, x_4\}^{(2,1)} + \{x_2, x_3\}^{(2,1)} + \{x_2, x_4\}^{(2,1)} + \{x_3, x_4\}^{(2,1)}$$

which can be written more concisely as:

$$M_{(2,1)} = [2, 1, 0, 0] \ + \ [2, 0, 1, 0] \ + \ [2, 0, 0, 1] \ + \ [0, 2, 1, 0] \ + \ [0, 2, 0, 1] \ + \ [0, 0, 2, 1]$$

instead of $M_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_1^2 x_4 + x_2^2 x_3 + x_2^2 x_4 + x_3^2 x_4$.

2.2. **Noncommutative symmetric functions.** The algebra of *noncommutative symmetric functions* [7] is the free associative algebra $\mathbf{NCSF} = \mathbb{Q}\langle S_1, S_2, \dots \rangle$ generated by an infinite sequence of noncommutative indeterminates S_k, called *complete* symmetric functions. For a composition $I = (i_1, i_2, \dots, i_r)$, one sets $S^I := S_{i_1} S_{i_2} \dots S_{i_r}$. The family (S^I) is a linear basis of $\mathbf{NCSF}$. A useful realization can be obtained by taking an infinite alphabet $A = \{ a_1, a_2, \dots \}$ and defining its complete homogeneous symmetric functions by the generating function

$$\sum_{n\geq 0} t^n S_n(A) = (1 - t\, a_1)^{-1} (1 - t\, a_2)^{-1} (1 - t\, a_3)^{-1} \dots \tag{4}$$

Then, $S_n(A)$ appears as the sum of all nondecreasing words of length n. Note that these functions are not symmetric in the usual sense. They are invariant for a more subtle action of the symmetric group of the alphabet due to Lascoux and Schützenberger [14], (see also [7]). The role of Schur functions is played by the noncommutative *ribbon Schur functions* R_I defined by

$$R_I := \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)} S^J . \tag{5}$$

The family (R_I) forms a basis of $\mathbf{NCSF}$. In the realization of $\mathbf{NCSF}$ given by Equation (4), R_I reduces to the sum of all words of shape I [7].

The duality pairing $\langle \cdot | \cdot \rangle$ between QSym and $\mathbf{NCSF}$ is defined by $\langle M_I | S^J \rangle = \delta_{IJ}$ or equivalently $\langle F_I | R_J \rangle = \delta_{IJ}$ (*cf.* [16, 7]). This duality can be interpreted as the canonical duality between the Grothendieck groups respectively associated with finite dimensional and projective modules over 0-Hecke algebras (*cf.* [5, 13]).

3. Local actions

3.1. **Definitions.**

Definition 1. A *local action* of $\mathfrak{S}_n$ on $\mathbb{Q}[X]$ is an action of $\mathfrak{S}_n$ on $\mathbb{Q}[X]$ *i.e.* a morphism $\rho : \mathfrak{S}_n \longrightarrow \mathrm{End}(\mathbb{Q}[X])$ which satisfies the following conditions:

(1) the elementary transposition σ_i acts only on the variables x_i and x_{i+1}, the other variables behaving as scalars;
(2) the action of an elementary transposition σ_i on a monomial $x_i^a x_{i+1}^b$ is independent of i and, depending on the values of a and b, either exchanges the variables or leaves the monomial invariant: for all i

$$\sigma_i(x_i^a x_{i+1}^b) = \begin{cases} x_i^b x_{i+1}^a & \text{for certain values of } a \text{ and } b\,, \\ x_i^a x_{i+1}^b & \text{for the other values}\,. \end{cases}$$

Consequently, the action of σ_1 on all monomials in x_1, x_2 fully determines the operation ρ.

Example 1. Let $m := [m_1, m_2, \dots m_n]$ be a monomial. Let also

$$\rho(\sigma_i)(m) := \begin{cases} [\dots, m_i, m_{i+1}, \dots] & \text{if } m_i \equiv m_{i+1} \pmod 2, \\ [\dots, m_{i+1}, m_i, \dots] & \text{otherwise.} \end{cases}$$

The hypothesis (1) and (2) are verified. We will see in the sequel that this actually defines a local action of the symmetric group.

The following proposition is an immediate consequence of the definition:

Proposition 3.1. *Let ρ a local action of $\mathfrak{S}_n$ on $\mathbb{Q}[x_1, \dots, x_n]$. Then the restriction of ρ to $\mathfrak{S}_{n-1}$ is a local action on $\mathbb{Q}[x_1, \dots, x_{n-1}]$.*

Let us give an other characterization of local actions:

Definition 2. Let ρ be a map of $\mathfrak{S}_n$ to $\operatorname{End}\mathbb{Q}[X]$ which satisfies conditions (1) and (2) of the definition 1. To ρ we associate the relation $\mathcal{R}_\rho$ on the integers defined by

$$u\,\mathcal{R}_\rho\,v \quad \text{iff} \quad \sigma_1[u,v] = [u,v]. \tag{6}$$

Note that we do not suppose that ρ is a morphism, consequently it does not surely define an action. The conditions (1) and (2) ensure that, for all i

$$\rho(\sigma_i)(m) = \begin{cases} [\dots, m_i, m_{i+1}, \dots] & \text{if } m_i\,\mathcal{R}_\rho\,m_{i+1}, \\ [\dots, m_{i+1}, m_i, \dots] & \text{otherwise.} \end{cases} \tag{7}$$

Conversely, if $\mathcal{R}$ is a reflexive relation on the set of integers, Equation (7) defines a map from $\mathfrak{S}_n$ to $\operatorname{End}\mathbb{Q}[X]$ which satisfies both conditions (1) and (2).

Proposition 3.2. *Let ρ be a map from $\mathfrak{S}_n$ to $\operatorname{End}\mathbb{Q}[X]$ which satisfies both conditions (1) and (2) of definition 1. Let $\mathcal{R}_\rho$ be the associated relation. The following properties are equivalent:*

(1) *ρ is a morphism from $\mathfrak{S}_n$ to $\operatorname{End}(\mathbb{Q}[X])$,*
(2) *$\mathcal{R}_\rho$ is an equivalence relation.*

Proof. One easily sees that the symmetry of the relation $\mathcal{R}_\rho$ is equivalent to the relation $\rho(\sigma_i)^2 = \operatorname{Id}_{\mathbb{Q}[X]}$.

Let us show that the braid relations implies the transitivity of $\mathcal{R}_\rho$. Let u, v, and w be three integers such that $u\,\mathcal{R}_\rho\,v$ and $v\,\mathcal{R}_\rho\,w$. Suppose that u is not related to w. One has consequently:

$$\rho(\sigma_1\,\sigma_2\,\sigma_1)[v,u,w] = \rho(\sigma_1\,\sigma_2)[v,w,u] = \rho(\sigma_1)[v,w,u] = [v,w,u]\,,$$

but also:

$$\rho(\sigma_2\,\sigma_1\,\sigma_2)[v,u,w] = \rho(\sigma_2\,\sigma_1)[v,w,u] = \rho(\sigma_2)[v,w,u] = [v,u,w]\,.$$

This is in contradiction with the hypothesis that ρ is a morphism from $\mathfrak{S}_n$ to $\operatorname{End}\mathbb{Q}[X]$. Thus, (1) implies (2).

To show the converse, rather than checking by hands the braid relation we will give an other construction of the action associated with $\mathcal{R}$ that makes obvious the fact that it is an action. Let $\mathcal{R}$ be an equivalence relation. Let $C = \mathbb{N}/\mathcal{R}$ be the quotient set, that is the set of equivalence classes for $\mathcal{R}$, and let us denote $C(i)$ the class of the integer i. Let m be a monomial. To m one associates, on one hand, the vector $C(m)$ on the alphabet C defined by

$$C(m) := [C(m_1), C(m_2), \ldots, C(m_n)]\,, \tag{8}$$

and, on the other hand, for all equivalence class c the sequence $E_c(m)$ of the exponents m_i that belongs to c, in the order of the variables. One gets a family $(E_c(m))_{c\in C}$, such that the length of the vector $E_c(m)$ is equal to the number of i's such that $C(m_i)$ is equal to c.

This clearly defines a bijective correspondence between the monomials on X and the couples

$$([c_1, c_2, \ldots, c_n], (E_c)_{c\in C}) \quad \text{such that} \quad \ell(E_c) = \#\{i \mid c_i = c\}\,. \tag{9}$$

The symmetric group $\mathfrak{S}_n$ acts naturally on the vector $[c_1, c_2, \ldots, c_n]$ by permuting the entries:

$$\sigma[c_1, c_2, \ldots, c_n] = [c_{\sigma^{-1}(1)}, c_{\sigma^{-1}(2)}, \ldots, c_{\sigma^{-1}(n)}] \tag{10}$$

If σ is a permutation, one sends the monomial associated to the couple $([c_1, \ldots, c_n], (E_c))$ to the monomial associated to $(\sigma[c_1, \ldots, c_n], (E_c))$. This clearly defines an action of the symmetric group on $\mathbb{Q}[X]$. It is easy to check that the action of the elementary transposition of these action agrees with definition 2. □

Example 2. Let us go back to the former example. There are two equivalence classes: E for even and O for odd. To the monomial

$$m = [\ 0,\ 2,\ 1,\ 4,\ 3,\ 5,\ 1,\ 1,\ 2\ ]$$

we associate the word

$$C(m) = [\ \mathrm{E},\ \mathrm{E},\ \mathrm{O},\ \mathrm{E},\ \mathrm{O},\ \mathrm{O},\ \mathrm{O},\ \mathrm{O},\ \mathrm{E}\ ]$$

and the two sequences of exponents

$$E_{\mathrm{E}} = (0, 2, 4, 2) \quad \text{and} \quad E_{\mathrm{O}} = (1, 3, 5, 1, 1).$$

Let $\sigma = 743652198$. Then,

$$\sigma(C(m)) = [\ \mathrm{O},\ \mathrm{O},\ \mathrm{O},\ \mathrm{E},\ \mathrm{O},\ \mathrm{E},\ \mathrm{E},\ \mathrm{E},\ \mathrm{O}].$$

We finally get

$$\rho(\sigma)m = [\ 1,\ 3,\ 5,\ 0,\ 1,\ 2,\ 4,\ 2,\ 1\ ].$$

Example 3. The classical action of $\mathfrak{S}_n$ corresponds to the trivial equivalence relation $u \mathcal{R} v$ if and only if $u = v$. The quasi-symmetrizing action [10, 11] corresponds to the relation $u \mathcal{R} v$ if and only if $u = v = 0$ or u and v are both non zero.

3.2. **Characteristic and generalized Temperley-Lieb algebras.** In the previous subsection, we defined a family of actions of the symmetric group $\mathfrak{S}_n$ on polynomials. In this section we describe more precisely these actions in terms of the representations theory. To describe a representation of $\mathfrak{S}_n$, rather than giving its character we prefer the equivalent but easier to handle Frobenius characteristic. We only describe briefly this tool and refer to [15] for notations and details.

Recall that the direct sum $\bigoplus_{n\geq 0} R(\mathfrak{S}_n)$ of Grothendieck rings of all symmetric groups is in natural isomorphism with the ring of symmetric functions, by the so-called Frobenius characteristic map *ch*. It sends the irreducible character χ^λ to the Schur function s_λ. The product of symmetric functions corresponds to induction from $\mathfrak{S}_n \times \mathfrak{S}_p$ to $\mathfrak{S}_{n+p}$. One can get the value of the character χ on the conjugacy class (cycle type) indexed by the partition $\mu = (\mu_1, \dots, \mu_p)$ by the scalar product:

$$\chi(\mu) = \langle \mathrm{ch}(\chi) | p_\mu \rangle \,. \tag{11}$$

where p_μ is the product of power sum symmetric functions.

The r-actions are compatible with the usual grading of polynomial rings. Hence, one can define the graded characteristic ch_t of the representation on $\mathbb{Q}[X]$ as the generating series of the characteristics of the representations on the homogeneous components $\mathbb{Q}_i[X]$:

$$\mathrm{ch}_t(\mathbb{Q}[X]) = \sum_{i=0}^{\infty} \mathrm{ch}(\mathbb{Q}_i[X]) t^i. \tag{12}$$

The main result of this section is the following

Theorem 3.3. *Let $(\rho_n)_n$ be the family of local actions of $\mathfrak{S}_n$ related to an equivalence relation $\mathcal{R}_\rho$. Denote by $C := \mathbb{N}/\mathcal{R}_\rho$ the set of equivalence classes. The generating series of the graded characteristic of the actions ρ_n of $\mathfrak{S}_n$ with respect to n is given by*

$$\sum_n \mathrm{ch}_t(\mathbb{Q}[X_n]_\rho) u^n = \prod_{c\in C} H\left(\sum_{i\in c} t^i u\right) \,, \tag{13}$$

where,

$$H(u) := \sum_j h_j u^j \,. \tag{14}$$

Proof. Recall that h_n is the character of the trivial representation of $\mathfrak{S}_n$. This formula is a translation in the language of symmetric functions, that the action of each $\mathfrak{S}_n$ is isomorphic to the induced representation on all the trivial representations on the equivalence classes $c \in C$.

Moreover, it is clear that the series of these trivial graded character on a class $c \in C$ is given by

$$\sum_n \left(\sum_{i\in c} t^i\right)^n u^n = H\left(\sum_{i\in c} t^i u\right). \tag{15}$$

Since induction of representations corresponds to product of symmetric functions we are done. □

In particular, if the number s of equivalence classes in C is finite, then only the products of at most s terms h_i appear in this characteristic. As a consequence, only the irreducible representation V_λ for λ of length smaller than s appear. Thus the action is in fact an action of a quotient of the algebra of $\mathfrak{S}_n$. This can be described explicitly:

Proposition 3.4. *Denote by $C := \mathbb{N}/\mathcal{R}_\rho$ the set of equivalence classes associated to a local action ρ of $\mathfrak{S}_n$. Suppose that C is finite of cardinal $s < n$. Then, the action of the element*

$$\nabla := \sum_{\sigma\in\mathfrak{S}_{s+1}} -1^{\ell(\sigma)}\sigma \tag{16}$$

where $\ell(\sigma)$ is the length (number of inversions) of σ, is zero.

Proof. The element ∇ verifies $\nabla\sigma_{ij} = -\nabla$ for each transposition σ_{ij} where $i < j \le s$. Recall that, in each orbit the action is isomorphic to the action on a word on the letters in C of length n. Since the cardinality s of C is smaller than n, in such a word w there is at least one equivalence class that appears twice, say at positions i and j. But then $\sigma_{ij}w = w$ and consequently $\nabla w = 0$. Hence we have proved that the action of ∇ is zero in each orbit. □

In fact, this is sufficient to prove the apparently stronger result which is a an easy generalization of Theorem 3.1 of [11] where the reader which is not familiar with pattern avoidance should refer.

Theorem 3.5. *The image of $\mathbb{Q}[\mathfrak{S}_n]$ in $\mathrm{End}(\mathbb{Q}[X])$ is the quotient of $\mathbb{Q}[\mathfrak{S}_n]$ by the ideal generated by*

$$\nabla_i := \sum_{\sigma\in\mathfrak{S}^i_{s+1}} -1^{\ell(\sigma)}\sigma, \tag{17}$$

where $\mathfrak{S}^i_{s+1}$ is the shifted symmetric group $\mathfrak{S}_{s+1}$ which acts on the set $\{i, i+1, \dots, i+s+1\}$.

The operators $\rho(\sigma)$ where σ runs along the sets of permutations that avoid the pattern $(s+1)s\dots21$ form a basis of the image of $\mathbb{Q}[\mathfrak{S}_n]$ in $\mathrm{End}(\mathbb{Q}[X])$.

Recall that the permutation avoiding $s+1s\dots21$ are also the permutations such that the shape of the tableaux obtained by Robinson-Schensted map is a partition with at most s rows.

The proof is then the same as in [11], by comparison on dimension.

3.3. **Fixed polynomials and main theorem.**

Definition 3. Let ρ be a local action. A polynomial f is *fixed by ρ* or *ρ-symmetric*, if it is invariant under the action of ρ, that is,

$$\rho(\sigma)f = f \qquad \text{for all } \sigma \in \mathfrak{S}_n. \tag{18}$$

The set of ρ-symmetric polynomials is denoted by $\mathbb{Q}[X_n]^{\rho(\mathfrak{S}_n)}$.

Due to Example 3, this a generalization of the notion of symmetric and quasi-symmetric polynomials.

We now want to investigate for which local actions the set of fixed polynomials form a sub-algebra of the polynomial algebra $\mathbb{Q}[x1, \dots, x_n]$. Equivalently, we want to know under which conditions on $\mathcal{R}$ the product of two fixed polynomials is always a fixed polynomial. Let us start with some simple remarks.

Remark 1. Let ρ be a local action of $\mathfrak{S}_n$ on the space of polynomials. Then any symmetric polynomial in the classical sense is ρ-symmetric.

Remark 2. Let ρ be a local action of $\mathfrak{S}_n$ on the space of polynomials. Let $f(x_1, \dots, x_n)$ be a ρ-symmetric polynomial. Then $f(x_1, \dots, x_{n-1}, 0)$ is μ-symmetric under the restriction μ of ρ to $\mathfrak{S}_{n-1}$.

We now want to classify the local-actions whose symmetric polynomials form an algebra.

Definition 4. Let r be an integer or infinite. The r-action of $\mathfrak{S}_n$ on polynomials denoted by ρ_r is defined by

$$\rho_r(\sigma_i)(m) = \begin{cases} [\dots, m_i, m_{i+1}, \dots] & \text{if } m_i \geq r \text{ and } m_{i+1} \geq r, \\ [\dots, m_{i+1}, m_i, \dots] & \text{otherwise.} \end{cases} \tag{19}$$

Note that the trivial action corresponds to $r = 0$, the classical action corresponds to r infinite and the quasi-symmetrizing action of [10, 11] corresponds to $r = 1$.

The main theorem says that they are the only actions whose invariants are stable by multiplication :

Theorem 3.6. *Let ρ be a local action of $\mathfrak{S}_n$. The set $\mathbb{Q}[X_n]^{\rho(\mathfrak{S}_n)}$ of fixed polynomials is a sub-algebra of $\mathbb{Q}[X]$ if an only if there exists an r, integer or infinite, such that $\rho = \rho_r$.*

Proof. Suppose that ρ is a local action whose fixed set forms an algebra. Let $\mathcal{R}_\rho$ the associated equivalence relation. To deal with $\mathcal{R}_\rho$ Remark 2 allows us to restrict ourself to the two variables case.

Suppose that there exists two integers $u < v$ such that $u\,\mathcal{R}_\rho\,v$ which means that the polynomial $f = x_1^u x_2^v = [u, v]$ is ρ-symmetric. By hypothesis, the product

$$(x_1 + x_2)f = x_1^{u+1}x_2^v + x_1^u x_2^{v+1} = [u + 1, v] + [u, v + 1]$$

is ρ-symmetric. Then

$$u+1\,\mathcal{R}_\rho\,v \quad \text{and} \quad u\,\mathcal{R}_\rho\,v+1.$$

Using $u\,\mathcal{R}_\rho\,v$ and transitivity, it follows that

$$u\,\mathcal{R}_\rho\,u+1.$$

By induction, for all $w \geq u$, on gets

$$u\,\mathcal{R}_\rho\,w\,.$$

Let r be the smallest N such that $N\,\mathcal{R}_\rho\,N+1$, then $\rho = \rho_r$, as desired.

Conversely, let r be an integer and ρ the associated action. It is easy to check that the product of two ρ-symmetric polynomial is again ρ-symmetric. Indeed, if the exponent are smaller than r, the polynomial is symmetric in the classical sense, otherwise the hypothesis is not restrictive. $\square$

Let $\mathrm{QSym}^r(X_n) := \mathbb{Q}[X_n]^{\rho_r(\mathfrak{S}_n)}$ denote the sub-algebra of $\mathbb{Q}[X_n]$ spanned by r-quasi-symmetric polynomial. It is clear that the restriction morphisms $(n > p)$

$$\begin{aligned} \phi_p^n \;:\; \mathrm{QSym}^r(x_1,\dots,x_n) &\longrightarrow \mathrm{QSym}^r(x_1,\dots,x_p) \\ f(x_1,\dots,x_n) &\longmapsto f(x_1,\dots,x_p,0,\dots,0) \end{aligned} \tag{20}$$

are compatible (*i.e.* $\phi_n^m \circ \phi_p^n = \phi_p^m$). Hence, it makes sense to take the reverse limit of this projective system in the category of *graded* algebras. We emphasize on the graded condition because we only want to deal with finite degree expressions rather than series. The limit is called the algebra of r-quasi-symmetric functions denoted by QSym^r. We will see in Section 4 than it can actually be equipped naturally with a Hopf algebra structure. For the moment we only concentrate on the algebra structure.

As in the case of classical symmetric and quasi-symmetric functions (*cf.* [15, 8]), QSym^r can be viewed as an algebra of power series of finite degree on an infinite set of variables.

The following inclusions hold:

$$\begin{aligned} \mathbb{Q}[X_n] = \mathrm{QSym}^0(X_n) \supset \mathrm{QSym}^1(X_n) \supset \mathrm{QSym}^2(X_n) \supset \\ \cdots \supset \mathrm{QSym}^r(X_n) \supset \cdots \supset \mathrm{QSym}^\infty(X_n) = \mathrm{Sym}(X_n)\,, \end{aligned} \tag{21}$$

and for the infinite alphabet case:

$$\mathrm{QSym}^0 \supset \mathrm{QSym}^1 \supset \cdots \supset \mathrm{QSym}^r \supset \cdots \supset \mathrm{QSym}^\infty = \mathrm{Sym}\ . \tag{22}$$

A basis of QSym^r is given by the analog of the quasi-monomial basis *i.e.* by the orbit-sums. It is indexed by pairs of composition I in parts at least r and a partition λ whose parts are strictly smaller than r. We call such a pair an r-*composition* and write it (I, λ). A monomial whose exponent is an r-composition with possibly added trailing zeroes is called a r-dominant monomial.

For example, here is the list of the 24 3-compositions of 8

$$(8),(7,1),(6,2),(6,11),(53),(5,21),(5,111),(44),(43,1),(4,22),$$
$$(4,211),(4,1111),(35),(34,1),(33,2),(33,11),(3,221),(3,2111),$$
$$(3,11111),(2222),(22211),(221111),(2111111),(11111111)\,.$$

For later reference let us state the following obvious lemma:

Lemma 3.7. *For a fixed r, there is only one r-dominant monomial in each orbit under the r-action.*

The r-dominant monomial corresponding to the orbit of the monomial $m = [m_1, \dots, m_i, \dots]$ is the monomial obtained by putting the part m_i strictly smaller than r in decreasing order and the end of m.

Take $I = (I_1, \dots, I_k)$ and $\lambda = (\lambda_1, \dots, \lambda_l)$. The monomial function $M^r_{(I,\lambda)}$ is defined as the sum of the orbit of the r-dominant monomial

$$X^{(I,\lambda)} := x_1^{I_1} x_2^{I_2} \dots x_k^{I_k} x_{k+1}^{\lambda_1} x_{k+2}^{\lambda_2} \dots x_{k+l}^{\lambda_l} \tag{23}$$

under the r-action, that is

$$M^r_{(I,\lambda)}(X) := \sum_{\substack{i_1 \neq i_2 \neq \dots \neq i_{k+l} \\ i_1 < i_2 < \dots < i_k}} x_{i_1}^{I_1} x_{i_2}^{I_2} \dots x_{i_k}^{I_k} x_{i_{k+1}}^{\lambda_1} x_{i_{k+2}}^{\lambda_2} \dots x_{i_{k+l}}^{\lambda_l} \tag{24}$$

One easily sees that

$$M^r_{(I,\lambda)} = \sum_{K \in I \sqcup\!\sqcup \lambda^\sigma} M_K\,, \tag{25}$$

where λ^σ denotes the set of all distinct reordering of λ and $M_K = M^1_K$ is the classical quasi-symmetric functions. For example

$$\begin{aligned} M^3_{(4,6),(2,1)} &= M_{4,6,1,2} + M_{4,1,6,2} + M_{4,1,2,6} + M_{1,4,6,2} + M_{1,4,2,6} + M_{1,2,4,6} \\ &\quad + M_{4,6,2,1} + M_{4,2,6,1} + M_{4,2,1,6} + M_{2,4,6,1} + M_{2,4,1,6} + M_{2,1,4,6} \end{aligned}$$

The previous expression allows to deduce the product rule of QSym^r from the one of QSym. Recall that the product of Classical quasi-monomial function is given by the quasi-shuffle:

$$M_I M_J = \sum_K \langle K | I \uplus J \rangle\, M_K\,, \tag{26}$$

where $I \uplus J$ is defined recursively by

$$\epsilon \uplus I \;=\; I \uplus \epsilon = I \tag{27}$$

$$I \uplus J \;=\; i_1 \triangleright (I' \uplus J) + j_1 \triangleright (I \uplus J') + (i_1 + j_1) \triangleright (I' \uplus J')\,, \tag{28}$$

where ϵ is the empty composition, $\triangleright$ denotes the operation of adding a part at the beginning of a composition, and finally $I = i_1 \triangleright I'$ and $J = j_1 \triangleright J'$. Then

Proposition 3.8. *Let $r > 0$ and (I, λ) and J, μ two r-compositions. Then,*

$$M^r_{(I,\lambda)} M^r_{(J,\mu)} = \sum_{(K,\nu)} \sum_{\substack{A \in I \sqcup\!\sqcup \lambda^\sigma \\ B \in J \sqcup\!\sqcup \mu^\sigma}} \langle (K,\nu) | A \uplus B \rangle \, M^r_{(K,\nu)} \tag{29}$$

where λ^σ and μ^σ run through the set of the reordering of λ and μ as in Equation (25) and $\langle (K,\nu)|A \uplus B\rangle$ is the coefficient of (K,ν) in the quasi-shuffle of A with B.

Proof. By Equation (25),

$$M^r_{(I,\lambda)} M^r_{(J,\mu)} = \sum_{A \in I \sqcup\!\sqcup \lambda^\sigma ; B \in J \sqcup\!\sqcup \mu^\sigma} M_A M_B \,. \tag{30}$$

However, the product of the quasi-monomial function M_I is given by the quasi-shuffle, consequently

$$M^r_{(I,\lambda)} M^r_{(J,\mu)} = \sum_C \sum_{A \in I \sqcup\!\sqcup \lambda^\sigma ; B \in J \sqcup\!\sqcup \mu^\sigma} \langle C | A \uplus B \rangle \, M_C \,. \tag{31}$$

Then, because we already know that this is r-quasi-symmetric, we can only consider the case where C is an r-composition in this expression. This gives the announced formula. □

3.4. **Generating series.**

Theorem 3.9. *The generating series of the graded characteristic of the r-action on $\mathfrak{S}_n$ with respect to n is given by*

$$\sum_n \mathrm{ch}_t(\mathbb{Q}[X_n]) a^n = H\left(\frac{at^r}{1-t}\right) \prod_{i=0}^{r-1} H(at^i) \,, \tag{32}$$

where as usual

$$H(u) := \sum_i h_i u^i = \sigma_u(A) = \sum_i h_i(A) u^i \,. \tag{33}$$

In λ-ring notation

$$\sum_n \mathrm{ch}_t(\mathbb{Q}[X_n]) a^n = \sigma_u\left(\left(\frac{1-t^r}{1-t} + q\right) A\right)\Bigg|_{q=\frac{t^r}{1-t}} \,. \tag{34}$$

Proof. This is a particular case of Theorem 3.3. □

A consequence of this theorem, which can also be proved directly, is the generating series of the homogeneous dimension of QSym^r.

Theorem 3.10. *The generating series of the dimensions of the homogeneous components of $\mathrm{QSym}^r(X_n)$ is given by:*

$$\sum_{d,n} \dim_d(\mathrm{QSym}^r(X_n)) a^n t^d = \frac{1-t}{1-t-at^r} \prod_{i=0}^{r-1} \frac{1}{1-at^i} \tag{35}$$

The generating series of the dimensions of the homogeneous components of QSym^r *is given by:*

$$\mathrm{Hilb}_t(\mathrm{QSym}^r) := \sum_{d,n} \dim_d(\mathrm{QSym}^r) t^d = \frac{1-t}{1-t-t^r} \prod_{i=1}^{r-1} \frac{1}{1-t^i} \tag{36}$$

Proof. The dimension $\dim_d(\mathrm{QSym}^r(X_n))$ is the multiplicity of the trivial character h_n is the r-action. This multiplicities is 1 in each product $h_{i_1} \dots h_{i_k}$ where $i_1 + \dots i_k = n$.

Another way to prove this is to see that the given series is the series of the r-compositions completed by trailing zeroes counted by their sums t^d and by their lengths a^n. Indeed the series

$$\prod_{i=0}^{r-1} \frac{1}{1-at^i} \tag{37}$$

is the series of partitions with trailing zeroes and parts at most $r-1$, and

$$\frac{1}{1-\frac{at^r}{1-t}} = \frac{1-t}{1-t-at^r} \tag{38}$$

is the series of compositions in part at least r.

Finally the second series is the same at the first one, when we forget about the length and do not add trailing zeroes. □

Here are the first values:

	0	1	1	3	4	5	6	7	8	9	10
QSym	1	1	2	4	8	16	32	64	128	256	512
QSym^2	1	1	2	3	5	8	13	21	34	55	89
QSym^3	1	1	2	3	5	7	11	16	24	35	52
QSym^4	1	1	2	3	5	7	11	15	22	31	44
Sym	1	1	2	3	5	7	11	15	22	30	42

4. Hopf algebra structure

There are two equivalent ways to endow QSym^r with a Hopf algebra structure. The first way is to prove that QSym^r is a sub-Hopf algebra of QSym; that is that the coproduct of an r-Quasi-Symmetric element is an r-quasi-symmetric element. We prefer a slightly less direct but in our opinion more instructive way, namely to use the classical alphabet doubling trick to define the coproduct.

Let $X = \{x_1, x_2, \dots\}$ and $Y = \{y_1, y_2, \dots\}$ be two totally ordered infinite alphabets. The ordered sum $X \sqcup Y$ of alphabet is, as a set, the union of X and Y, where we keep the order inside X and Y and we choose that the elements of X are smaller than the elements of

Y. Clearly, as soon as an alphabet Z is totally ordered and infinite $\mathrm{QSym}^r \simeq \mathrm{QSym}^r(Z)$; in particular,

$$\mathrm{QSym}^r(X) \simeq \mathrm{QSym}^r(X \sqcup Y)\,. \tag{39}$$

But now its clear from the restriction property (Remark 2) that a r-quasi-symmetric function in $X \sqcup Y$ is in particular r-quasi-symmetric in X and in Y. Thus one has a natural algebra morphism

$$\mathrm{QSym}^r(X) \mapsto \mathrm{QSym}^r(X) \otimes \mathrm{QSym}^r(Y)\,. \tag{40}$$

which can be seen as a coproduct. The co-associativity is an easy consequence of the associativity of the ordered sum of alphabets. The co-unit corresponds to the restriction to an empty alphabet, *i.e.* the mapping which extracts the coefficient of the constant monomial.

Hence the following theorem

Theorem 4.1. *The r-quasi-symmetric functions form an infinite hierarchy of graded sub-Hopf-algebras:*

$$\mathrm{QSym} = \mathrm{QSym}^1 \supset \cdots \supset \mathrm{QSym}^r \supset \cdots \supset \mathrm{QSym}^\infty = \mathrm{Sym}\,. \tag{41}$$

Let us express this explicitly.

Proposition 4.2. *Let $r > 0$ and (I, λ) be a r-composition. Then, the coproduct of $M^r_{(I,\lambda)}$ is given by*

$$\delta(M^r_{(I,\lambda)}) = \sum_{(K,K')=I;\ \lambda\in\nu\sqcup\!\sqcup\nu'} M^r_{(K,\nu)} \otimes M^r_{(K',\nu')}\,. \tag{42}$$

Proof. Suppose that $I = (I_1, \dots, I_k)$ and $\lambda = (\lambda_1, \dots, \lambda_l)$. We have to find the r-dominant monomials in X and Y in $M^r_{(I,\lambda)}(X \sqcup Y)$. The expansion of the r-quasi-monomial function (Equation 24) writes

$$M^r_{(I,\lambda)}(X \sqcup Y) = \sum_{\mu\in\lambda^\sigma} \sum_{r=0}^{k} \sum_{s=0}^{l} \tag{43}$$
$$\sum_{\substack{i_1<i_2<\cdots<i_r\\ j_1<j_2<\cdots<j_{k-r}}} x_{i_1}^{I_1} \dots x_{i_r}^{I_r} y_{j_1}^{I_{r+1}} \dots y_{j_{k-r}}^{I_k} x_{i_{r+1}}^{\mu_1} \dots x_{i_{r+s}}^{\mu_s} y_{j_{k-r+1}}^{\mu_{s+1}} \dots y_{j_{k+l-(r+s)}}^{\mu_l}\,,$$

where μ runs through the set of all the distinct reordering of λ and the fourth sum is over the set of all the tuples of pairwise distinct indices $i_1 \neq i_2 \neq \cdots \neq i_{r+s}$ and $j_1 \neq j_2 \neq \cdots \neq j_{k+l-(r+s)}$. A monomial of this sum is dominant if and only if

$$i_1 < i_2 < \cdots < i_{r+s} \quad \text{and} \quad j_1 < j_2 < \cdots < j_{k+l-(r+s)} \tag{44}$$

on one hand and

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s \quad \text{and} \quad \mu_{s+1} \geq \mu_{s+2} \geq \cdots \geq \mu_l\,, \tag{45}$$

on the other hand. Since λ is in decreasing order, this second condition can be expressed as

$$\lambda \in (\mu_1\mu_2 \dots \mu_s \sqcup\!\sqcup \mu_{s+1}\mu_{s+2} \dots \mu_l)\,. \tag{46}$$

Hence if we call $K := (i_1, \dots, i_r)$ and $K' := (I_{r+1} \dots, I_k)$ on one hand and $\nu := (\mu_1\mu_2\dots\mu_s)$ and $\nu' := (\mu_{s+1}\mu_{s+2}\dots\mu_l)$ on the other hand, Equation 43 rewrites as

$$M^r_{(I,\lambda)}(X \sqcup Y) = \sum_{(K,K')=I;\ \lambda\in\nu ш\nu'} M^r_{(K,\nu)}(X)\, M^r_{(K',\nu')}(Y)\,. \tag{47}$$

which, through translation $F(X)G(Y) \mapsto F\otimes G$ gives finally the desired formula. □

For example

$$\begin{aligned}\delta(M^2_{635211}) = M^2_{635211}\otimes 1 + M^2_{63521}\otimes M^2_{1} + M^2_{63511}\otimes M^2_{2} + M^2_{6351}\otimes M^2_{21} +\\ M^2_{635}\otimes M^2_{211} + M^2_{63211}\otimes M^2_{5} + M^2_{6321}\otimes M^2_{51} + M^2_{6311}\otimes M^2_{52} +\\ M^2_{631}\otimes M^2_{521} + M^2_{63}\otimes M^2_{5211} + M^2_{6211}\otimes M^2_{35} + M^2_{621}\otimes M^2_{351} +\\ M^2_{611}\otimes M^2_{352} + M^2_{61}\otimes M^2_{3521} + M^2_{6}\otimes M^2_{35211} + M^2_{211}\otimes M^2_{635} +\\ M^2_{21}\otimes M^2_{6351} + M^2_{11}\otimes M^2_{6352} + M^2_{1}\otimes M^2_{63521} + 1\otimes M^2_{635211}\end{aligned}$$

4.1. **Dual Hopf algebra and NCSF.** Since the Hopf algebra QSym^r is a sub-algebra the algebra of quasi-symmetric functions, its dual is naturally a quotient of **NCSF**. The goal of this section is to describe explicitly this duality.

Recall that **NCSF** is the free associative algebra over an infinite set of generators S_i with basis (S_i) where

$$S^I = S^{(I_1,I_2,\dots,I_k)} := S_{I_1}S_{I_2}\dots S_{I_k}\,. \tag{48}$$

The bases (M_I) and (S^I) are two dual bases of respectively QSym and **NCSF**. Since the basis $M^r_{(I,\lambda)}$ of QSym^r is formed by disjoints sums of elements of (M_I), the dual of QSym^r can be described by identifying some elements of the basis (S^I).

The dual of QSym^r can be described explicitly as follows:

Theorem 4.3. *The dual of* QSym^r *is the quotient of the Hopf algebra* **NCSF** *by the relations*

$$S_iS_j = S_jS_i \qquad \text{for all } i < r \text{ and } j \in \mathbb{N}. \tag{49}$$

A basis of this quotient is given by $(S^{(I,\lambda)})$, where (I,λ) goes along the set of all r-compositions.

Proof. Suppose that $i < r$ and $j \geq i$. Then, Equation (25) expands the r-quasi-monomial function as $M^r_{(j,i)} = M_{(i,j)} + M_{(j,i)}$. Moreover the two terms $M_{(i,j)}$ and $M_{(j,i)}$ appear in M^r_K only in the case $K = (i,j)$. Otherwise stated, for all K,

$$\left\langle M^r_K | S^{(i,j)}\right\rangle = \left\langle M^r_K | S^{(j,i)}\right\rangle = \delta_{K,(j,i)}\,. \tag{50}$$

As a consequence in the dual of QSym^r one has $S_iS_j = S_jS_i$ if $i < r$. Moreover, it easy to see that $(S^{(I,\lambda)})$ is a basis of the quotient of **NCSF** by these relations. Since both the dual and the quotient have the same finite homogeneous dimension they coincide. □

4.2. **Primitive Elements.** The Hopf algebra QSym^r is graded connected and co-commutative. Hence, by the Milnor-Moore theorem it is isomorphic to the universal enveloping algebra of the Lie algebra of its primitive elements. The goal of this section is to explicitly describe these primitive elements.

In [7] (see also [16]), an explicit construction of several generating families, called noncommutative power sums, of the Lie algebra $\mathrm{Prim}(\mathbf{NCSF})$ is given. Let us recall briefly one of them. We use a slightly modified base: our Φ_n corresponds to $n\Phi_n$ of [7] or equivalently to the base P_n^* of [16]. They are defined by means of generating series, the relation

$$\sum_{k\geq 1} t^k \Phi_k = \log\left(1 + \sum_{k\geq 1} S_k t^k\right) \tag{51}$$

defining uniquely a sequence of primitive elements $(\Phi_k)_{k\geq 1}$ in $\mathbf{NCSF}$. Explicitly this gives:

$$\Phi_n = \sum_{K\vDash n} \frac{(-1)^{\ell(K)-1}}{\ell(k)} S^K\,. \tag{52}$$

An important consequence is that $\mathbf{NCSF}$ is freely generated by (Φ_n); moreover as a Hopf algebra it is isomorphic to the universal enveloping algebra $\mathrm{U}(\mathcal{L}(\Phi_i; i > 0))$ where $\mathcal{L}(\Phi_i; i > 0)$ is the free Lie algebra generated by $(\Phi_i)_{i>0}$.

By projecting these elements into $\mathbf{NCSF}^r$ we get an explicit description of the structure of the primitive Lie algebra $\mathrm{Prim}(\mathbf{NCSF}^r)$ and of the Hopf algebra $\mathbf{NCSF}^r$.

Theorem 4.4. $\mathbf{NCSF}^r$ *is the quotient of the Hopf algebra* $\mathbf{NCSF}$ *by the relations*

$$[\Phi_i, \Phi_j] = 0 \qquad \textit{for all } i < r \textit{ and } j \in \mathbb{N}. \tag{53}$$

The Lie algebra $\mathrm{Prim}(\mathbf{NCSF}^r)$ *is generated by* $(\Phi_i)_{i>0}$. *Moreover it is isomorphic to the central extension of the free Lie algebra with generators* $(\Phi_i)_{i\geq r}$ *by the trivial commutative Lie algebra with basis* $(\Phi_i)_{i<r}$*:*

$$\mathrm{Prim}(\mathbf{NCSF}^r) \simeq \mathbb{Q}[\Phi_1, \ldots, \Phi_{r-1}] \otimes \mathcal{L}(\Phi_r, \Phi_{r+1}, \ldots)\,. \tag{54}$$

Proof. Due to Equation (52), the ideal generated by $[S_i, S_j]$ for $i < r$ is equal to the ideal generated by $[\Phi_i, \Phi_j]$ for $i < r$. We deduce that as an algebra $\mathbf{NCSF}^r$ is the tensor product of the free commutative algebra with generators $(\Phi_i)_{i<r}$ by the free noncommutative algebra with generators $(\Phi_i)_{i\geq r}$. Moreover all these generators are primitive. However, it is known that the free commutative algebra on a set of

generators G is the enveloping algebra of the trivial commutative Lie algebra on the vector space spanned by G, and that the free algebra on G is the enveloping algebra of the free Lie algebra on G. Thus $\mathbf{NCSF}^r$ is the enveloping algebra of the Lie algebra

$$K[\Phi_1, \dots, \Phi_{r-1}] \otimes \mathcal{L}(\Phi_r, \Phi_{r+1}, \dots\,)\,. \tag{55}$$

which shows that $\mathrm{Prim}(\mathbf{NCSF}^r)$ is generated by the (Φ_i)'s. □

Hence we get that the set of the elements

$$\Phi^{(I,\lambda)} = \Phi_{I_1} \dots \Phi_{I_k} \Phi_{\lambda_1} \dots \Phi_{\lambda_k}\,, \tag{56}$$

where (I, λ) goes along all r-compositions, is a multiplicative basis of $\mathbf{NCSF}^r$ generated by primitive elements.

5. Internal product and descent algebras

Recall that there is a second product structure on **NCSF** which is the analogue of the internal product $*$ of symmetric functions. Although this operation is usually defined in terms of characters of symmetric groups, it also has a more fundamental characterization as the dual of the natural comultiplication $\delta(f) = f(XY)$, where, as usual, XY is the set of products $x_i y_j$ and $u(X)v(Y)$ is identified with $u \otimes v \in \mathrm{Sym} \otimes \mathrm{Sym}$.

The comultiplication $f \mapsto f(XY)$ makes sense as well for quasi-symmetric functions, if ones takes care of the order on the product set XY. That is, X and Y need to be totally ordered, and one can set $x_i y_j < x_k y_l$ iff $(i, j) < (k, l)$ for the lexicographic order. Let us denote by $X \hat{\times} Y$ the product XY endowed with this order. Then, we can extend δ from Sym to QSym by setting [8]

$$\delta(f) = f(X \hat{\times} Y)\,. \tag{57}$$

The internal product $*$ of **NCSF** can now be defined as the dual of δ, that is,

$$\langle f | P * Q \rangle = \langle \delta(f) | P \otimes Q \rangle\ . \tag{58}$$

Clearly, each homogeneous component $\mathbf{NCSF}_n$ is a $*$-subalgebra. In fact, it follows from results of Gessel [8] that this algebra is *anti*-isomorphic to the so-called *descent algebra* Σ_n of $\mathfrak{S}_n$ introduced by Solomon [19].

From one of Solomon's formulas, one obtains the following multiplication rule. Let $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$ be two compositions of n. Then,

$$S^I * S^J = \sum_{M \in \mathrm{Mat}(I,J)} S^M \tag{59}$$

where $\mathrm{Mat}(I, J)$ denotes the set of matrices of nonnegative integers $M = (m_{ij})$ of order $p \times q$ such that $\sum_s m_{rs} = i_r$ and $\sum_r m_{rs} = j_s$ for

$r \in [1,p]$ and $s \in [1,q]$, and where

$$S^M = S_{m_{11}}\, S_{m_{12}}\, \cdots\, S_{m_{1p}}\, \cdots\, S_{m_{q1}} \cdots S_{m_{qp}}\ . \tag{60}$$

Note that by definition, if F and G are homogeneous of different degrees, $F * G = 0$, and that S_n is the unit element of the $*$-subalgebra $\mathbf{NCSF}_n$. Moreover it is known from the formula giving the internal product of classical homogeneous functions h_λ that the commutative image morphism $\mathbf{NCSF} \mapsto \mathrm{Sym}$ is a homomorphism for the internal products.

It is a natural question to know whether the construction of the present paper is compatible with the internal structure. The answer is positive. We begin by the following lemma:

Proposition 5.1. *Let r be a positive integer. The ideal $\mathcal{I}_r$ generated by*

$$S_iS_j - S_jS_i \qquad \text{for all } i < r \text{ and } j \in \mathbb{N}. \tag{61}$$

for the usual product and is also an ideal for the internal product.

Proof. Let $I = (i_1, \ldots, i_p)$ and $I' = (i_1, \ldots, i_{k+1}, i_k, \ldots, i_p)$ for some k such that $i_k < r$. We have to prove that

$$(S^I - S^{I'}) * S^J \in \mathcal{I}_r \quad \text{and} \quad S^J * (S^I - S^{I'}) \in \mathcal{I}_r \tag{62}$$

for any composition J.

According to Solomon's product rule (Equation 59), to compute $S^I * S^J$ we have to enumerate the matrices $M \in \mathrm{Mat}(I,J)$ whose row sum is I and column sum J. The exchange τ_k of the k-th and the $(k+1)$-th row is a bijection between $\mathrm{Mat}(I,J)$ and $\mathrm{Mat}(I',J)$. Then for any matrix $M \in \mathrm{Mat}(I,J)$, the integers appearing in the k-th row are of sum i_k and therefore are all smaller than r because i_k itself is. As a consequence, the readings S^M and $S^{M'}$ of M and $M' := \tau_k(M)$ only differ by the position of some integers smaller than r. Thus $S^M - S^{M'}$ belongs to $\mathcal{I}_r$. So we have proved that $(S^I - S^{I'}) * S^J$ belongs to $\mathcal{I}_r$. Similarly we prove that $S^J * (S^I - S^{I'})$ belongs to $\mathcal{I}_r$. □

Recall that the n-th homogeneous component of $\mathcal{I}_\infty$ is the Jacobson radical of the descent algebra Σ_n. This amount to say that $\mathrm{Sym}_n = \Sigma_n/\mathcal{I}_\infty$ is semi-simple which is true because it is isomorphic to the character ring of the symmetric group itself. So we have built a filtration

$$\mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots \subset \mathcal{I}_r \subset \cdots \subset \mathcal{I}_\infty = \mathrm{Rad}(\Sigma_n) \tag{63}$$

of the radical of the descent algebra which is not the iterated radical filtration. It would be interesting to find an interpretation of this filtration in the representation theory of the descent algebra.

For the time being, we are interested in the following reformulation.

Corollary 5.2. *For all $r > 0$, the Hopf algebra morphisms*

$$\mathbf{NCSF} \longmapsto \mathbf{NCSF}^r \quad \textit{and} \quad \mathbf{NCSF}^r \longmapsto \mathrm{Sym} \tag{64}$$

are also algebra morphisms for the internal product.

As a consequence, we have an infinite sequence of epimorphisms, both for the Hopf algebra and the internal structure such that the following diagram commute

$$\begin{tikzcd} \mathbf{NCSF} \arrow[r, two heads] \arrow[d] & \mathbf{NCSF}^2 \arrow[r, two heads] \arrow[d] & \mathbf{NCSF}^3 \arrow[r, two heads] \arrow[d] & \cdots \\ \mathrm{Sym} \arrow[r, equal] & \mathrm{Sym} \arrow[r, equal] & \mathrm{Sym} \end{tikzcd} \tag{65}$$

The Dual corollary 5.2 is the following statement:

Corollary 5.3. QSym^r *is stable by the coproduct δ, that is if f is r-quasi-symmetric then $f(X \hat{\times} Y)$ is r-quasi-symmetric in X and Y.*

6. Free and Cohen-Macauley algebra structure

This last property shows that the dual basis $\phi_{(I,\lambda)}$ has a very nice product rule. As a consequence, we elucidate the structure of QSym^r. First let us give explicitly the dual basis of Φ^I of $\mathbf{NCSF}^r$. In the classical case of the duality $(\mathrm{QSym}, \mathbf{NCSF})$ the dual bases of the Φ^I is the basis ϕ_I given by

$$\phi_I = \sum_{I \succeq K} \frac{1}{(\#(I,K))!} M_K\,, \tag{66}$$

where $(\#(I,K))! = \ell(J_1)!\,\ell(J_2)! \dots \ell(J_r)!$, and $J_1 \dots J_r$ are compositions such that $I = I_1 \cdot I_2 \cdots I_r$ and $K = (|I_1|, |I_2|, \dots, |I_r|)$. In particular ϕ_i is equal to the symmetric power sum $p_i = \sum x_k^i$. Since the Φ_i's are primitive the product of the ϕ_I is given by the shuffle product

$$\phi_I \phi_J = \sum_K \langle K | I \sqcup\!\sqcup J \rangle \, \phi_K\,. \tag{67}$$

Let $\phi'^r_{(I,\lambda)}$ be the dual basis of $\Phi^{(I,\lambda)}$. The formula which expresses $\phi_{(I,\lambda)}$ in terms of the ϕ_J is the same as the formula which expresses $M_{(I,\lambda)}$ in terms of the M_J's $\phi'^r_{(I,\lambda)} = \sum_{K \in I \sqcup\!\sqcup \lambda^\sigma} \phi_K$. We prefer a slightly different normalization: write the partition λ in exponential notation $\lambda = (1^{m_1} 2^{m_1} \dots)$, and set $z_\lambda := \prod m_i!$. The basis $\phi^r_{(I,\lambda)}$ is defined as

$$\phi^r_{(I,\lambda)} := z_\lambda \phi'^r_{(I,\lambda)} = z_\lambda \sum_{K \in I \sqcup\!\sqcup \lambda^\sigma} \phi_K = \phi_I \phi_{\lambda_1} \phi_{\lambda_2} \dots \phi_{\lambda_l}\,. \tag{68}$$

Then the product of two such functions is given by

$$\phi^r_{(I,\lambda)} \phi^r_{(J,\mu)} = \sum_K \langle K | I \sqcup\!\sqcup J \rangle \, \phi^r_{(K,\nu)}\,, \tag{69}$$

where ν is the partition obtained by sorting the concatenation of the partitions λ and μ.

Proof. The algebra $\mathbf{NCSF}^r$ is the tensor product

$$\text{Prim}(\mathbf{NCSF}^r) \simeq \mathbb{Q}[\Phi_1, \dots, \Phi_{r-1}] \otimes \mathbb{Q}\langle \Phi_r, \Phi_{r+1}, \dots \rangle \,. \tag{70}$$

hence the dual is the tensor product

$$\text{QSym}^r) \simeq \mathbb{Q}[\Phi_1^*, \dots, \Phi_{r-1}^*] \otimes \text{Shuff}(\Phi_r^*, \Phi_{r+1}^*, \dots) \,. \tag{71}$$

where $\text{Shuff}(a, b, c, \dots)$ is the shuffle algebra on the variable $a, b, c, \dots$ This translate to the previous product rule. □

Now it is known that the shuffle algebra is free on the Lyndon words (see [17] for a definition of Lyndon words). Hence we have the following theorem

Theorem 6.1. *The algebra* QSym^r *is the free algebra generated by the set*

- ϕ_k *for* $k < r$*;*
- ϕ_L *for all Lyndon words* $L = (l_1 \dots l_m)$ *with* $l_i \geq r$*.*

As a simple consequence:

Corollary 6.2. QSym^r *is a free* Sym*-module with generators* $(\phi_L)_L$ *where* L *goes along the set of Lyndon words* $L = (l_1 \dots l_m)$ *with* $m > 1$ *and* $l_i \geq r$*.*

Proof. This is obvious since ϕ_i is equal to the symmetric power sum $p_i = \sum x_k^i$. And the latter are a set of free generators of Sym over the rationals. □

It seems that the same property holds when the number of variables is finite. This is our main conjecture.

Conjecture 1. Let n be finite and $X_n := \{x_1, \dots, x_n\}$. As a $\text{Sym}(X_n)$-module $\text{QSym}^r(X_n)$ is free of dimension $n!$. Otherwise stated, the algebra $\text{QSym}^r(X_n)$ is Cohen-Macaulay.

As proved recently by Garsia and Wallach [6], this conjecture holds for in the case of QSym that is for $r = 1$. It seems that their proof can be adapted to the generalized case, but we have not worked out this proof. However, we believe that the knowledge of the generalized case and the following proposition allows us to simplify the proof. Assuming this conjecture, we get the following Hilbert series for the quotient:

Proposition 6.3. *The quotient of the Hilbert series of* QSym^r *and* Sym *is given by*

$$\text{Hilb}_t(\text{QSym}^r(X_n)) / \text{Hilb}_t(\text{Sym}(X_n)) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{Maj}(\sigma) + r(n - \text{Fix}(\sigma))} \,, \tag{72}$$

where $\text{Fix}(\sigma)$ *is the number of fixed points of the permutation* σ *and* $\text{Maj}(\sigma)$ *is the* Major index *of* σ *that is the sum if the descents of* σ*.*

As far as we now, such a formula was unknown even for QSym. It follows from the following result of Gessel and Reutenauer [9]. Let

$$D_n(q,\alpha) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{Maj}(\sigma)}\alpha^{\mathrm{Fix}(\sigma)}\,. \tag{73}$$

Then [9], Formula (8.5), Theorem (8.4) states that

$$\sum_{n=0}^{\infty} D_n(q,\alpha)\frac{z^n}{(q;q)_n} = \left(1-\frac{z}{1-q}\right)^{-1}\frac{(z;q)_\infty}{(\alpha z;q)_\infty}\,, \tag{74}$$

where $(x;q)_n = (1-x)(1-xq^2)\dots(1-xq^{n-1})$.

Proof. The formula to prove is equivalent to

$$\mathrm{Hilb}_t(\mathrm{QSym}^r(X_n)) = \mathrm{Hilb}_t(\mathrm{Sym}(X_n)) \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{Maj}(\sigma)+r(n-\mathrm{Fix}(\sigma))}\,. \tag{75}$$

But $\mathrm{Hilb}_t(\mathrm{Sym}(X_n)) = 1/(t;t)_n$. So, using generating series with respect to n conjugate with the variable a, we have to evaluate This identity strongly suggests that the freeness of as well as the $n!$ result for each $Qsym^r[X_n]$ should be derivable by means of some action of S_n, in the same manner these results are established for the the polynomial ring. To this date no such a proof has been found even in the $r=1$ case since the Garsia-Wallach proof follows a completely different path. Identifying the action that yields such a proof would be an outstanding result that would deeply increase our understanding of these remarkable modules.

$$f(a,t) := \sum_{n=0}^{\infty}\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{Maj}(\sigma)+r(n-\mathrm{Fix}(\sigma))}\frac{a^n}{(t;t)_n}\,. \tag{76}$$

Now, remark that

$$f(a,t) = \sum_{n=0}^{\infty}\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{Maj}(\sigma)}(t^{-r})^{\mathrm{Fix}(\sigma)}\frac{(at^r)^n}{(t;t)_n} = \sum_{n=0}^{\infty} D_n(t,t^{-r})\frac{(at^r)^n}{(t;t)_n}\,. \tag{77}$$

Thus by formula 74, with $z=at^r$, $q=t$ and $\alpha=t^{-r}$, one finds

$$f(a,t) = \left(1-\frac{at^r}{1-t}\right)^{-1}\frac{(at^r;t)_\infty}{(a;t)_\infty}\,, \tag{78}$$

which simplifies as

$$f(a,t) = \frac{1-t}{1-t-at^r}\frac{1}{(1-a)(1-at)\dots(1-at^{r-1})}\,. \tag{79}$$

This is exactly the series $\sum_n \mathrm{Hilb}_t(\mathrm{QSym}^r(X_n))a^n$ as given by Formula 35. Hence we are done. □

7. Noncommutative variables

The goal of this section is to discuss the analogue of r-quasi symmetric function in noncommutative variables. It is worth noting that most of the structure fully generalize including the internal product.

7.1. **Basic properties.**

Let $A = \{a_1, \dots a_n\}$ be a alphabet of noncommutative variables. In the following $|w|_i$ denote the number of occurrences of the letter a_i is the word w.

The notion of local action of the symmetric group on monomials naturally extends to noncommutative monomials that is words. An elementary transposition σ_i leave a word invariant or replace the letters a_i by the letter a_{i+1} and the a_{i+1} letters by the letter a_i. Then it is clear that the commutative image of a local action on words is a local action on polynomials. Consequently, the only local actions on words whose fixed noncommutative polynomials can be a sub-algebra are the following natural lifting of the r-action to the free algebra $\mathbb{Q}\langle A\rangle$.

Definition 5. Let r be an integer or infinite. The r-action of $\mathfrak{S}_n$ on words denoted by ρ_r is defined by

$$\rho_r(\sigma_i)(w) = \begin{cases} w & \text{if } |w|_i \geq r \text{ and } |w|_{i+1} \geq r, \\ w' & \text{otherwise,} \end{cases} \tag{80}$$

where w' is the word w where the a_i letters are replaced by the letter a_{i+1} and the a_{i+1} letters by the letter a_i.

It is clear that this defines an action of $\mathfrak{S}_n$ on $\mathbb{Q}\langle A\rangle$ which commute with the commutative image $a_i \to x_i$ morphism. Moreover, for the same reason that in the commutative case, it's very easy to check that the fixed points form a sub-algebra.

Theorem 7.1. *The linear span of the noncommutative polynomials $\mathbb{Q}\langle A\rangle^{\rho_r(\mathfrak{S}_n)}$ fixed by the r-action of $\mathfrak{S}_n$ is a subalgebra of $\mathbb{Q}\langle A\rangle$ denoted by $\mathbf{WQSym}^r(A)$. The projective limit when the number of variable tends to the infinity is a Hopf algebra called the algebra of r-word-quasi-symmetric functions denoted by $\mathbf{WQSym}^r$*

The case $r = \infty$ is called the algebra of r-word-symmetric functions denoted by $\mathbf{WSym}$.

The quasi-symmetric case ($r = 1$) was first considered by the author in [10] and more recently by [12, 1]. The symmetric ($r = \infty$) case has been first defined by Wolf [20] and has been further recently extensively studied [18, 3, 2].

Sketch of the proof. The proof are done in a very similar way to the commutative case. In particular the construction of the co-product is done using the alphabet doubling technique (see for example [10, 4] for analogous proofs). □

We are thus lead to an infinite hierarchy of Hopf algebras such that the following diagram commute (the vertical arrows correspond to the commutative image $a_i \to x_i$).

$$\begin{tikzcd} \mathbf{WQSym} \arrow[d] & \mathbf{WQSym}^2 \arrow[l, hook'] \arrow[d] & \cdots \arrow[l, hook'] & \mathbf{WSym} \arrow[l, hook'] \arrow[d] \\ \mathrm{QSym} & \mathrm{QSym}^2 \arrow[l, hook'] & \cdots \arrow[l, hook'] & \mathrm{Sym} \arrow[l, hook'] \end{tikzcd} \tag{81}$$

A natural basis for $\mathbf{WQSym}^r(A)$ is given by the analogue of the so-called monomial functions defined as the orbit sum under the r-quasi-symmetrizing action. It can be conveniently indexed by pairs of ordered and unordered set-partitions. Let $[n] := \{1, 2, \ldots, n\}$. The set-partition $\{\{1, 3\}, \{2, 5, 6\}, \{4\}\}$ is conveniently denoted by $\{13.256.4\}$, where the part are in the order of their smaller integer (this is called the canonical order). An ordered set composition is a set-partition together with a total order on the part. We denote the ordered set-partitions (or set-compositions) by brackets as [256.4.13].

For an ordered set-partition $I = [I_1, \ldots, I_k] \vDash [n]$ we denote by A^I the so called *dominant word* $w = w_1 \ldots w_n$ such that $w_i = a_j$ if and only if $i \in I_j$. For example $A^{[256.4.13]} = a_3a_1a_3a_2a_1a_1$. Let r be a positive integer. For a set-partition (or a set-composition) λ we denote $\lambda \geq r$ (resp. $\lambda < r$) if the part of λ are of cardinality greater or equal that r (resp. smaller than r).

Definition 6. Let (I, λ) be a pair of an ordered set-partition $I \geq r$ and a set partition $\lambda < r$. Let $I.\lambda$ be the ordered set-partition obtained by gluing λ at the end of I in the canonical order and $A^{I.\lambda}$ the associated dominant word. Then we define the quasi-word function by

$$\mathbf{WQM}^r_{(I,\lambda)} = \sum_{w \in \mathcal{O}^r(A^{I.\lambda})} w \tag{82}$$

where $\mathcal{O}^r(A^{I.\lambda})$ denote the orbit of the word $A^{I.\lambda}$ under the r-quasi-symmetrizing action.

It's clear that the family $(\mathbf{WQM}^r_{(I,\lambda)})$ is a basis for $\mathbf{WQSym}^r(A)$ where (I, λ) goes along the set of pairs of ordered set-partition $I \geq r$ and a set partition $\lambda < r$. Moreover there is an analogue of Equation (25) which explicit the inclusion of $\mathbf{WQSym}^r(A)$ in $\mathbf{WQSym}(A)$:

$$\mathbf{WQM}^r_{(I,\lambda)} = \sum_{K \in I \shuffle \lambda^\sigma} \mathbf{WQM}^1_K\,, \tag{83}$$

where λ^σ denotes the set of all distinct reordering of λ, the shuffle $K \in I \shuffle \lambda^\sigma$ being a shuffle of ordered partition considered as word whose letter are subsets of $[n]$. For example

$$\begin{aligned}
&\mathbf{WQM}^3_{[2\,4\,6\,13\,.\,1\,3\,5\,7\,11\,12],\{8\,10\,.\,9\}} = \\
&\quad \mathbf{WQM}^1_{[2\,4\,6\,13\,.\,1\,3\,5\,7\,11\,12\,.\,8\,10\,.\,9]} + \mathbf{WQM}^1_{[2\,4\,6\,13\,.\,8\,10\,.\,1\,3\,5\,7\,11\,12\,.\,9]} + \\
&\quad \mathbf{WQM}^1_{[2\,4\,6\,13\,.\,8\,10\,.\,9\,.\,1\,3\,5\,7\,11\,12]} + \mathbf{WQM}^1_{[8\,10\,.\,2\,4\,6\,13\,.\,1\,3\,5\,7\,11\,12\,.\,9]} + \\
&\quad \mathbf{WQM}^1_{[8\,10\,.\,2\,4\,6\,13\,.\,9\,.\,1\,3\,5\,7\,11\,12]} + \mathbf{WQM}^1_{[8\,10\,.\,9\,.\,2\,4\,6\,13\,.\,1\,3\,5\,7\,11\,12]} + \\
&\quad \mathbf{WQM}^1_{[2\,4\,6\,13\,.\,1\,3\,5\,7\,11\,12\,.\,9\,.\,8\,10]} + \mathbf{WQM}^1_{[2\,4\,6\,13\,.\,9\,.\,1\,3\,5\,7\,11\,12\,.\,8\,10]} + \\
&\quad \mathbf{WQM}^1_{[2\,4\,6\,13\,.\,9\,.\,8\,10\,.\,1\,3\,5\,7\,11\,12]} + \mathbf{WQM}^1_{[9\,.\,2\,4\,6\,13\,.\,1\,3\,5\,7\,11\,12\,.\,8\,10]} + \\
&\qquad\qquad \mathbf{WQM}^1_{[9\,.\,2\,4\,6\,13\,.\,8\,10\,.\,1\,3\,5\,7\,11\,12]} + \mathbf{WQM}^1_{[9\,.\,8\,10\,.\,2\,4\,6\,13\,.\,1\,3\,5\,7\,11\,12]}\,.
\end{aligned}$$

The basis $\mathbf{WQM}^1_K$ has already been considered in [10] and in [1] under the notation $\mathbf{M}_\Phi$. They explicit the algebra structure and in particular the product rule in term of quasi-shuffle of ordered partitions. It's therefore easy to get the non-commutative analogue of the product rule of Proposition 3.8 as well as the co-product rule of Proposition 4.2. This is left to the reader.

We now briefly investigate the dual Hopf algebras. Let $(\mathbf{WS}^r_{(I,\lambda)})$ the dual basis of $(\mathbf{WQM}^r_{(I,\lambda)})$. The dual of Equation (83) leads to the following proposition

Proposition 7.2. *The transpose of the inclusion map*

$$\mathbf{WQSym}^r \longmapsto \mathbf{WQSym}^1 \tag{84}$$

is a projection

$$\Theta_r \;:\; \mathbf{WQSym}^{1*} \longmapsto \mathbf{WQSym}^{r*} \tag{85}$$

which is explicitly given by

$$\Theta_r(\mathbf{WS}^1_I) = \mathbf{WS}^r_{(J,\lambda)} \tag{86}$$

where J is to ordered set partition obtained from I by removing the part smaller than r and λ is the (unordered) set partition formed by these parts.

7.2. **Internal product and the Solomon-Tits algebras.** As in the commutative case, it makes sense to define the specialization of $\mathbf{WQSym}^r$ to a an ordered product of mutually commuting alphabet. The algebra $\mathbf{WQSym}^r(X\hat{\times}Y)$ is contained in $\mathbf{WQSym}^r(X) \otimes \mathbf{WQSym}^r(Y)$. Consequently on get a coproduct δ which occur inside the homogeneous component. As noticed in [1] in the $r = 1$ case, the dual internal product is anti isomorphic to product of the so-called Solomon-Tits algebra in the same way as the internal product of **NCSF** is anti-isomorphic to the product of the descent algebra. We show here that the r-case is in a complete analogy to the commutative case.

The internal product of the Solomon-Tits algebra ($r = 1$) is the following: let $I = [I_1, \dots, I_k]$ and $J = [J_1, \dots, J_l]$ be two ordered set-partition. Then

$$\mathbf{WS}^1_I * \mathbf{WS}^1_J = \mathbf{WS}^1_K \tag{87}$$

where K is the ordered set partition obtained by removing the empty parts in the ordered tuple of subset $[((I_u \cup J_v)_{u=1\dots k})_{v=1\dots l}]$. For example

$$\mathbf{WS}^1_{[12578.3469]} * \mathbf{WS}^1_{[1457.26.389]} = \mathbf{WS}^1_{[157.2.8.4.6.39]}$$

We are now in position to state the main result of this part

Theorem 7.3. *The Hopf morphism* $\Theta_r : \mathbf{WQSym}^{1*} \longmapsto \mathbf{WQSym}^{r*}$ *is also a morphism for the internal product.*

Proof. This is quite trivial since by definition of the internal co-product $f(X) \mapsto f(X \hat{\times} Y)$, the inclusion map is a morphism for the co-algebra structures of the internal co-product. □

It is worth noting that as in the commutative case, the map $\Theta_\infty : \mathbf{WQSym}^* \longmapsto \mathbf{WSym}^*$, can be interpreted as the projection on the canonical semi-simple algebra associated to the Solomon-Tits algebra. More precisely, the kernel of Θ_∞ restricted to the n-th homogeneous component is anti-isomorphic to the radical of the Solomon-Tits algebra. Thus we have build a filtration of the radical of the Solomon-Tits algebra.

We conclude by saying that finally we have build a commutative diagram of morphisms for both the Hopf and the internal structure as follows

$$\begin{tikzcd}
\mathbf{WQSym}^* \arrow[r, two heads] & \mathbf{WQSym}^{2*} \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathbf{WSym}^* \\
\mathbf{NCSF} \arrow[u, hook] \arrow[r, two heads] & \mathbf{NCSF}^2 \arrow[u, hook] \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathrm{Sym} \arrow[u, hook]
\end{tikzcd} \tag{88}$$

So it seems that many of the properties of the commutative case are shared by the non-commutative one.

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