Noncommutative Symmetric Functions VI: Free Quasi-Symmetric Functions and Related Algebras

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Abstract

This article is devoted to the study of several algebras related to symmetric functions, which admit linear bases labelled by various combinatorial objects: permutations (free quasi-symmetric functions), standard Young tableaux (free symmetric functions) and packed integer matrices (matrix quasi-symmetric functions). Free quasi-symmetric functions provide a kind of noncommutative Frobenius characteristic for a certain category of modules over the 0-Hecke algebras. New examples of indecomposable $H_n(0)$-modules are discussed, and the homological properties of $H_n(0)$ are computed for small n. Finally, the algebra of matrix quasi-symmetric functions is interpreted as a convolution algebra.

Contents

1 Introduction

This article is devoted to the study of several algebras closely related to symmetric functions. By ‘closely related’, we mean that these algebras can be fitted into a diagram of homomorphisms

$$\begin{tikzcd}
\mathbf{FSym} \arrow[rr, tail] \arrow[dd] & & \mathbf{FQSym} \arrow[d] \arrow[dr] & \\
& \mathbf{Sym} \arrow[ul] \arrow[ur] \arrow[dl, two heads] \arrow[dr] & QSym_q \arrow[d] & \mathbf{MQSym} \arrow[dl, two heads] \\
Sym \arrow[rr, tail] & & Qsym &
\end{tikzcd}$$

along which most of the interesting structure can pulled back or pushed forward.

Our notation is summarized in the following table, which indicates also the combinatorial objects labelling the natural bases of the various algebras:

Symbol	Algebra	Basis
Sym	Symmetric functions	Partitions
$Qsym$	Quasi-symmetric functions	Compositions
$\mathbf{Sym}$	Noncommutative symmetric functions	Compositions
$QSym_q$	Quantum quasi-symmetric functions	Compositions ($\mathbb{C}(q)$-basis)
$\mathbf{FSym}$	Free symmetric functions	Standard Young tableaux
$\mathbf{FQSym}$	Free quasi-symmetric functions	Permutations
$\mathbf{MQSym}$	Matrix quasi-symmetric functions	Packed integer matrices

The starting point of the construction is the triangular diagram formed by the embedding of Sym in $QSym$, and the abelianization map from $\mathbf{Sym}$ onto Sym [11]. Since the maps preserve the natural gradations, we have, for the homogeneous components of degree n of these algebras, a commutative diagram

$$\begin{tikzcd}
Sym_n \arrow[rr, "d"] & & QSym_n \\
& \mathbf{Sym}_n \arrow[ul, "e"] \arrow[ur, "c"'] &
\end{tikzcd}$$

which has a neat interpretation in representation theory: it is the Cartan-Brauer triangle of $H_n(0)$, the Hecke algebra of type A_{n-1} at $v = 0$ [9, 6, 18]. This means that $QSym_n$ is to be be interpreted as $G_0(H_n(0))$, the Grothendieck group of the category of finitely generated $H_n(0)$-modules, $\mathbf{Sym}_n$ as $K_0(H_n(0))$, the Grothendieck group of finitely generated projective $H_n(0)$-modules, and Sym_n as the Grothendieck group $R(H_n(v)) = G_0(H_n(v)) = K_0(H_n(v))$ of the semi-simple algebra $H_n(v)$, for generic v. Moreover, the inclusion map $d : Sym_n \to QSym_n$ is the decomposition map. Indeed, the simple $H_n(v)$ modules $V_\lambda(v)$ correspond to the Schur functions s_λ, with $\lambda \vdash n$, and the coefficients of the quasi-symmetric expansion

$$s_\lambda = \sum_{|I|=n} d_{\lambda I} F_I \qquad (1)$$

are the multiplicities of the simple $H_n(0)$ modules $\mathbf{S}_I$ (parametrized by compositions I of n) as composition factors of the specialized module $V_\lambda(0)$.

This interpretation leads to a q-analogue of $QSym$: the algebra $QSym_q$ of quantum quasi-symmetric functions, defined in [31]. Here, the indeterminate q is introduced to record a certain filtration on $H_n(0)$-modules. For generic complex values of q, $QSym_q$ is non commutative, and in fact isomorphic to $\mathbf{Sym}$, but for $q = 1$ one recovers the commutative algebra of quasi-symmetric functions $QSym$.

This construction can be somewhat clarified by the introduction of the larger algebra $\mathbf{FQSym}$, a subalgebra of the free associative algebra $\mathbb{C}\langle A\rangle$ (whence the name free quasi-symmetric functions) which admits $\mathbf{Sym}$ as a subalgebra, and is mapped onto $QSym_q$ when one imposes the q-commutation relations of the quantum affine space ($a_j a_i = q a_i a_j$ for $j > i$) on the letters of A.

This algebra turns out to be isomorphic to the convolution algebra of symmetric groups studied by Malvenuto and Reutenauer [23]. It contains a subalgebra whose bases are naturally labelled by standard Young tableaux, which provides a concrete realization of the algebras of tableaux of Poirier and Reutenauer [27]. We call it $\mathbf{FSym}$, the algebra of free symmetric functions. To illustrate the relevance of the realization of $\mathbf{FSym}$ as an algebra of noncommutative polynomials, we use it to present a complete proof of the Littlewood-Richardson rule within a dozen of lines (the idea of the proof is not new, but the formalism makes it quite compact and transparent). In the same vein, we show that the use of $\mathbf{FQSym}$ allows one to give simple presentations of Stanley's QS-distribution [30] and of the Hopf algebra of planar binary trees of Loday and Ronco [21].

The next step is to look for a representation theoretical interpretation of $\mathbf{FQSym}$. It turns out that $\mathbf{FQSym}_n$ can be interpreted as a kind of Grothendieck group for a certain category $\mathcal{N}_n$ of $H_n(0)$-modules, which contains in particular simple, projective, and skew Specht modules. However, this is far from exhausting all the $H_n(0)$-modules, since we prove that for $n \geq 4$, $H_n(0)$ is not representation finite. As a step towards a more exhaustive study of the 0-Hecke algebras, we determine their quivers for all n, and discuss their homological properties for small values of n.

Finally, we show that $\mathbf{FQSym}$ can be embedded into a larger algebra, $\mathbf{MQSym}$, whose bases are labelled by packed integer matrices, or, if one prefers, by double cosets of symmetric groups modulo parabolic subgroups. This is a self-dual bialgebra, which accommodates all the previous ones as quotients or subalgebras, and in which most of the structure of symmetric functions survives. It is not known whether $\mathbf{MQSym}$ can be interpreted as a sum of Grothendieck groups. It has, however, some representation theoretical meaning, as the centralizer algebra of $GL(N, \mathbb{C})$ in a certain infinite dimensional representation (these results have been announced in [7, 8]).

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2 Background

2.1 Hypoplactic combinatorics

Our notations will be essentially as in [11]. In this paper, we will use the realization of $\mathbf{Sym}$ as a subalgebra of the free associative algebra $\mathbb{C}\langle A\rangle$ over an infinite ordered noncommutative alphabet $A = \{a_i \mid i \geq 1\}$. Then, the ribbon Schur function R_I is identified with the sum of all words whose shape is encoded by the composition I.

We recall the notion of quasi-ribbon words and tableaux. A quasi-ribbon tableau of shape I is a ribbon diagram r of shape I filled by letters of A in such a way that each row of r is nondecreasing from left to right, and each column of r is strictly increasing from top to bottom. A word is said to be a quasi-ribbon word of shape I if it can be obtained by reading from bottom to top and from left to right the columns of a quasi-ribbon diagram of shape I.

The hypoplactic Robinson-Schensted correspondence is a bijection between words w and pairs $(\mathsf{Q}(w), \mathsf{R}(w))$, where $\mathsf{Q}(w)$ and $\mathsf{R}(w)$ are respectively a quasi-ribbon tableau and a standard ribbon tableau of the same shape [18]. The equivalence relation on words u and v defined by

$$u \equiv v \iff \mathsf{Q}(u) = \mathsf{Q}(v) \tag{2}$$

can be shown to coincide with the hypoplactic congruence of the free monoid A^*, which is generated by the plactic relations

$$\begin{cases} aba \equiv baa\,, \quad bba \equiv bab & \text{for } a < b, \\ acb \equiv cab\,, \quad bca \equiv bac & \text{for } a < b < c. \end{cases}$$

and the quartic hypoplactic relations

$$\begin{cases} baba \equiv abab\,, \quad baca \equiv abac & \text{for } a < b < c, \\ cacb \equiv acbc\,, \quad cbab \equiv bacb & \text{for } a < b < c, \\ badc \equiv dbca\,, \quad acbd \equiv cdab & \text{for } a < b < c < d. \end{cases}$$

Despite the apparent complexity of these relations, it can be shown that $u \equiv v$ if and only if u and v have the same evaluation and the permutations $\mathrm{Std}(u)^{-1}$ and $\mathrm{Std}(v)^{-1}$ have the same descents. Here, $\mathrm{Std}(w)$ denotes the standardized of the word w, *i.e.* the permutation obtained by iteratively scanning w from left to right, and labelling $1, 2, \ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. Alternatively, $\sigma = \mathrm{Std}(w)^{-1}$ can be characterized as the unique permutation of minimal length such that $w\sigma$ is a nondecreasing word.

Quasi-symmetric functions can be lifted to the hypoplactic algebra. The hypoplactic quasi-ribbon $F_I(A)$ is defined as the sum of all quasi-ribbon words of shape I in the hypoplactic algebra. It is shown in [18] that these elements span a commutative $\mathbb{Z}$-subalgebra, and that the image of $F_I(A)$ in $\mathbb{Z}[X]$ by the natural homomorphism is the usual quasi-symmetric function $F_I(X)$.

2.2 0-Hecke algebras

The 0-Hecke algebra $H_n(0)$ is the $\mathbb{C}$-algebra generated by $n-1$ elements $T_1, \dots, T_{n-1}$ satisfying the braid relations and $T_i^2 = -T_i$. It will be convenient to introduce a special notation for the generators $\xi_i = 1 + T_i$ and $\eta_i = -T_i$, which also satisfy the braid relations, $\xi_i^2 = \xi_i$ and $\eta_i^2 = \eta_i$. To a permutation $\sigma \in \mathfrak{S}_n$, we can therefore associate three elements T_σ, ξ_σ and η_σ by the usual process of taking the products of generators labelled by a reduced word for σ.

The irreducible $H_n(0)$ modules are denoted by $\mathbf{S}_I$ and the unique indecomposable projective module M such that $M/\operatorname{rad}(M) = \mathbf{S}_I$ is denoted by $\mathbf{P}_I$. Its socle is simple and isomorphic to $\mathbf{S}_{\bar{I}}$, where $\bar{I}$ is the mirror composition of I.

The dimension of $\mathbf{P}_I$ is equal to the cardinality of the descent class D_I, the set of permutations having I as descent composition. This set is an interval $[\alpha(I), \omega(I)]$ of the (left) weak order on $\mathfrak{S}_n$. As shown by Norton [25], one can realize $\mathbf{P}_I$ as the left ideal generated by

$$\epsilon_I = \eta_{\alpha(I)} \xi_{\alpha(\bar{I}^\sim)} \tag{3}$$

where $J^\sim$ denotes the conjugate of a composition J.

For a module M over $H_n(0)$, let us say that M is a combinatorial module if there exists a basis m_j of M such that $\eta_i m_j$ is either 0 or some m_k (this generalizes the notion of permutation representation of a group).

Projective $H_n(0)$ modules are combinatorial. The relevant bases are subsets of a basis of $H_n(0)$ which can also be found in [25]. Here we will denote it by g_σ. We set $g_{\alpha(I)} = \epsilon_I$, and if $\sigma = \tau\alpha(I)$ with $\ell(\sigma) = \ell(\tau) + \ell(\alpha(I))$ and $\sigma \in D_I$,

$$g_\sigma = \eta_\tau \epsilon_I \, . \tag{4}$$

It is important to mention that the generators ϵ_I are not idempotents. The corresponding orthogonal idempotents are denoted by e_I. One way to compute them is to express the identity of $H_n(0)$ in the basis g_σ. If

$$1 = \sum_{\sigma \in \mathfrak{S}_n} a_\sigma g_\sigma \tag{5}$$

then

$$e_I = \sum_{\sigma \in [\alpha(I), \omega(I)]} a_\sigma g_\sigma \, . \tag{6}$$

2.3 Quantum quasi-symmetric functions and quantum shuffles

It is known that $H_n(0)$-modules are endowed with a natural filtration, which can be taken into account in the description of the composition factors of the induced modules

$$\mathbf{S}_I \,\widehat{\otimes}\, \mathbf{S}_J = \mathbf{S}_I \otimes \mathbf{S}_J \uparrow_{H_n(0)\otimes H_m(0)}^{H_{m+n}(0)} \, . \tag{7}$$

The multiplicity c_{IJ}^K of $\mathbf{S}_K$ as a composition factor of this module is equal to the coefficient of F_K in the product $F_I F_J$. The rule to evaluate this product is as follows: take any permutation u of $1, \dots, n$ with descent composition $C(u) = I$ and any permutation v of $n+1, \dots, n+m$

such that $C(v) = J$. Then the shuffle of the two words u and v is a sum of permutations of $\{1, \ldots, n+m\}$

$$u \sqcup\!\sqcup v = \sum_{w \in \mathfrak{S}_{m+n}} c_w w \tag{8}$$

and the product is given by

$$F_I F_J = \sum_{w \in \mathfrak{S}_{m+n}} c_w F_{C(w)} . \tag{9}$$

There exists a q-analogue of the shuffle product, which is defined by

$$\text{if } u = au' \text{ and } v = bv' \text{ with } a, b \in A, \quad \text{then} \quad u \sqcup\!\sqcup_q v = a(u' \sqcup\!\sqcup_q v) + q^{|u|} b(u \sqcup\!\sqcup_q v') \tag{10}$$

where $|u|$ is the length of u. It can be shown that this operation is associative, and that when q is not a root of unity, the q-shuffle algebra is isomorphic to the concatenation algebra, which corresponds to the case $q = 0$ [6].

The induced representation $\mathbf{S}_I \,\widehat{\otimes}\, \mathbf{S}_J$ is generated by a single vector u. There is a filtration of this module whose k-th slice M_k is spanned by the elements $T_\sigma u$ for permutations σ of length k. Now, if one computes the product $F_I F_J$ by using the q-shuffle instead of the ordinary one in formula (9), the coefficient of $q^k F_H$ in the result is the multiplicity of $\mathbf{S}_H$ at level k of the filtration. The algebra $QSym_q$ of quantum quasi-symmetric functions is defined accordingly as the algebra with generators F_I and multiplication rule

$$F_I F_J = \sum_w \langle\, w \,|\, u \sqcup\!\sqcup_q v \,\rangle\, F_{C(w)} \tag{11}$$

for permutations u and v as above, $\langle\, w \,|\, u \sqcup\!\sqcup_q v \,\rangle$ being the coefficient of w in $u \sqcup\!\sqcup_q v$.

All the usual bases of $QSym$, in particular (M_I), are defined in $QSym_q$ by the same expressions in terms of the F_I as in the classical case.

For generic values of q, $QSym_q$ is freely generated by the one-part quasi-ribbons F_n, or as well by the power-sums M_n, or any sequence corresponding to a free set of generators of the algebra of symmetric functions in the classical case. This means that if we define for a composition $I = (i_1, \ldots, i_r)$

$$F^I = F_{i_1} F_{i_2} \cdots F_{i_r} \quad \text{and} \quad M^I = M_{i_1} M_{i_2} \cdots M_{i_r} \in QSym_q \tag{12}$$

then the F^I (resp. the M^I) form a basis of $QSym_q$. This is clearly not true for $q = 1$, for in this case these elements are symmetric functions.

Thus, for generic q, $QSym_q$ is isomorphic to the algebra of noncommutative symmetric functions. Actually, it can be obtained by specializing the formal variables of the polynomial realization $\mathbf{Sym}(A)$ of $\mathbf{Sym}$ (see [11], Sec. 7.3) to the generators of the (infinite dimensional) quantum affine space $\mathbb{C}_q[X] = \mathbb{C}_q[x_1, x_2, \ldots]$, the associative algebra generated by an infinite sequence of elements x_i subject to the q-commutation relations

$$\text{for } j > i, \quad x_j x_i = q x_i x_j . \tag{13}$$

More precisely, let $\mathbf{Sym}(X)$ be the subalgebra of $\mathbb{C}_q[X]$ generated by the specialization $a_i \to x_i$ of the noncommutative symmetric functions. Then, $\mathbf{Sym}(X)$ is isomorphic as an algebra to $QSym_q$, the correspondence being given by

$$M_I \longleftrightarrow \overline{M}_I = \sum_{j_1<\cdots<j_r} x_{j_1}^{i_1} \cdots x_{j_r}^{i_r} \,. \tag{14}$$

That is, if ones defines

$$\overline{F}_I = \sum_{J \succeq I} \overline{M}_J \,, \tag{15}$$

one has for u a permutation of $1, \dots, n$ and v a permutation of $n+1, \dots, n+m$

$$\overline{F}_{C(u)} \overline{F}_{C(v)} = \sum_w \langle\, w \,|\, u \sqcup\!\sqcup_q v \,\rangle \, \overline{F}_{C(w)} \,. \tag{16}$$

Thus, $QSym_q$ provides a kind of unification of both generalizations $QSym$ and $\mathbf{Sym}$ of Sym.

2.4 Convolution algebras

Let H be a bialgebra with multiplication μ and comultiplication Δ. The convolution product of two endomorphisms ϕ, ψ of H is given by

$$\phi \star \psi = \mu \circ (\phi \otimes \psi) \circ \Delta \,. \tag{17}$$

This is an associative operation, as soon as μ is associative and Δ coassociative. Actually, (17) makes sense, and is still associative, without assuming any compatibility between μ and Δ, and such expressions will arise in the sequel. When no bialgebra structure is assumed, we speak of pseudo-convolution.

Interesting examples of convolution algebras are provided by the centralizer algebras of group actions on tensor spaces. Let V be a representation of some group G. Then, the tensor algebra $T(V)$ is a representation of G, and one can consider its centralizer algebra

$$H = \mathrm{End}_G T(V) \,. \tag{18}$$

It is clearly stable under composition, but also under convolution since

$$\begin{aligned}
(\phi \star \psi)(gx) &= \mu \circ (\phi \otimes \psi) \circ \Delta(gx) \\
&= \mu \circ (\phi \otimes \psi)(g \otimes g)\Delta(x) \\
&= \mu \circ (g \otimes g) \circ (\phi \otimes \psi) \circ \Delta(x) \\
&= g(\phi \star \psi)(x) \,.
\end{aligned}$$

When one takes $G = GL(N, \mathbb{C})$ and $V = \mathbb{C}^N$, H is a homomorphic image of the direct sum $\mathbb{C}\mathfrak{S}$ of all $\mathbb{C}\mathfrak{S}_n$. By letting $N \to \infty$, one obtains a convolution structure on $\mathbb{C}\mathfrak{S}$. The resulting algebra has been extensively studied by Reutenauer and his students [28, 23, 27]. In the following, we will propose a new approach leading to a generalization of this algebra.

3 Free quasi-symmetric functions

Our first generalization is obtained by lifting the multiplication rule (9) to the free associative algebra, where it becomes multiplicity free. One arrives in this way to an algebra with basis labelled by all permutations, which turns out to be isomorphic to the algebra studied by Malvenuto and Reutenauer in [23], Sec. 3.

3.1 Free quasi-symmetric functions in a free algebra

Definition 3.1 *The* free quasi-ribbon $\mathbf{F}_\sigma$ *labelled by a permutation* $\sigma \in \mathfrak{S}_n$ *is the noncommutative polynomial*

$$\mathbf{F}_\sigma = \sum_{\mathrm{Std}(w)=\sigma^{-1}} w \quad \in \mathbb{Z}\langle A\rangle \tag{19}$$

where $\mathrm{Std}(w)$ *denotes the standardized of the word* w*.*

The hypoplactic version of the Robinson-Schensted correspondence shows that the commutative image of $\mathbf{F}_\sigma$ is the quasi-symmetric function F_I, where $I = C(\sigma)$. Indeed, the standard ribbon playing the role of the insertion tableau is equal to $\mathrm{Std}(w)^{-1}$, so that $\mathbf{F}_\sigma$ contains exactly one representative of each hypoplactic class of shape I.

For a word $w = x_1 x_2 \cdots x_n$ in the letters $1, 2, \ldots$ and an integer k, denote by $w[k]$ the shifted word $(x_1 + k)(x_2 + k) \cdots (x_n + k)$, e.g., $312[4] = 756$. The shifted concatenation of two words u, v is defined by

$$u \bullet v = u \cdot v[k] \tag{20}$$

where k is the length of u.

Proposition 3.2 *Let* $\alpha \in \mathfrak{S}_k$ *and* $\beta \in \mathfrak{S}_l$*. Then,*

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{\sigma \in \alpha ш \beta[k]} \mathbf{F}_\sigma \tag{21}$$

Therefore, the free quasi-ribbons span a $\mathbb{Z}$*-subalgebra of the free associative algebra.*

Proof– A word $w = a_{i_1} a_{i_2} \ldots a_{i_n}$ can be represented by a monomial in commuting "biletters" $\binom{a_i}{j}$ (which are just a convenient notation for doubly indexed indeterminates x_{ij}). We identify w with any monomial $\binom{a_{i_1}}{j_1}\binom{a_{i_2}}{j_2} \cdots \binom{a_{i_n}}{j_n}$ such that $j_1 < j_2 < \ldots < j_n$, and in particular with the product $\binom{a_{i_1}}{1}\binom{a_{i_2}}{2} \cdots \binom{a_{i_n}}{n}$, which we also denote by

$$\begin{pmatrix} a_{i_1} a_{i_2} \cdots a_{i_n} \\ 12 \cdots n \end{pmatrix} = \begin{pmatrix} w \\ \mathrm{id} \end{pmatrix} = \begin{pmatrix} w' \\ \tau \end{pmatrix} \tag{22}$$

whenever τ is a permutation such that $w'\tau = w$. Such a representation if of course not unique. Then, $\sigma = \mathrm{Std}(w)^{-1}$ is the unique permutation of minimal length such that $\binom{w}{\mathrm{id}} = \binom{w^+}{\sigma}$, where w^+ denotes the non-decreasing rearrangement of w. The correspondence $w \leftrightarrow \binom{w^+}{\sigma}$ is a bijection between words and pairs (u, α) where u is a nondecreasing word and α is a permutation of the same length such that $\alpha_i < \alpha_{i+1}$ when $u_i = u_{i+1}$. In this case, we say that u is α-compatible, and we write $u \uparrow \alpha$. The concatenation product corresponds to an operation $\circ$ on biwords, given by the rule

$$\binom{u}{\alpha} \circ \binom{v}{\beta} = \binom{uv}{\alpha \bullet \beta}. \tag{23}$$

Now, if we write

$$\mathbf{F}_\alpha = \sum_{u \uparrow \alpha} \binom{u}{\alpha}, \qquad \mathbf{F}_\beta = \sum_{v \uparrow \beta} \binom{v}{\beta}, \tag{24}$$

where u and v run over nondecreasing words of respective lengths k and l, we see that

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{u \uparrow \alpha, v \uparrow \beta} \binom{uv}{\alpha \cdot \beta[k]} = \sum_{\sigma \in \alpha ш \beta[k]} \sum_{w \uparrow \sigma} \binom{w}{\sigma}, \tag{25}$$

whence the proposition. ∎

Definition 3.3 *The subalgebra of* $\mathbb{C}\langle A \rangle$

$$\mathbf{FQSym} = \bigoplus_{n \geq 0} \bigoplus_{\sigma \in \mathfrak{S}_n} \mathbb{C}\,\mathbf{F}_\sigma \tag{26}$$

is called the algebra of free quasi-symmetric functions.

It will be convenient to define a scalar product on $\mathbf{FQSym}$ by setting

$$\langle \mathbf{F}_\sigma, \mathbf{F}_\tau \rangle = \delta_{\sigma^{-1},\tau} \tag{27}$$

and to introduce the notation

$$\mathbf{G}_\sigma = \mathbf{F}_{\sigma^{-1}} \tag{28}$$

for the adjoint basis of $(\mathbf{F}_\sigma)$.

Since the convolution of permutations is related to the shifted shuffle by

$$(\alpha^\vee \star \beta^\vee)^\vee = \alpha ш \beta[k] \tag{29}$$

where $f \to f^\vee$ is the linear involution defined on permutations by $\alpha \to \alpha^\vee = \alpha^{-1}$, we see that $\mathbf{FQSym}$ is isomorphic to the convolution algebra of permutations of [23]. The interesting point is that the natural map $\sigma \mapsto F_{C(\sigma)}$ from this algebra to $QSym$ becomes simply the commutative image $a_i \mapsto x_i$.

The quasi-symmetric generating function of a set of permutations in the sense of [12] can now be regarded as the commutative image of an element of $\mathbf{FQSym}$. We shall see that in certain special cases, such as linear extensions of posets, the free quasi-symmetric function can be more interesting (*cf.* Section 3.8).

Another property of $\mathbf{FQSym}$ is that it contains a subalgebra with a distinguished basis labelled by standard Young tableaux, which maps to ordinary Schur functions under abelianization, and to which the Littlewood-Richardson rule can be lifted to a multiplicity free formula (see Proposition 3.12). The kind of argument used to establish this formula can also be used to prove Proposition 3.2. Recall that we denote by $w \mapsto (\mathsf{Q}(w), \mathsf{R}(w))$ the hypoplactic Robinson-Schensted correspondence. An alternative definition of $\mathbf{F}_\sigma$ is

$$\mathbf{F}_\sigma = \sum_{\mathsf{R}(w)=\sigma} w \tag{30}$$

and Proposition 3.2 can be derived exactly in the same way as Proposition 3.12, from the fact that the hypoplactic congruence is compatible to restriction to intervals (see [20]).

3.2 Duality

One can define on $\mathbf{FQSym}$ a bialgebra structure imitated from the case of ordinary quasi-symmetric functions.

Let A' and A'' be two mutually commuting ordered alphabets. Identifying $F \otimes G$ with $F(A')G(A'')$, we set $\Delta(F) = F(A' \oplus A'')$, where $\oplus$ denotes the ordered sum. Clearly, this is an algebra homomorphism.

Proposition 3.4 $\mathbf{FQSym}$ *is a bialgebra for* Δ*, and on the basis* $\mathbf{F}_\sigma$*, the comultiplication is given by*

$$\Delta\mathbf{F}_\sigma = \sum_{u\cdot v=\sigma} \mathbf{F}_{\mathrm{Std}(u)} \otimes \mathbf{F}_{\mathrm{Std}(v)} \,. \tag{31}$$

where $u \cdot v$ *denotes the concatenation of* u *and* v*.*

Proof – Like Proposition 3.2, this formula is easily obtained in the biword notation. Indeed, $\mathbf{F}_\sigma(A' \oplus A'')$ is the image of the element

$$\sum_{w\uparrow\sigma} \binom{w}{\sigma} = \sum_{w=w'w''\uparrow\sigma} \binom{w'w''}{\sigma} \tag{32}$$

of the free algebra $\mathbb{C}\langle A' \cup A''\rangle$ under the map

$$\pi : \mathbb{C}\langle A' \cup A''\rangle \to \mathbb{C}\langle A', A''\rangle \simeq \mathbb{C}\langle A\rangle \otimes \mathbb{C}\langle A\rangle. \tag{33}$$

The sum runs over all nondecreasing σ-compatible words w, which are necessarily of the form $w'w''$ with $w' \in A'^*$ and $w'' \in A''^*$, since $A' < A''$. Let $k = |w'|$ and $l = |w''|$. As a word, σ can be factorized as $\sigma = uv$, where $|u| = k$ and $|v| = l$. Let $\sigma' = \mathrm{Std}(u)$ and $\sigma'' = \mathrm{Std}(v)$. Since A' and A'' are disjoint, w' and w'' have to be respectively σ' and σ'' compatible, and actually, the sum (32) runs exactly over all such words. Since

$$\pi\binom{w'w''}{\sigma} = \binom{w'}{\sigma'} \circ \binom{w''}{\sigma''} \tag{34}$$

the image under π of this sum factorizes into

$$\sum_{uv=\sigma} \sum_{w'\uparrow\mathrm{Std}(u)} \binom{w'}{\mathrm{Std}(u)} \sum_{w''\uparrow\mathrm{Std}(v)} \binom{w''}{\mathrm{Std}(v)} \tag{35}$$

whence the proposition. ∎

Corollary 3.5 $\mathbf{FQSym}$ *is a self-dual bialgebra. That is, for all* $F, G, H \in \mathbf{FQSym}$,

$$\langle F \otimes G, \Delta H \rangle = \langle FG, H \rangle \ . \tag{36}$$

Proof – Denote by $\odot$ the multiplication adjoint to Δ, that is, such that

$$\langle F \otimes G, \Delta H \rangle = \langle F \odot G, H \rangle \ , \tag{37}$$

and consider the structure constants

$$\mathbf{G}_\alpha \odot \mathbf{G}_\beta = \sum_\gamma g_{\alpha\beta}^\gamma G_\gamma \, . \tag{38}$$

Then, $g_{\alpha\beta}^\gamma = 1$ if $\alpha = \mathrm{Std}(u)$ and $\beta = \mathrm{Std}(v)$ for some factorization $\gamma = uv$, and $g_{\alpha\beta}^\gamma = 0$ otherwise. Therefore, these structure constants coincide with those of the convolution product on permutations:

$$\alpha \star \beta = \sum_\gamma g_{\alpha\beta}^\gamma \gamma \, . \tag{39}$$

We have therefore interpreted the two multiplications and comultiplications of [23] as operations on labels of two different bases of the same subalgebra of the free associative algebra.

3.3 Algebraic structure

We can now apply to $\mathbf{FQSym}$ the results of Poirier and Reutenauer [27] and we see that $\mathbf{FQSym}$ is freely generated by the $\mathbf{G}_\sigma$, where σ runs over *connected permutations* (see [3]), *i.e.* permutations such that $\sigma([1,k]) \neq [1,k]$ for all intervals $[1,k] \subseteq [1, n-1]$. Actually, this result holds for a one-parameter family of algebras, and we shall now reprove it in this context.

We denote by $\mathcal{C}$ the set of connected permutations, and by $c_n = |\mathcal{C}_n|$ the number of such permutations in $\mathfrak{S}_n$. For later reference, we recall that the generating series of c_n is

$$\begin{aligned}\sum_{n\geq 1} c_n t^n =& 1 - \left(\sum_{n\geq 0} n! t^n\right)^{-1} \\ =& t + t^2 + 3\,t^3 + 13\,t^4 + 71\,t^5 + 461\,t^6 + 3447\,t^7 + 29093\,t^8 \\ &+ 273343\,t^9 + 2829325\,t^{10} + 31998903\,t^{11} + 392743957\,t^{12} + O(t^{13}) \, .\end{aligned}$$

For $\alpha \in \mathfrak{S}_k$ and $\beta \in \mathfrak{S}_l$, recall that $\alpha \bullet \beta = \alpha \cdot \beta[k]$ is the shifted concatenation of α and β. Any permutation $\sigma \in \mathfrak{S}_n$ has a unique maximal factorization $\sigma = \sigma_1 \bullet \cdots \bullet \sigma_r$ into connected permutations. Then, the elements

$$\mathbf{G}^\sigma = \mathbf{G}_{\sigma_1} \cdots \mathbf{G}_{\sigma_r} \tag{40}$$

and

$$\mathbf{F}^\sigma = \mathbf{F}_{\sigma_1} \cdots \mathbf{F}_{\sigma_r} \tag{41}$$

form two bases of $\mathbf{FQSym}$. Since $(\alpha^{-1} \bullet \beta^{-1})^{-1} = \alpha \bullet \beta$, we have $\mathbf{F}^\sigma = \mathbf{G}^{\sigma^{-1}}$, and the multiplication of $\mathbf{FQSym}$ is given in both bases by the same formula: $\mathbf{G}^\alpha \mathbf{G}^\beta = \mathbf{G}^{\alpha \bullet \beta}$ and $\mathbf{F}^\alpha \mathbf{F}^\beta = \mathbf{F}^{\alpha \bullet \beta}$.

The operations on permutations $\alpha \bullet \beta$ and $\alpha ⧢ \beta[k]$ describing the multiplication in the bases $\mathbf{F}^\sigma$ and $\mathbf{F}_\sigma$ are the cases $q = 0$ and $q = 1$ of the shifted q-shuffle $\alpha ⧢_q \beta[k]$. This suggests the consideration of a q-deformed algebra $\mathbf{FQSym}_q$, defined as the (abstract) algebra with generators F_σ and relations $\mathsf{F}_\alpha \mathsf{F}_\beta = \mathsf{F}_{\alpha ⧢_q \beta[k]}$ (where linearity of the symbol F with respects to subscripts is understood). As above, let $\mathsf{F}^\sigma = \mathsf{F}^{\sigma_1} \cdots \mathsf{F}^{\sigma_r} = \mathsf{F}_\sigma + O(q)$. For each n, the $n! \times n!$ matrix expressing the elements F^σ on the basis $\mathbf{F}_\sigma$ is of the form $I + O(q)$, and is therefore invertible over $\mathbb{C}[[q]]$. Moreover, it is unitriangular with respect to the lexicographic order on permutations, so that it is actually invertible over $\mathbb{C}[q]$. This proves that the algebras $\mathbf{FQSym}_q$ are actually isomorphic to each other for all values of q. For $q \neq 0$, the isomorphism $\mathbf{FQSym} \to \mathbf{FQSym}_q$ is realized by $\mathbf{F}_\sigma \mapsto q^{l(\sigma)} \mathsf{F}_\sigma$, and for $q = 0$, by $\mathbf{F}_\sigma \mapsto \mathsf{F}^\sigma$.

3.4 Primitive elements

Let $\mathcal{L}$ be the primitive Lie algebra of $\mathbf{FQSym}$. Since Δ is not cocommutative, $\mathbf{FQSym}$ cannot be the universal enveloping algebra of $\mathcal{L}$. Let $l_n = \dim \mathcal{L}_n$.

Let us recall that $\mathbf{G}^\sigma = \mathbf{G}_{\sigma_1} \cdots \mathbf{G}_{\sigma_r}$ where $\sigma = \sigma_1 \bullet \cdots \bullet \sigma_r$ is the unique maximal factorization of $\sigma \in \mathfrak{S}_n$ into connected permutations.

Proposition 3.6 *Let $\mathbf{V}_\sigma$ be the adjoint basis of $\mathbf{G}^\sigma$. Then, the family $(\mathbf{V}_\alpha)_{\alpha \in C}$ is a basis of $\mathcal{L}$. In particular, we have $l_n = c_n$.*

Proof – If α is connected, then

$$\begin{aligned} \Delta \mathbf{V}_\alpha &= \sum_{\sigma,\tau} \langle \Delta \mathbf{V}_\alpha, \mathbf{G}^\sigma \otimes \mathbf{G}^\tau \rangle \, \mathbf{V}_\sigma \otimes \mathbf{V}_\tau \\ &= \sum_{\sigma,\tau} \langle \mathbf{V}_\alpha, \mathbf{G}^{\sigma \bullet \tau} \rangle \, \mathbf{V}_\sigma \otimes \mathbf{V}_\tau = \mathbf{V}_\alpha \otimes 1 + 1 \otimes \mathbf{V}_\alpha \end{aligned}$$

since the only possible factorization of α is $\alpha = \emptyset \bullet \alpha = \alpha \bullet \emptyset$, where $\emptyset$ denotes the empty word.

Conversely, let $Z = \sum_\alpha c_\alpha \mathbf{V}_\alpha$ be a primitive element. If α is not connected, let $\alpha = \sigma \bullet \tau$ be a non-trivial factorization. Then,

$$\langle \Delta Z, \mathbf{G}^\sigma \otimes \mathbf{G}^\tau \rangle = \langle Z, \mathbf{G}^{\sigma \bullet \tau} \rangle = \langle Z, \mathbf{G}^\alpha \rangle = c_\alpha \tag{42}$$

which has to be zero since the left-hand side is the coefficient of $\mathbf{V}_\sigma \otimes \mathbf{V}_\tau$ in ΔZ. ∎

Example 3.7 In degree 3 we have

$$\begin{aligned} \mathbf{V}_{312} &= \mathbf{F}_{312} - \mathbf{F}_{213} \\ \mathbf{V}_{231} &= -\mathbf{F}_{132} + \mathbf{F}_{231} \\ \mathbf{V}_{321} &= \mathbf{F}_{123} - \mathbf{F}_{132} - \mathbf{F}_{213} + \mathbf{F}_{321} \end{aligned}$$

and in degree 4

$$\begin{array}{lcl}
\mathbf{V}_{4123} & = & \mathbf{F}_{4123} - \mathbf{F}_{3124} \\
\mathbf{V}_{4132} & = & \mathbf{F}_{4132} - \mathbf{F}_{3124} + \mathbf{F}_{2134} - \mathbf{F}_{2143} \\
\mathbf{V}_{3412} & = & -\mathbf{F}_{1423} + \mathbf{F}_{1324} + \mathbf{F}_{3412} - \mathbf{F}_{2314} \\
\mathbf{V}_{3142} & = & \mathbf{F}_{3142} - \mathbf{F}_{2143} \\
\mathbf{V}_{4312} & = & -\mathbf{F}_{1423} + \mathbf{F}_{1324} + \mathbf{F}_{4312} - \mathbf{F}_{3214} \\
\mathbf{V}_{2413} & = & -\mathbf{F}_{1423} + \mathbf{F}_{1324} + \mathbf{F}_{2413} - \mathbf{F}_{2314} \\
\mathbf{V}_{4213} & = & \mathbf{F}_{4213} - \mathbf{F}_{3214} \\
\mathbf{V}_{2431} & = & -\mathbf{F}_{1432} + \mathbf{F}_{2431} \\
\mathbf{V}_{2341} & = & -\mathbf{F}_{1342} + \mathbf{F}_{2341} \\
\mathbf{V}_{4231} & = & \mathbf{F}_{1243} - \mathbf{F}_{1342} - \mathbf{F}_{3124} + \mathbf{F}_{2134} - \mathbf{F}_{2143} + \mathbf{F}_{4231} \\
\mathbf{V}_{3421} & = & \mathbf{F}_{1324} - \mathbf{F}_{1432} - \mathbf{F}_{2314} + \mathbf{F}_{3421} \\
\mathbf{V}_{3241} & = & \mathbf{F}_{1243} - \mathbf{F}_{1342} - \mathbf{F}_{2143} + \mathbf{F}_{3241} \\
\mathbf{V}_{4321} & = & -\mathbf{F}_{1234} + \mathbf{F}_{1243} + \mathbf{F}_{1324} - \mathbf{F}_{1432} + \mathbf{F}_{2134} - \mathbf{F}_{2143} - \mathbf{F}_{3214} + \mathbf{F}_{4321}
\end{array}$$

The Hilbert series of the universal enveloping algebra $U(\mathcal{L})$ (the domain of cocommutativity of Δ) is

$$\begin{aligned}
\prod_{n\geq 1}(1-t^n)^{-c_n} &= 1 + t + 2\,t^2 + 5\,t^3 + 19\,t^4 + 93\,t^5 \\
&\quad + 574\,t^6 + 4134\,t^7 + 34012\,t^8 + 313231\,t^9 + 3191402\,t^{10} \\
&\quad + 35635044\,t^{11} + 432812643\,t^{12} + O\left(t^{13}\right) .
\end{aligned}$$

Conjecture 3.8 *$\mathcal{L}$ is a free Lie algebra.*

Assuming the conjecture, denote by d_n the number of generators of degree n of $\mathcal{L}$. Then, using the λ-ring notation, since $\sigma_1 \circ L = (1-p_1)^{-1}$ (where $\sigma_1 = \sum_{n\geq 0} h_n$, $L = \sum_{n\geq 1} \ell_n$, and $\ell_n = \frac{1}{n}\sum_{d|n}\mu(d)p_d^{n/d}$ are the Lie characters, or Witt symmetric functions), we have the equivalent plethystic equations

$$L\left[\sum_{n\geq 1} d_n t^n\right] = \sum_{n\geq 1} c_n t^n \,, \tag{43}$$

and

$$d(t) = \sum_{n\geq 1} d_n t^n = 1 - \lambda_{-1}\left[\sum_{n\geq 1} c_n t^n\right] = 1 - \prod_{n\geq 1}(1-t^n)^{c_n} \,. \tag{44}$$

Numerical calculation gives for the first terms

$$\begin{aligned}
d(t) &= t + t^2 + 2\,t^3 + 10\,t^4 + 55\,t^5 + 377\,t^6 \\
&\quad + 2892\,t^7 + 25007\,t^8 + 239286\,t^9 + 2514113\,t^{10} + 28781748\,t^{11} \\
&\quad + 356825354\,t^{12} + O(t^{13}) \,.
\end{aligned}$$

We shall now give a formula for the projector $\pi: \mathbf{FQSym} \to \mathcal{L}$ such that

$$\pi(\mathbf{F}_\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is not connected} \\ \mathbf{V}_\alpha & \text{if } \alpha \text{ is connected .} \end{cases}$$

Let p_n denote the projection onto the homogeneous component $\mathbf{FQSym}_n$ of $\mathbf{FQSym}$, and let $\mu_q: \mathbf{F}_\alpha \otimes \mathbf{F}_\beta \mapsto \mathbf{F}_{\alpha \sqcup\!\sqcup_q \beta[k]}$ be the multiplication map of $\mathbf{FQSym}_q$. The q-convolution of two graded linear endomorphisms f, g of $\mathbf{FQSym}$ is defined by

$$f \odot_q g = \mu_q \circ (f \otimes g) \circ \Delta \, . \tag{45}$$

For $q = 1$, this reduces to ordinary convolution, otherwise, it is an example of pseudo-convolution as defined in 2.4. We shall be interested in the case $q = 0$. For a composition $I = (i_1, \ldots, i_m)$, let

$$p_I = p_{i_1} \odot_0 \cdots \odot_0 p_{i_m} \, . \tag{46}$$

Lemma 3.9 *The p_I are mutually commuting projectors. More precisely we have*

$$p_I \circ p_J = \begin{cases} 0 & \textit{if } |I| \neq |J|. \\ p_{I \vee J} & \textit{otherwise .} \end{cases}$$

where $I \vee J$ is the composition with descent set $\mathrm{Des}(I) \cup \mathrm{Des}(J)$.

Proof – The result is clear when $|I| \neq |J|$. Otherwise, we suppose $|I| = |J| = n$ and proceed by induction on $d = \min(l(I \vee J) - l(I), l(I \vee J) - l(J))$. If $d = 0$ it is easy to check that $p_I \circ p_{I \vee J} = p_{I \vee J} \circ p_I = p_{I \vee J}$ otherwise, the induction step is a consequence of the standardization inertia $\mathrm{Std}(\sigma \bullet \tau) = \mathrm{Std}(\sigma) \bullet \mathrm{Std}(\tau)$ ∎

Before stating the main proposition we need some notation: For a word of length n, $w = a_1 a_2 \cdots a_n$ and $S = \{s_1, s_2 \cdots s_k\} \subset [1..n]$ a subset in increasing order, we denote the corresponding subword by $w|_S = a_{s_1} a_{s_2} \cdots a_{s_k}$. Let $I = (i_1, i_2, \cdots i_m)$ be a composition of weight n. The factorization-standardization operator sfact_I is defined by

$$\mathrm{sfact}_I(w) = \begin{cases} \mathrm{Std}(w|_{[1..i_1]}) \otimes \mathrm{Std}(w|_{[i_1+1..i_1+i_2]}) \otimes \cdots \mathrm{Std}(w|_{[n-i_m+1..n]}) & \text{if } |w| = n, \\ 0 & \text{otherwise} \end{cases}$$

For example $\mathrm{sfact}_{(2,3)}(53412) = \mathrm{Std}(53) \otimes \mathrm{Std}(412) = 21 \otimes 312$. We can now state:

Proposition 3.10 *(i) The operator*

$$\pi = \sum_{|I| \geq 1} (-1)^{l(I)-1} p_I \tag{47}$$

is the projector onto the primitive Lie algebra with the span of $(\mathbf{F}_\alpha)_{\alpha \notin C}$ as kernel.

(ii) Moreover, one has $\mathbf{V}_\alpha = \pi(\mathbf{F}_\alpha)$ for α connected.

Proof – The m-fold shifted concatenation $\mathrm{sconc}^{(m)}$ is defined in the obvious way. Then, for $l(I)=m$,

$$p_I(\mathbf{F}_\alpha)=\begin{cases}\mathbf{F}_{\mathrm{sconc}^{(m)}\circ\,\mathrm{sfact}_I(\alpha)} & \text{if } \alpha\in\mathfrak{S}_n\,,\\ 0 & \text{otherwise.}\end{cases}$$

We first prove that, if $l(I_0)=2$, one has $\pi\circ p_{I_0}=0$. For $i=0,1$, let

$$\mathcal{I}_i=\{|I|=n \mid \delta(I_0\prec I)=i\}\,,$$

it is easy to check that $\sharp(\mathcal{I}_i)=2^{n-2}$ and that $I\to I\vee I_0$ induces a bijection $\mathcal{I}_0\to\mathcal{I}_1$. Hence

$$\begin{aligned}\pi\circ p_{I_0}&=\sum_{l(I)\geq 1}(-1)^{l(I)-1}p_I\circ p_{I_0}=\sum_{|I|=n}(-1)^{l(I)-1}p_{I\vee I_0}\\&=\sum_{I\in\mathcal{I}_0}(-1)^{l(I)-1}p_{I\vee I_0}+\sum_{I\in\mathcal{I}_1}(-1)^{l(I)-1}p_{I\vee I_0}\\&=\sum_{I\in\mathcal{I}_1}(-1)^{l(I)}p_I+\sum_{I\in\mathcal{I}_1}(-1)^{l(I)-1}p_I=0\end{aligned}$$

If $\alpha\notin\mathcal{C}_n$, then for some composition I_0 of n of length 2, we have $p_{I_0}(\mathbf{F}_\alpha)=\mathbf{F}_\alpha$. Hence $\mathbf{F}_\alpha\in\ker(\pi)$. Now, if $\alpha\in\mathcal{C}_n$, the construction of π shows that

$$\pi(\mathbf{F}_\alpha)=\mathbf{F}_\alpha+\sum_{\beta\notin\mathcal{C}_n}c_\beta\mathbf{F}_\beta \tag{48}$$

and then $\pi^2(\mathbf{F}_\alpha)=\pi(\mathbf{F}_\alpha)$. This finishes to prove that π is a projector and from (48), we get that the generating series of $Im(\pi)$ is exactly $\sum_n c_nt^n$.

The comultiplication on $\mathbf{FQSym}$ can be rewritten as

$$\Delta=Id\otimes 1+1\otimes Id+\sum_{l(I)=2}\mathrm{sfact}_I \tag{49}$$

so, to get $\mathrm{Im}(\pi)\subset\mathcal{L}$, it suffices to prove

$$\left(\sum_{l(I)=2}\mathrm{sfact}_I\right)\circ\pi=0. \tag{50}$$

But, from the construction of sfact_I, one has $\mathrm{sfact}_I=\mathrm{sfact}_I\circ p_I$. Now, if $l(I)=2$, we get

$$\mathrm{sfact}_I\circ\pi=\mathrm{sfact}_I\circ p_I\circ\pi=\mathrm{sfact}_I\circ(\pi\circ p_I)=0 \tag{51}$$

which proves that $Im(\pi)\subset\mathcal{L}$, the equality of these two spaces follows from the fact that the generating series are equal.

Equation (48) says that $(\pi(F_\alpha))_{\alpha\in\mathcal{C}}$ is the unique basis of $\mathcal{L}$ such that

$$\pi(\mathbf{F}_\alpha)=\mathbf{F}_\alpha+\sum_{\beta\notin\mathcal{C}_n}c_\beta\mathbf{F}_\beta\,. \tag{52}$$

Since $\mathbf{V}_\alpha$ also have this property, $\mathbf{V}_\alpha=\pi(\mathbf{F}_\alpha)$. ∎

3.5 Free symmetric functions and the Littlewood-Richardson rule

Definition 3.11 *Let t be a standard tableau of shape λ. The* free Schur function *labelled by t is*

$$\mathbf{S}_t = \sum_{P(\sigma)=t} \mathbf{F}_\sigma = \sum_{Q(w)=t} w\,, \tag{53}$$

where $w \mapsto (P(w), Q(w))$ is the usual Robinson-Schensted map.

As pointed out in [20], Schützenberger's version of the Littlewood-Richardson rule is equivalent to the following statement, which shows that the free Schur functions span a subalgebra of $\mathbf{FQSym}$. We will call it the algebra of *free symmetric functions* and denote it by $\mathbf{FSym}$. It provides a realization of the algebra of tableaux introduced by Poirier and Reutenauer [27] as a subalgebra of the free associative algebra. A representation theoretical interpretation will be given in the sequel.

Proposition 3.12 (LRS rule) *Let t', t'' be standard tableaux, and let k be the number of cells of t'. Then,*

$$\mathbf{S}_{t'}\mathbf{S}_{t''} = \sum_{t\in \mathrm{Sh}(t',t'')} \mathbf{S}_t \tag{54}$$

where $\mathrm{Sh}(t',t'')$ is the set of standard tableaux in the shuffle of t' (regarded as a word via its row reading) with the plactic class of $t''[k]$.

Proof – This follows from Proposition 3.2, and the fact that the plactic congruence is compatible with restriction to intervals. Indeed, denote by $\equiv$ the plactic congruence on the free algebra $\mathbb{Z}\langle A\rangle$, for some ordered alphabet $A = \{a_1 < a_2 < \cdots < a_n\}$. For a word $w \in A^*$ and an interval $I = [a_i, a_j]$ of A, denote by $w|_I$ the word obtained by erasing in w the letters not in I. Then, since the plactic relations $xzy \equiv zxy$ $(z \le y < z$ and $yxz \equiv yzx$ $(x < y \le z)$ reduce to equalities after erasing x or z, $w \equiv w' \Rightarrow w|_I \equiv w'|_I$. From this, we see that

$$\mathbf{S}_{t'}\mathbf{S}_{t''} = \sum_{P(\sigma')=t, P(\sigma'')=t''} \mathbf{F}_{\sigma'}\mathbf{F}_{\sigma''} = \sum_{t\in\mathrm{Sh}(t',t'')} \sum_{P(w)=t} \mathbf{F}_\sigma\,, \tag{55}$$

since the set of permutations $\{\sigma' \sqcup\!\sqcup \sigma''[k] \mid P(\sigma') = t, P(\sigma'') = t''\}$ is, by the above remark, a union of plactic classes. Each class contains a unique tableau, and since the restriction of a tableau to an initial segment of the alphabet has to be a tableau, such a tableau can appear only in the shuffles $t' \sqcup\!\sqcup \sigma''[k]$. ∎

The original Littlewood-Richardson rule, as well as its plactic version, are immediate corollaries of Proposition 3.12 (see [20]).

Example 3.13 The smallest interesting example occurs for the shape $(2,1)$, e.g., with

$$t' = t'' = \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}\,,$$

the product $\mathbf{S}_{t'}\mathbf{S}_{t''}$ is equal to $\sum_t \mathbf{S}_t$ where t ranges over the following tableaux:

$$
\begin{array}{llll} 3 & 6 & & \\ 1 & 2 & 4 & 5 \end{array}
\qquad
\begin{array}{lll} 3 & 4 & 6 \\ 1 & 2 & 5 \end{array}
\qquad
\begin{array}{llll} 6 & & & \\ 3 & & & \\ 1 & 2 & 4 & 5 \end{array}
\qquad
\begin{array}{lll} 4 & & \\ 3 & 6 & \\ 1 & 2 & 5 \end{array}
$$

$$
\begin{array}{lll} 6 & & \\ 3 & 4 & \\ 1 & 2 & 5 \end{array}
\qquad
\begin{array}{ll} 4 & 6 \\ 3 & 5 \\ 1 & 2 \end{array}
\qquad
\begin{array}{lll} 6 & & \\ 4 & & \\ 3 & & \\ 1 & 2 & 5 \end{array}
\qquad
\begin{array}{ll} 6 & \\ 4 & \\ 3 & 5 \\ 1 & 2 \end{array}
$$

The scalar product of two free Schur functions is equal to 1 whenever the corresponding tableaux have the same shape, and to 0 otherwise. Indeed,

$$\langle \mathbf{S}_{t'}, \mathbf{S}_{t''} \rangle = \sum_{P(\sigma)=t', Q(\sigma)=t''} \langle \mathbf{F}_\sigma, \mathbf{G}_\sigma \rangle = 1 \tag{56}$$

since a permutation is uniquely determined by its P and Q symbols.

Note that the algebra of noncommutative symmetric functions $\mathbf{Sym}(A)$ is a subalgebra of $\mathbf{FSym}$, since

$$R_I(A) = \sum_{\mathrm{Rec}(t)=\mathrm{Des}(I)} \mathbf{S}_t \,, \tag{57}$$

where $\mathrm{Rec}(t)$ denotes the recoil (or descent) set of the tableau t.

3.6 An example: the QS-distribution on symmetric groups

The definitions of this section are well illustrated by a certain probability distribution on symmetric groups investigated by Stanley in [30]. Let $x = (x_i)_{i\geq 1}$ be a probability distribution on our infinite alphabet $A = \{a_1, a_2, \ldots\}$, that is, $\mathrm{Prob}(a_i) = x_i$, $x_i \geq 0$, and $\sum x_i = 1$. From this, one defines a probability distribution $QS(x)$ on each symmetric group $\mathfrak{S}_n$ by the formula

$$\mathrm{Prob}(\sigma) = \mathbf{G}_\sigma(x) \,. \tag{58}$$

That this is actually a probability distribution follows from the identity $\mathbf{G}_1^n = \sum_{\sigma\in\mathfrak{S}_n} \mathbf{G}_\sigma$. Then, Theorem 2.1 of [30] states that $\mathrm{Prob}(\sigma) = F_{C(\sigma^{-1})}(x)$, which follows from the equalities $\mathbf{G}_\sigma = \mathbf{F}_{\sigma^{-1}}$ and $\mathbf{F}_\sigma(x) = F_{C(\sigma)}(x)$.

Next, Stanley introduces the operator

$$\Gamma_n(x) = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{Prob}(\sigma)\sigma \;\in\; \mathbb{C}\mathfrak{S}_n \,. \tag{59}$$

Actually, $\Gamma_n(x)$ is in the descent algebra Σ_n, and the corresponding noncommutative symmetric function is $S_n(xA)$. Therefore [17], the eigenvalues of $\Gamma_n(x)$ are the $p_\lambda(x)$, with multiplicities $n!/z_\lambda$. Also, the convolution formula $\Gamma_n(x)\Gamma_n(y) = \Gamma_n(xy)$ amounts to the identity $S_n(xA) * S_n(yA) = S_n(xyA)$ of [17].

Another result of [30] is that the probability $M_n(k)$ that a random permutation (chosen from the QS-distribution) has k inversions is equal to the probability that it has major index k (Theorem 3.2). This is equivalent to the identity

$$\sum_{\sigma\in\mathfrak{S}_n} q^{l(\sigma)}\mathbf{G}_\sigma(x) = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)}\mathbf{G}_\sigma(x). \tag{60}$$

The right-hand side can be rewritten as

$$\sum_{|I|=n} q^{\mathrm{maj}(I)} r_I(x) = \sum_{I,J} q^{\mathrm{maj}(I)} \langle r_I, r_J\rangle F_J(x)$$
$$= \sum_J \left(\sum_{C(\sigma^{-1})=J} q^{\mathrm{maj}(\sigma)}\right) F_J(x) = \sum_J \left(\sum_{C(\sigma^{-1})=J} q^{l(\sigma)}\right) F_J(x) = \sum_{\sigma\in\mathfrak{S}_n} q^{l(\sigma)}\mathbf{G}_\sigma(x)$$

since $l(\sigma)$ and $\mathrm{maj}(\sigma^{-1})$ have the same distribution on a descent class (*cf.* [29, 10]).

Finally, we note that the specialization $\mathbf{S}_t(x)$ of a free Schur function is the probability that a QS-random permutation has t as insertion tableau, and that $R_I(x)$ is the probability that a random permutation has shape I (Theorems 3.4 and 3.6 of [30]).

3.7 Quantum quasi-symmetric functions again

Recall that we denote by $\mathbb{C}_q[X]$ the algebra of quantum polynomials, generated by letters x_i subject to the relations $x_jx_i = qx_ix_j$ when $j > i$. The following proposition clarifies the constructions of [31].

Proposition 3.14 *The natural homomorphism $\varphi_q : a_i \mapsto x_i$ from $\mathbb{C}\langle A\rangle$ to the algebra of quantum polynomials $\mathbb{C}_q[X]$ maps $\mathbf{F}_\sigma$ to the quantum quasi-symmetric function $q^{\ell(\sigma)}F_{C(\sigma)}$.*

Proof – For any word w, one has

$$\varphi_q(w) = q^{\ell(\sigma)}\varphi_q(w^+)$$

where $\sigma = \mathrm{Std}(w)^{-1}$ and w^+ is the nondecreasing rearrangement of w. ∎

Therefore, $QSym_q$ is a quotient of $\mathbf{FQSym}$. The multiplication formula (11) appears now as an immediate consequence of Proposition 3.2. The q-generating function $\Gamma_q(P)$ of a poset, introduced in [31] to derive (11), can also be regarded as the image under φ_q of of a free generating function $\mathbf{\Gamma}(P)$ described in the forthcoming section. Most formulas of [31] are easy consequences of Proposition 3.14. For example, formula (38) of [31], which can be stated as

$$\varphi_q(R_I(A)) = \sum_J c_{IJ}(q)\overline{F}_J \tag{61}$$

where

$$c_{IJ}(q) = \sum_{C(\sigma)=I,\ C(\sigma^{-1})=J} q^{\ell(\sigma)} \tag{62}$$

follows from the expression

$$R_I(A) = \sum_{C(\sigma)=I} \mathbf{G}_\sigma \,. \tag{63}$$

3.8 Posets, P-partitions, and the like

Here, by a *poset*, we mean any partial order P on the set $[n] = \{1, 2, \ldots, n\}$. We write $<_P$ for the order of P and $<$ for the usual total order on $[n]$. Stanley [29] defines a P-partition as a function $f : [n] \to X$ for some totally ordered set of variables X, such that

$$i <_P j \Rightarrow f(i) \le f(j) \quad \text{and} \quad i <_P j \text{ and } i > j \Rightarrow f(i) < f(j) \,. \tag{64}$$

In [12], Gessel associates to a poset P a generating function

$$\Gamma(P) = \sum_{f \in \mathcal{A}(P)} f(1) f(2) \cdots f(n) \tag{65}$$

where $\mathcal{A}(P)$ denotes the set of all P-partitions. This generating function turns out to be quasi-symmetric, actually,

$$\Gamma(P) = \sum_{\sigma \in L(P)} \Gamma(\sigma) = \sum_{\sigma \in L(P)} F_{C(\sigma)} \tag{66}$$

where $L(P)$ denotes the set of linear extensions of P, which can be identified with permutations $\sigma \in \mathfrak{S}_n$ such that $i <_P j \Rightarrow \sigma^{-1}(i) < \sigma^{-1}(j)$. Identifying a P-partition f with the word $w_f = a_{f(1)} a_{f(2)} \cdots a_{f(n)}$, we arrive at the following

Definition 3.15 *The free quasi-symmetric generating function* $\mathbf{\Gamma}(P) \in \mathbf{FQSym}$ *of a poset* P *is*

$$\mathbf{\Gamma}(P) = \sum_{\sigma \in L(P)} \mathbf{F}_\sigma \,. \tag{67}$$

This amounts to encode a poset by the set of its linear extensions. It is well known, and easy to see, that if P_1 is an order on $[k]$ and P_2 an order on $[l]$, the order $P = P_1 \sqcup P_2$ on $[n] = [k+l]$ defined by $i <_P j \Leftrightarrow i <_{P_1} j$ or $i - k <_{P_2} j - k$ has for linear extensions the shifted shuffles of those of P_1 and P_2:

$$\sum_{\sigma \in L(P)} \sigma = \sum_{\alpha \in L(P_1)} \sum_{\beta \in L(P_2)} \alpha \sqcup\!\sqcup \beta[k] \,. \tag{68}$$

Thus, for the free generating functions, one has as well

$$\mathbf{\Gamma}(P_1 \sqcup P_2) = \mathbf{\Gamma}(P_1) \mathbf{\Gamma}(P_2) \tag{69}$$

in $\mathbb{Z}\langle A \rangle$.

It will be convenient to introduce the notation $P_2[k]$ for the order on $[k+1, k+l]$ defined above, so that $L(P_1 \sqcup P_2) = L(P_1) \sqcup\!\sqcup L(P_2[k])$.

Example 3.16 The free Schur functions $\mathbf{S}_t$ are of the form $\mathbf{\Gamma}(P)$ for the posets associated to plane partitions. Malvenuto [22] has shown that if $\mathbf{\Gamma}(P) \in \mathbf{FSym}$, then P is associated to a plane partition. This is a step towards a famous conjecture of Stanley, asserting that the conclusion remains valid as soon as the commutative image of $\mathbf{\Gamma}(P)$ is symmetric.

Example 3.17 The concatenation $P_1 \sqcup P_2$ is not the only interesting poset which can be constructed from P_1 and P_2. One can also define $P = P_1 \wedge P_2$ as the poset obtained by adjoining a maximal element to the juxtaposition of P_1 and P_2. The correct way to do this is to take as maximal element $h = k+1$ if P_1 is a poset on $[k]$. Therefore, $i <_P j$ iff $i, j \leq k$ and $i <_{P_1} j$ or $i, j > k+1$ and $i-k-1 <_{P_2} i-k-1$, or $j = k+1$. The linear extensions of P are clearly

$$L(P_1 \wedge P_2) = (L(P_1) \sqcup\!\sqcup L(P_2[h])) \cdot h\,. \tag{70}$$

The posets generated from $\bullet = [1]$ by the operation $\wedge$ are in one-to-one correspondence with binary trees, since they correspond to all possible bracketings of the words $\bullet\bullet\cdots\bullet$. Let $\mathbf{F}(T) = \mathbf{\Gamma}(T)$ be the free quasi-symmetric generating functions of such posets. We will see that they span a subalgebra of $\mathbf{FQSym}$, which is precisely the Hopf algebra of binary trees introduced by Loday and Ronco [21]. Indeed, let $T = T_1 \wedge T_2$ and $T' = T_1' \wedge T_2'$ be two binary trees. From the above considerations, we see that

$$\begin{aligned} L(T \sqcup T') = L(T) \sqcup\!\sqcup L(T'[n]) &= \sum_{\substack{\alpha \in L(T_1)\\ \beta \in L(T_2)}} \sum_{\substack{\alpha' \in L(T_1'[n])\\ \beta' \in L(T_2'[n])}} [(\alpha \sqcup\!\sqcup \beta[h])h] \sqcup\!\sqcup [(\alpha' \sqcup\!\sqcup \beta'[h'])h'] \\ &= \sum [(\alpha \sqcup\!\sqcup \beta[h]) \sqcup\!\sqcup (\alpha' \sqcup\!\sqcup \beta'[h'])h']\, h + \sum [((\alpha \sqcup\!\sqcup \beta[h])h) \sqcup\!\sqcup (\alpha' \sqcup\!\sqcup \beta'[h'])]\, h' \end{aligned}$$

(using the formula $(ua) \sqcup\!\sqcup (vb) = (u \sqcup\!\sqcup vb)a + (ua \sqcup\!\sqcup v)b$, valid for $a, b \in A$ and $u, v \in A^*$). Therefore,

$$L(T \sqcup T') = L(T_1 \wedge (T_2 \sqcup T')) + L((T \sqcup T_1') \wedge T_2) \tag{71}$$

which proves that $\mathbf{F}(T)\mathbf{F}(T')$ is a sum of elements $\mathbf{F}(T'')$ which are given by the above recursion.

The connection with the algebra of Loday and Ronco comes for the fact that $\mathbf{F}(T) = \sum_{\mathbf{T}(\sigma)=T} \mathbf{G}_\sigma$, where $\mathbf{T}(\sigma)$ is the underlying binary tree of σ, defined as follows: if $n = 1$ (σ is the empty word), $\mathbf{T}(\sigma) = \bullet$, otherwise, write $\sigma = unv$, $\alpha = \mathrm{Std}(u)$, $\beta = \mathrm{Std}(v)$. Then, $\mathbf{T}(\sigma) = \mathbf{T}(\alpha) \wedge \mathbf{T}(\beta)$, where $T_1 \wedge T_2$ is the binary tree having T_1 as left subtree and T_2 as right subtree.

3.9 Posets as 0-Hecke modules

There is a striking similarity between the behavior of the quasi-symmetric generating functions of posets under concatenation, and the characteristic quasi-symmetric functions of 0-Hecke modules under induction product. Actually, the former is a special case of the latter:

Definition 3.18 *The* 0-Hecke module M_P associated with a poset P *is the (right)* 0*-Hecke* *module with basis the set of linear extensions $L(P)$ and structure defined by*

$$\sigma T_i = \begin{cases} \sigma\sigma_i & \textit{if } i \notin \mathrm{Des}(\sigma) \textit{ and } \sigma\sigma_i \in L(P)\,, \\ 0 & \textit{if } i \notin \mathrm{Des}(\sigma) \textit{ but } \sigma\sigma_i \notin L(P)\,, \\ -\sigma & \textit{if } i \in \mathrm{Des}(\sigma)\,. \end{cases} \tag{72}$$

Proof – We have to prove that M_P is actually a 0-Hecke module. Here we need some definitions.

Definition 3.19 *A poset P is* rise free *if there is no $i < j$ such that $i <_P j$.*

Let $E(P)$ be the set of all minimal linear extensions of P for the right weak order.

Proposition 3.20 *Let P be a rise free poset. The set of permutations that are larger for the right weak order than any permutation of $E(P)$ is exactly the set of linear extensions of P.*

It has for consequence that if P is a rise free poset, the submodule of the regular representation generated by its minimal linear extensions has the structure defined above. Then M_P is a module for P rise free.

Now, if P is not rise free, let $\mathrm{RiseFree}(P)$ be its associated rise free poset defined by

$$i <_{\mathrm{RiseFree}(P)} j \qquad \text{if and only if} \qquad (i > j \text{ and } i <_P j)\,. \tag{73}$$

Note that P and $\mathrm{RiseFree}(P)$ have the same minimal linear extensions. Consider the module $M_{\mathrm{RiseFree}(P)}$. It has for basis the set of permutations that are greater than the minimal linear extensions of $\mathrm{RiseFree}(P)$ and P. If a permutation σ in this set is not a linear extension of P then there is a $i < j$ such that $i <_P j$ but $\sigma_i > \sigma_j$. And then all the permutations bigger than σ are not linear extensions of P. This means that the set of permutations larger than any element of $E(P)$ but that are not linear extensions of P span a sub-module N of $M_{\mathrm{RiseFree}(P)}$. Now it is easy to see that

$$M_P \equiv M_{\mathrm{RiseFree}(P)}/N\,. \tag{74}$$

is a realisation of M_P. And hence M_P is a module. ∎

Then one has following proposition.

Proposition 3.21 *(i) Let P be a poset. Then* $\mathrm{ch}(M_P) = \Gamma(P)$.

(ii) $M_{P\sqcup P'} = M_P \hat{\otimes} M_{P'}$ and consequently

$$\mathbf{ch}(M_{P\sqcup P'}) = \mathbf{\Gamma}(P)\,\mathbf{\Gamma}(P') = \mathbf{ch}(M_P)\,\mathbf{ch}(M_{P'}) = \mathbf{ch}(M_P \hat{\otimes} M_{P'}) \tag{75}$$

See Figure 1 for an example.

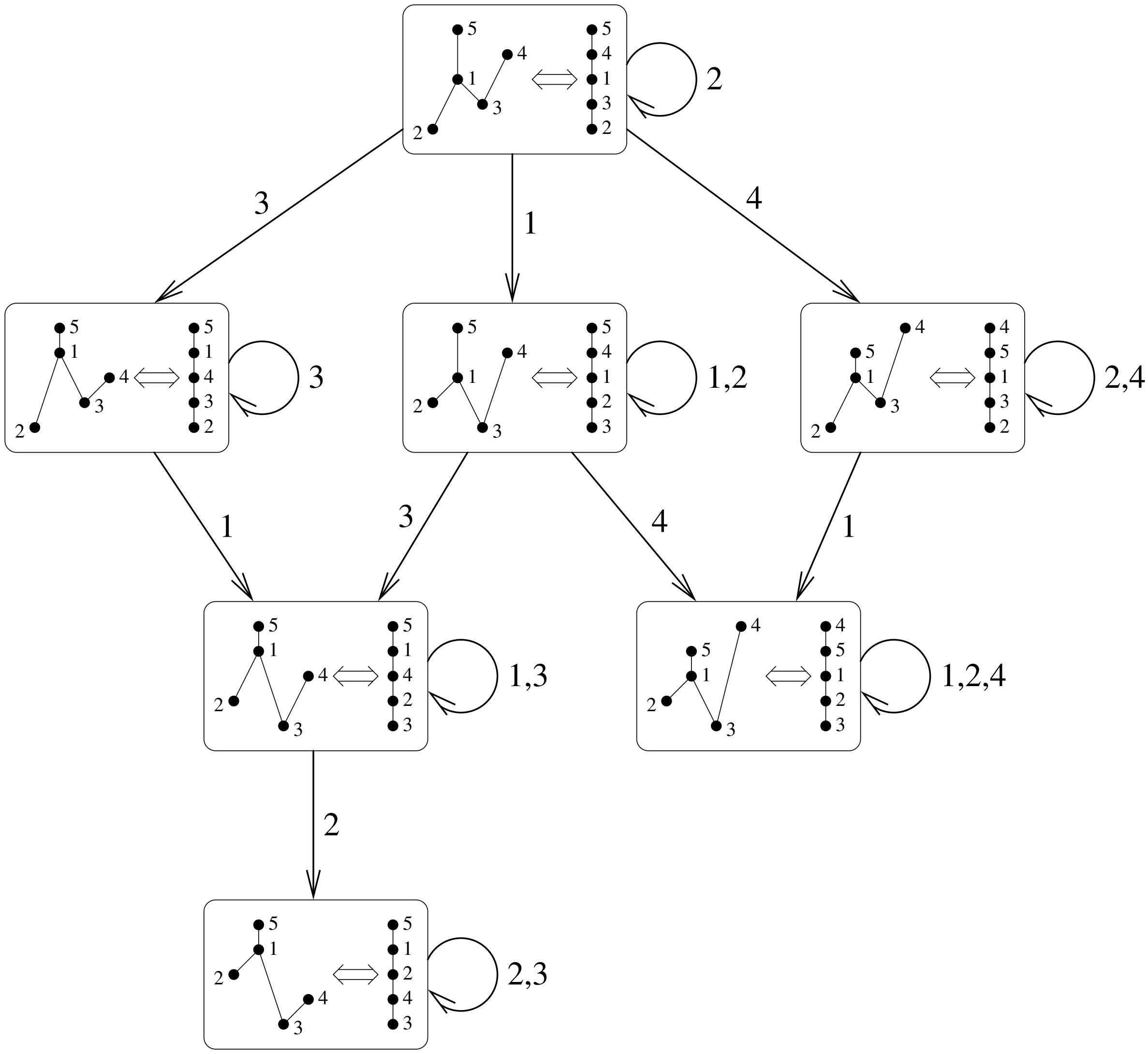


Figure 1: Example of module associated with a poset

To each vertex of the graph enclosed in a box corresponds a basis element associated with the depicted linear extension. There is a straigth arrow labelled i from u to v if $uT_i = v$. A loop labelled i means that $uT_i = -u$. If there is no arrow labelled i leaving the vertex u, then $uT_i = 0$.

3.10 Shuffle and pseudo-convolution

In this section, we will encounter another example of pseudo-convolution, in the sense of 2.4. Let $\Box$ be the operation on $\bigoplus \mathbb{C}\mathfrak{S}_n$ defined on permutations by

$$\alpha\Box\beta = \sum_{I\sqcup J=[1,k+l]} i_{\alpha(1)}\cdots i_{\alpha(k)} \sqcup\!\sqcup j_{\beta(1)}\cdots j_{\beta(l)} \tag{76}$$

This operation arises naturally in the problem of calculating the orthogonal projection onto the free Lie algebra. One can show that this problem boils down to the inversion of the element

$$T_n = \sum_{k=0}^{n} (1\cdots k)\Box(1\cdots n-k) \tag{77}$$

of $\mathbb{Q}\mathfrak{S}_n$ [5, 16]. No closed formula is known for T_n^{-1}, but numerical experiments suggest that it should be possible to give a combinatorial description of its characteristic polynomial.

Example 3.22 The characteristic polynomial of T_4 as an operator on the regular representation of $\mathfrak{S}_4$ is

$$(x-2)^6(x-6)^4(x-14)^3(x-18)^3(x-42)^3(x-70)(x^2-28x+84)^2\,.$$

It is natural to introduce *q-pseudo-convolution* $\Box_q$, which is defined similarly, with $\sqcup\!\sqcup$ replaced by $\sqcup\!\sqcup_q$. Actually, this operation can be interpreted in $\mathbf{FQSym}$ in the same way as the ordinary convolution. Let $\langle\,\sigma\,|\,\tau\,\rangle = \delta_{\sigma,\tau}$ be the scalar product on the group algebra for which permutations form an orthonormal basis, so that

$$\alpha\Box_q\beta = \sum_\sigma \langle\,\sigma\,|\,\alpha\Box_q\beta\,\rangle\,\sigma\,. \tag{78}$$

Proposition 3.23 *The algebra of free quasi-symmetric functions is a q-shuffle subalgebra of $\mathbb{C}(q)\langle A\rangle$, and in the $\mathbf{G}$-basis, the structure constants coincide with those of q-pseudo-convolution*

$$\mathbf{G}_\alpha \sqcup\!\sqcup_q \mathbf{G}_\beta = \sum_\sigma \langle\,\sigma\,|\,\alpha\Box_q\beta\,\rangle\,\mathbf{G}_\sigma\,. \tag{79}$$

Proof – (sketch) We proceed as for Proposition 3.2. In the biword notation, we have

$$\mathbf{G}_\alpha \sqcup\!\sqcup_q \mathbf{G}_\beta = \sum_{\substack{u\uparrow\alpha^{-1}\\ v\uparrow\beta^{-1}}} \binom{u}{\alpha^{-1}} \sqcup\!\sqcup_q \binom{v}{\beta^{-1}} = \sum_{\substack{u\uparrow\alpha^{-1}\\ v\uparrow\beta^{-1}}} \binom{uv}{(\alpha\sqcup\!\sqcup_q\beta)^\vee}$$
$$= \sum_{\sigma\in\alpha\Box_q\beta}\ \sum_{w\uparrow\sigma^{-1}} \binom{w}{\sigma} = \sum_\sigma \langle\,\sigma\,|\,\alpha\Box_q\beta\,\rangle\,\mathbf{G}_\sigma\,.$$

In particular, for $q=1$, identifying $\mathbf{G}_{(12\cdots n)}$ to the noncommutative complete function S_n, we see that

$$T_n = S_n + S_1 \sqcup\!\sqcup S_{n-1} + \cdots + S_{n-1} \sqcup\!\sqcup S_1 + S_n = h_n(2X) \tag{80}$$

if we identify the $\sqcup\!\sqcup$-subalgebra generated by the S_n with the algebra of commutative symmetric functions of some alphabet X. At this point, it is natural to introduce q-analogues $T_n(q)$. If we define them as

$$T_n(q) = S_n + qS_1 \sqcup\!\sqcup_q S_{n-1} + \cdots + q^n S_n \tag{81}$$

we see that $T_n(q) = S_n((1+q)B)$, if we now identify the $\sqcup\!\sqcup_q$-subalgebra of $\mathbf{FQSym}$ generated by the S_n with $\mathbf{Sym}(B_q)$ for a noncommutative alphabet B_q. That is, we have a one-parameter family of identifications of the S^I with elements of the group algebra.

Example 3.24 The characteristic polynomial of $T_3(q)$ is

$$(x-2)^2(x-4-4q-2q^2)^2(x-8-6q-6q^2)(x-4+2q-2q^2)\,.$$

It makes sense to consider the quasi-symmetric generating functions of the elements T_n, which amounts to take the commutative images of the corresponding elements of $\mathbf{FQSym}$ (here it does not matter whether one interprets σ as $\mathbf{F}_\sigma$ or $\mathbf{G}_\sigma$ since T_n is self-adjoint. One finds that

$$\underline{T}_n = \sum_{i+j=n} \binom{n}{i} h_i h_j = [t^n]\frac{1}{1-t}\sum_{m=0}^{n}\left(\frac{t}{1-t}\right)^m h_m h_{n-m}\,. \tag{82}$$

The first values are, on the Schur basis

$$2s_1,\ 4s_2+2s_{11},\ 8s_3+6s_{21},\ 16s_4+14s_{31}+6s_{22}\ \ldots$$

The elementary symmetric functions of the $\sqcup\!\sqcup$-algebra generated by the $S_n = \mathbf{G}_{(12\cdots n)}$ also seem to be interesting. It would be interesting to investigate the structure of $\mathbf{FQSym}$ as a $\sqcup\!\sqcup$-module over this commutative subalgebra, and also the q-analogue of this situation.

This suggests the possibility of using the machinery of noncommutative symmetric functions to invert $T_n(q)$. The problem is to interpret the internal product of $\mathbf{Sym}(B_q)$ in terms of the structure of $\mathbf{FQSym}$, and more precisely to connect it to the ordinary composition of permutations. That is, if one defines $*_q$ on $\mathbf{Sym}(B_q)$ by the standard formulas giving $S^I * S^J$, for example, does there exist an automorphism ϕ_q of $\mathbb{C}(q)\langle A\rangle$ such that $F*G = \phi_q^{-1}(\phi_q(F)\circ\phi_q(G))$? (here $\circ$ is the composition of permutations).

3.11 Identities

A few identities between series of free quasi-symmetric functions (mainly conjectures) can be found in [32]. For example, the inverses of the series

$$\begin{aligned}
H_1 &= \sum_I (-1)^{\ell(I)}\mathbf{F}_{\omega(I)} \\
H_2 &= \sum_{n\geq 0} (-1)^n \mathbf{F}_{\omega(2^n)} \\
H_3 &= \sum_I (-1)^{\ell(I)}\mathbf{F}_{\omega(2I)}
\end{aligned}$$

are conjectured to be as follows. For a permutation σ of shape I, let $\hat{\sigma} = \sigma\alpha(I)$. Then,

$$\begin{aligned} H_1^{-1} &= \sum_{\alpha} \mathbf{G}_{\hat{\alpha}} \\ H_2^{-1} &= \sum_{\beta} \mathbf{G}_{\hat{\beta}} \\ H_3^{-1} &= \sum_{\gamma} \mathbf{G}_{\hat{\gamma}} \end{aligned}$$

where α runs over all permutations, $\beta \in \mathfrak{S}_{2p}$ runs over permutations of shape 2^{2p}, and $\gamma \in \mathfrak{S}_{2p}$ runs over permutations with descent set contained in $\{2, 4, \ldots, 2p-2\}$.

4 The 0-Hecke algebra revisited

4.1 $H_n(0)$ as a Frobenius algebra

Recall that a bilinear form $(\ ,\ )$ on a $\mathbb{K}$-algebra A is said to be associative if $(ab, c) = (a, bc)$ for all $a, b, c \in A$, and that A is called a *Frobenius algebra* whenever it has a nondegenerate associative bilinear form. Such a form induces an isomorphism of left A-modules between A and the dual A^* of the right regular representation. Frobenius algebras are in particular self-injective, so that finitely generated projective and injective modules coincide (see [4]).

For a basis (Y_σ) of $H_n(0)$, we denote by (Y_σ^*) the dual basis. We set $\chi = T_\omega^*$, where $\omega = (n\, n-1 \ldots 1)$ is the longest permutation of $\mathfrak{S}_n$.

Proposition 4.1 *(i) The associative bilinear form defined by*

$$(f, g) = \chi(fg) \tag{83}$$

is non-degenerate on $H_n(0)$. Therefore, $H_n(0)$ is a Frobenius algebra.

(ii) $(\eta_\sigma, \eta_{\tau^{-1}\omega}) = \delta(\sigma \geq \tau)$, where $\geq$ is the Bruhat order on $\mathfrak{S}_n$, and for a statement P, $\delta(P)$ is 1 when P is true and 0 otherwise.

(iii) The elements $\zeta_\sigma = (-1)^{\ell(\omega\sigma^{-1})}\xi_{\omega\sigma^{-1}}$ satisfy

$$(\zeta_\sigma, \eta_\tau) = \delta_{\sigma,\tau}\,. \tag{84}$$

Proof – The bilinear form defined in *(i)* is clearly associative. That it is non-degenerate follows from *(ii)*, which implies that the matrix (η_σ, η_τ) is, up to a permutation of columns, the incidence matrix of the Bruhat order, which is obviously invertible. The proof of *(ii)* is a simple induction on $\ell(\sigma)$. Finally, *(iii)* follows from *(ii)* and [19], Lemme 1.13, which says that

$$\xi_\alpha = \sum_{\beta \leq \alpha} T_\beta \quad \text{and} \quad \eta_\alpha = \sum_{\beta \leq \alpha} (-1)^{\ell(\beta)} \xi_\beta\,. \tag{85}$$

Remark 4.2 As recently shown by L. Abrams [1], a Frobenius algebra is endowed with a comultiplication $\delta : A \to A \otimes A$ which is a morphism of A-bimodules, that is, $\delta(axb) = a\delta(x)b$. It can be defined by the formula

$$\delta = (\lambda^{-1} \otimes \lambda^{-1}) \circ (\mu \circ T)^* \circ \lambda \tag{86}$$

where $\lambda : A \to A^*$ is an isomorphism of left A-modules, $\mu : a \otimes b \mapsto ab$ is the multiplication map, and $T : a \otimes b \mapsto b \otimes a$ is the exchange operator. Since δ is a bimodule map, it is completely specified by the element $\delta(1_A)$, which we will now calculate explicitly for $H_n(0)$. Let λ be defined by $\lambda(x)(y) = (y, x)$. Then,

$$\lambda(\eta_\sigma) = \sum_{\omega\tau^{-1} \le \sigma} \eta_\tau^*$$

so that $\lambda^{-1}(\eta_\sigma^*) = \zeta_\sigma$. If we define the permutation $\{\alpha, \beta\}$ by the rule $\eta_\alpha \eta_\beta = \eta_{\{\alpha,\beta\}}$, then,

$$\begin{aligned}
\delta(1) &= \sum_{\{\alpha,\beta\}=\omega} \zeta_\beta \otimes \zeta_\alpha \\
&= \sum_\alpha \left(\sum_{\{\alpha,\beta\}=\omega} (-1)^{\ell(\omega\beta^{-1})} \xi_{\omega\beta^{-1}} \right) \otimes \zeta_\alpha \\
&= \sum_\alpha \left(\sum_{\gamma \le \alpha} (-1)^{\ell(\gamma)} \xi_\gamma \right) \otimes \zeta_\alpha = \sum_\alpha \eta_\alpha \otimes \zeta_\alpha \,.
\end{aligned}$$

Therefore, the canonical comultiplication of $H_n(0)$ is given by

$$\delta(1) = \sum_{\sigma \in \mathfrak{S}_n} \eta_\sigma \otimes \zeta_\sigma \tag{87}$$

and $\delta(x) = x\delta(1) = \delta(1)x$.

4.2 FQSym as a Grothendieck ring

Let (g_σ) be the basis of $H_n(0)$ defined by

$$g_\sigma = T_{\sigma\alpha(I)^{-1}} \epsilon_I \tag{88}$$

where $I = C(\sigma)$ is the descent composition of σ and ϵ_I the generator of the principal indecomposable projective module $\mathbf{P}_I$. Then, $\{g_\sigma | \sigma \in [\alpha(I), \omega(I)]\}$ is a basis of $\mathbf{P}_I$ (the interval is taken with respect to the weak order).

Definition 4.3 *For any permutation $\sigma \in \mathfrak{S}_n$, we denote by $\mathbf{N}_\sigma$ the submodule of $\mathbf{P}_I$ (where $I = C(\sigma)$) generated by g_σ.*

All the $\mathbf{N}_\sigma$ are indecomposable $H_n(0)$-modules, since any submodule of a $\mathbf{P}_I$ must contain its one-dimensional socle, and therefore cannot be a direct summand. The simple $H_n(0)$ modules are the $\mathbf{N}_{\omega(I)}$, and $\mathbf{P}_I = \mathbf{N}_{\alpha(I)}$.

Of course, the $\mathbf{N}_\sigma$ do not exhaust all submodules of the $\mathbf{P}_I$, but, as we will see, they generate an interesting subcategory $\mathcal{N}_n$ of $H_n(0)-\mathrm{mod}$. In particular, all the specializations $q=0$ of the Specht modules $V_\lambda(q)$ of $H_n(q)$, as well as their skew versions $V_{\lambda/\mu}(q)$, with λ/μ connected, are of the form $\mathbf{N}_\sigma$, where σ is the row reading of the hyperstandard tableau of shape λ (or λ/μ) i.e., the tableau whose columns are filled with consecutive integers. As a consequence, all the $V_{\lambda/\mu}(0)$, with λ/μ connected, are indecomposable.

Define a characteristic map with values in $\mathbf{FQSym}$ by

$$\mathbf{ch}(\mathbf{N}_\sigma) = N_\sigma = \sum_{\tau\in[\sigma,\omega(I)]} \mathbf{G}_\sigma\,. \tag{89}$$

This definition is compatible with the former one for projective modules, since $\mathbf{ch}(\mathbf{P}_I) = R_I$. More generally, the characteristic of a Specht module is a free symmetric function: $\mathbf{ch}(V_\lambda(0)) = \mathbf{S}_t$, where t is the tableau congruent to the contretableau of shape $\omega(\lambda)$ whose rows consist of consecutive integers (e.g., 456 23 1 for $\lambda=(321)$).

Proposition 4.4 *The characteristic map is compatible with induction product, that is, we have an exact sequence*

$$0 \to \mathbf{N}_\beta \to \mathbf{N}_\sigma\hat{\otimes}\mathbf{N}_\tau \to \mathbf{N}_\alpha \to 0 \tag{90}$$

where $\alpha=\sigma\bullet\tau$, and if as words $\sigma^{-1}=ukv$, $\tau^{-1}[k]=u'(k+1)v'$ then $\beta^{-1}=uu'(k+1)kvv'$, and also

$$\mathbf{ch}(\mathbf{N}_\sigma\hat{\otimes}\mathbf{N}_\tau) = N_\sigma N_\tau = N_\alpha + N_\beta\,. \tag{91}$$

In the case of skew Specht modules indexed by connected skew diagrams D, D', the formula reads

$$0 \to V_{D_2} \to V_D\hat{\otimes}V_{D'} \to V_{D_1} \to 0 \tag{92}$$

where D_1 and D_2 are the two ways of glueing the first box of D' to the last box of D.

Proof – Remark first that if $C(\sigma)=I$ and $C(\tau)=J$, $M=\mathbf{N}_\sigma\hat{\otimes}\mathbf{N}_\tau$ is a submodule of $\mathbf{P}_I\hat{\otimes}\mathbf{P}_J = \mathbf{P}_{I\triangleright J}\oplus\mathbf{P}_{I\cdot J}$. Also, M is a combinatorial module. It is generated by the element $g_\sigma\otimes g_\tau$, which can be represented by the skew ribbon r_0 obtained by making the upper left corner of the first cell of the ribbon of $\tau[k]$ coincide with the bottom right corner of the last cell of the ribbon of σ. The combinatorial basis of M is formed by those skew ribbon of the same shape as r_0 which can be obtained from r_0 by application of a chain of operators $\eta_i=-T_i$. Their action is given by the same formulas as for the case of connected ribbons representing the bases of the projective indecomposable modules: if i is a recoil of r, then $\eta_i(r)=r$. If $i+1$ is in the same row as i, then $\eta_i(r)=0$, and otherwise, $\eta_i(r)=r'$, the skew ribbon obtained from r by exchanging i and $i+1$.

Now, the skew ribbons generated from r_0 can be converted into connected ribbons of shape IJ or $I\triangleright J$, according to whether the first entry of the right connected component is greater or smaller than the last entry of the left component. The generator r_0 corresponds to the shape $I\triangleright J$, filled with the permutation α. According to the above rules, the action of $H_n(0)$ will generate all permutations of this shape which are greater than α for the weak order, plus some other ones of shape IJ.

All the permutations of shape IJ are greater than those of shape $I\triangleright J$, and span therefore a submodule, which is easily seen to be generated by β. Indeed, define β as the smallest (for the weak order) permutation of shape IJ which is greater than α. Set $\beta = st$ as a word, with $|s| = k$. Since $\beta > \alpha$, we have $\mathrm{Std}(s) = \sigma$ and $\mathrm{Std}(t) = \tau$. This means that the letters $1, \ldots, k$ occur in the same order in σ^{-1} and in β^{-1}, and also, $k+1, \ldots, k+l$ occur in the same order in τ^{-1} and β^{-1}. Hence, $\beta^{-1} \in \sigma^{-1} ⧢ \tau^{-1}[k]$. Also, k must be a descent of β. Hence, in β^{-1}, the letter $k+1$ appears on the left of k. The smallest permutation with these properties is $\beta^{-1} = uu'(k+1)kvv'$, as claimed.

Hence, $\mathbf{N}_\beta$ is a submodule (even a subgraph) of M, and the quotient is isomorphic to $\mathbf{N}_\alpha$. Now, the permutations obtained by applying the η_i to r_0 can also be described as those γ which, as words, satisfy $\gamma = uv$ with $\mathrm{Std}(u) \in [\sigma, \omega(I)]$ and $\mathrm{Std}(v) \in [\tau, \omega(J)]$. These are exactly the standardizations of the words occurring in the product $N_\sigma N_\tau$. ∎

In particular, we obtain a description of the induction products of simple modules, which is much more precise than the one given by the product of quasi-symmetric functions:

Corollary 4.5 *Any induction product of simple modules $\mathbf{S}_{I_1} \hat{\otimes} \cdots \hat{\otimes} \mathbf{S}_{I_r}$ has a filtration by modules $\mathbf{N}_\sigma$, which can be explicitely computed.*

By using a standard result on self-injective algebras, we can now define another family of indecomposable modules. Indeed, for any self-injective Artin algebra A, and any exact sequence

$$0 \to N \to P \to M \to 0\,, \tag{93}$$

of left A-modules, with P projective, N is indecomposable non injective, and $N \to P$ an injective hull, iff M is indecomposable non projective, and $P \to M$ a projective cover (*cf.* [4]). It is customary to set $N = \Omega M$ and $M = \Omega^{-1} N$. Ω is called the syzygy functor (as defined here it is only a map on the set of modules, but it becomes a functor in the stable category; here it is well defined as a map because of the unicity of the minimal projective resolution).

Since the inclusion $\mathbf{N}_\sigma \to \mathbf{P}_I$ is clearly an injective hull, we have:

Lemma 4.6 *For $\sigma \in]\alpha(I), \omega(I)]$, $\mathbf{M}_\sigma = \mathbf{P}_I / \mathbf{N}_\sigma$ is indecomposable.*

Starting with M simple, next taking a projective cover of N, and iterating the process, one can construct a sequence of indecomposable modules $\Omega^n M$. In this way, one can see that for $n > 3$, $H_n(0)$ is not representation finite: the sequences $\Omega^n \mathbf{S}_I$ are neither finite nor periodic for $I \neq (n), (1^n)$, and $\dim_{\mathbb{C}} \Omega^n \mathbf{S}_I \to \infty$.

4.3 Homological properties of $H_n(0)$ for small n

Being a finite dimensional elementary $\mathbb{C}$-algebra, $H_n(0)$ can be presented in the form $\mathbb{C}Q/\mathcal{I}$, where Q is a quiver, $\mathbb{C}Q$ its path algebra, and $\mathcal{I}$ an ideal contained in $\mathcal{J}^2$ where $\mathcal{J}$ is the ideal generated by all the arrows of Q [2]. The vertices of Q are the simple modules $\mathbf{S}_I$, and the number e_{IJ} of arrows $\mathbf{S}_I \to \mathbf{S}_J$ is equal to $\dim \mathrm{Ext}^1(\mathbf{S}_I, \mathbf{S}_J) = [\mathrm{rad}\, \mathbf{P}_I / \mathrm{rad}^2\, \mathbf{P}_I : \mathbf{S}_J]$.

Therefore, $e_{IJ} = c_{IJ}^{(1)}$, where $c_{IJ}^{(k)} = [\mathrm{rad}^k\, \mathbf{P}_I / \mathrm{rad}^{k+1}\, \mathbf{P}_I : \mathbf{S}_J]$ are the coefficients of the q-Cartan invariants

$$c_{IJ}(q) = \sum_{k \geq 0} c_{IJ}^{(k)} q^k \tag{94}$$

associated to the radical series. Let $C_n(q) = (c_{IJ}(q))_{I,J\models n}$. For $n \leq 4$, these matrices are as follows.

	3	21	12	1111
3	1	0	0	0
21	0	1	q	0
12	0	q	1	0
111	0	0	0	1

	4	31	22	211	13	121	112	1111
4	1	0	0	0	0	0	0	0
31	0	1	q	0	q^2	0	0	0
22	0	q	$1+q^2$	0	q	q	0	0
211	0	0	0	1	0	q	q^2	0
13	0	q^2	q	0	1	0	0	0
121	0	0	q	q	0	$1+q^2$	q	0
112	0	0	0	q^2	0	q	1	0
1111	0	0	0	0	0	0	0	1

The corresponding quivers are given on Figures 2 and 3.

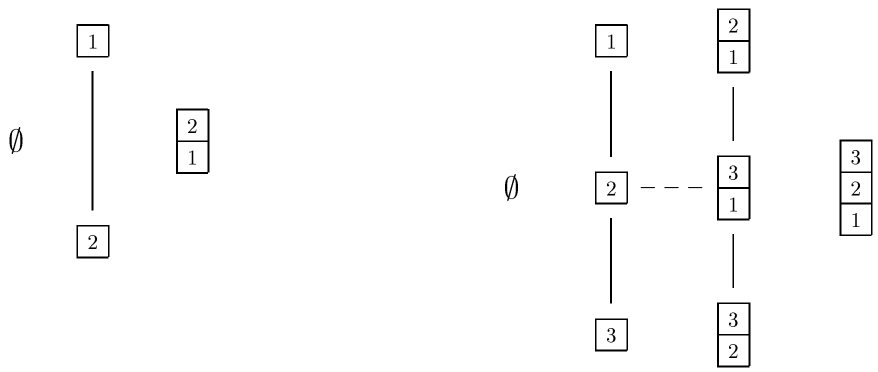


Figure 2: The quivers of $H_3(0)$ and $H_4(0)$.

The vertices of the quivers are labelled by descent sets, depicted as column shaped tableaux, instead of the corresponding compositions. This is to emphasize the curious fact that the subgraph on tableaux of a given height can be interpreted as the crystal graph of a fundamental representation of $\mathfrak{g}l_n$, or as the graph of the Bruhat order on the Schubert cells of a Grassmannian.

For $n \geq 3$, $H_n(0)$ has always three blocks, a large non trivial one, corresponding to the central connected component of the quiver, and two one-dimensional blocks, corresponding to the two simple projective modules $\mathbf{S}_n$ and $\mathbf{S}_{1^n}$. We denote by Γ_n the quiver of the non trivial block.

	5	41	32	311	23	221	212	2111	14	131	122	1211	113	1121	1112	11111
5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
41	0	1	q	0	q^2	0	0	0	q^3	0	0	0	0	0	0	0
32	0	q	$1+q^2$	0	q^4+q	q	0	0	q^2	q^2	q^3	0	0	0	0	0
311	0	0	0	1	0	q	q^2	0	0	q^2	q^3	0	q^4	0	0	0
23	0	q^2	q^4+q	0	$1+q^2$	q^3	0	0	q	q^2	q	0	0	0	0	0
221	0	0	q	q	q^3	$2\,q^2+1$	$q+q^3$	0	0	$q+q^3$	$q^4+2\,q^2$	q	q^3	q^3	0	0
212	0	0	0	q^2	0	$q+q^3$	$1+q^4$	0	0	q^2	$q+q^3$	q^2	q^2	q^2	0	0
2111	0	0	0	0	0	0	0	1	0	0	0	q	0	q^2	q^3	0
14	0	q^3	q^2	0	q	0	0	0	1	0	0	0	0	0	0	0
131	0	0	q^2	q^2	q^2	$q+q^3$	q^2	0	0	$1+q^4$	$q+q^3$	0	q^2	0	0	0
122	0	0	q^3	q^3	q	$q^4+2\,q^2$	$q+q^3$	0	0	$q+q^3$	$2\,q^2+1$	q^3	q	q	0	0
1211	0	0	0	0	0	q	q^2	q	0	0	q^3	$1+q^2$	0	q^4+q	q^2	0
113	0	0	0	q^4	0	q^3	q^2	0	0	q^2	q	0	1	0	0	0
1121	0	0	0	0	0	q^3	q^2	q^2	0	0	q	q^4+q	0	$1+q^2$	q	0
1112	0	0	0	0	0	0	0	q^3	0	0	0	q^2	0	q	1	0
11111	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 1: The q-Cartan matrix of $H_5(0)$

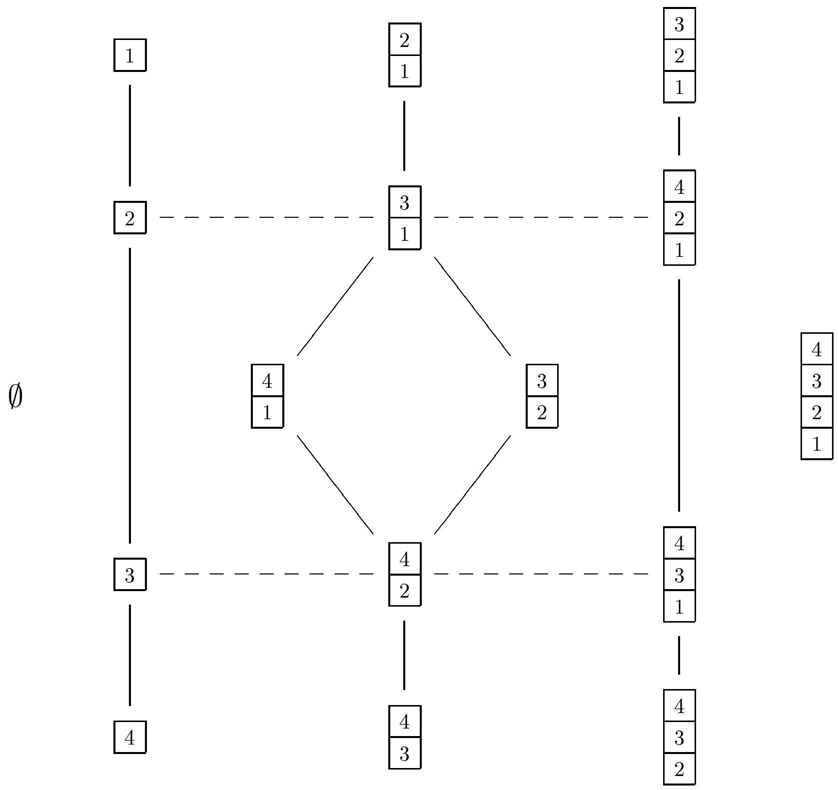
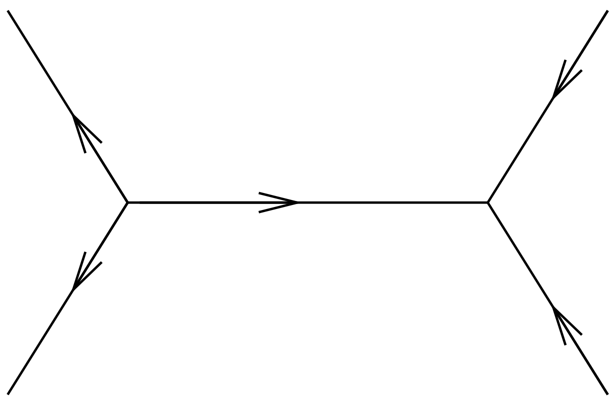


Figure 3: The quiver of $H_5(0)$.

For $n = 2$, $H_2(0) = \mathbb{C}\mathfrak{S}_2$ is semi-simple. For $n = 3$, Γ_3 is of type A_2. From the well-known representation theory of such quivers, we see that $H_3(0)$ has only 6 indecomposable modules: the 4 simple modules $\mathbf{S}_I$, $I \models 4$, and the two non-simple indecomposable projective modules $\mathbf{P}_{21}$ and $\mathbf{P}_{12}$.

For $n = 4$, Γ_4 is of type $\tilde{D}_5$. This allows us to conclude that $H_4(0)$ is not of finite representation type. Indeed, choosing an orientation of $\tilde{D}_5$ such that the corresponding path algebra is a quotient of $H_4(0)$, for example



(no path of length > 1), and according to a result of Kac [15], there is at least one indecomposable representation of dimension α for each positive root α, and there is an infinite number of them.

For $n \geq 5$, then Γ_n is considerably more complicated, and does not belong to any familiar class of quivers. Anyway, since $H_4(0)$ is a quotient of $H_n(0)$ for $n > 4$, all these algebras are of infinite representation type.

The quiver Γ_n can nevertheless be described for all n. Indeed, since the simple modules are one-dimensional, non trivial extensions

$$0 \to \mathbf{S}_J \to M \to \mathbf{S}_I \to 0$$

are in one-to-one correspondence with indecomposable two-dimensional modules M such that $\operatorname{soc} M = \mathbf{S}_J$ and $M/\operatorname{rad} M = \mathbf{S}_I$.

Let M be such a module, and denote by t_i the matrix of T_i in some basis $\{u, v\}$ of M. Then, M is decomposable if and only if all the t_i commute. If it is not the case, let i be the smallest integer such that t_i does not commute with t_{i+1}. The restriction of M to the subalgebra $H_3(0)$ generated by T_i and T_{i+1} is indecomposable, and must therefore be isomorphic to $\mathbf{P}_{21}$ or to $\mathbf{P}_{12}$. In both cases, it is possible to choose the basis such that the matrix of T_i be

$$t_i = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \tag{95}$$

and t_{i+1} is either

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

Next, t_{i+2} commutes with t_i and satisfies the braid relation with t_{i+1}. This implies that it is either scalar (with eigenvalue 0 or -1) or equal to t_i. For $j > 2$, t_j commutes with t_i and t_{i+1}, so it must be a scalar matrix, again with eigenvalue 0 or -1. Also, the matrices t_k for $k < i$ commute with t_{i+1} and have to be scalar.

From these considerations, one obtains a complete list of indecomposable two dimensional modules, and the following description of Γ_n

Theorem 4.7 *There is an arrow $A \to B$ between two subsets A, B of $\{1, \dots, n-1\}$ if and only if one of the two subsets is obtained from the other*

(1) by replacing an element i either by $i + 1$ or $i - 1$,

(2) by deleting i and inserting $i - 1$ and $i + 1$ if none of them were already present,

(3) by deleting a pair $i - 1$, $i + 1$ and inserting i, if it was not already there.

From this, it is easy to see that the total number of 2-dimensional indecomposable $H_n(0)$ modules is $(3n - 7) \cdot 2^{n-3}$ for $n \geq 3$.

4.4 Syzygies

A way to generate infinite families of non isomorphic indecomposable modules is to calculate the syzygies $\Omega^k \mathbf{S}_I$ and $\Omega^{-k}\mathbf{S}_I$ of the simple modules $\mathbf{S}_I$. The dimensions and composition factors of these modules can be read off from the q-Euler characteristics $\chi_q(\mathbf{S}_I, \mathbf{S}_J)$, where

$$\chi_q(M,N) = \sum_{k\geq 0} q^k \dim \operatorname{Ext}^k_{H_n(0)}(M,N) \tag{96}$$

Indeed, if

$$0 \leftarrow \mathbf{S}_I \leftarrow P^0 \leftarrow P^1 \leftarrow P^2 \leftarrow \cdots \leftarrow P^k \leftarrow \cdots \tag{97}$$

is a minimal projective resolution of $\mathbf{S}_I$, and if we write

$$P^k \simeq \bigoplus \mathbf{P}_J^{\oplus m^k_{IJ}} \tag{98}$$

then

$$\sum_{k\geq 0} m^k_{IJ} q^k = \chi_q(\mathbf{S}_I, \mathbf{S}_J)\,. \tag{99}$$

Also, since $\Omega^k\mathbf{S}_I = \ker(P^k \to P^{k-1})$, we have

$$\operatorname{ch}(\Omega^k \mathbf{S}_I) = \operatorname{ch}(P^{k-1}) - \operatorname{ch}(\Omega^{k-1}\mathbf{S}_I)\,. \tag{100}$$

Moreover, for $n \leq 4$, we have the more precise information

$$\operatorname{ch}_q(\Omega^k \mathbf{S}_I) = q^{-1}(\operatorname{ch}_q(P^{k-1}) - \operatorname{ch}_q(\Omega^{k-1}\mathbf{S}_I)) \tag{101}$$

where $\Omega^0\mathbf{S}_I = \mathbf{S}_I$ and $P^0 = \mathbf{P}_I$. This formula is equivalent to the following property. Let

$$A(q) = (a_{IJ}(q))_{I,J} = C(q)^{-1}\,. \tag{102}$$

Proposition 4.8 *For $n \leq 4$, the Poincaré series of $\operatorname{Ext}^*_{H_n(0)}(\mathbf{S}_I, \mathbf{S}_J)$ is given by*

$$\chi_q(\mathbf{S}_I, \mathbf{S}_J) = a_{IJ}(-q)\,. \tag{103}$$

Proof – For $n = 2$, this is trivial, and for $n = 3$ the direct calculation of both sides is straightforward. So let us suppose $n = 4$. As we have seen, $H_4(0)$ is a self injective algebra of infinite representation type, with radical cube 0, but radical square nonzero. Hence, we can apply Theorem 1.5 of [24], and conclude that $H_4(0)$ is a Koszul algebra. Also, we know that the ideal I such that $H_4(0) = \mathbb{C}Q/I$ is graded (it is generated by a set of words of lengths 2 or 3). Then, Theorem 5.6 of [13] implies the required equality. ∎

Example 4.9 For $n = 4$, the nontrivial part of the q-Cartan matrix, corresponding to the compositions $(31), (22), (211), (13), (121), (112)$ (in this order) is

$$C(q) = \begin{bmatrix} 1 & q & 0 & q^2 & 0 & 0 \\ q & 1+q^2 & 0 & q & q & 0 \\ 0 & 0 & 1 & 0 & q & q^2 \\ q^2 & q & 0 & 1 & 0 & 0 \\ 0 & q & q & 0 & 1+q^2 & q \\ 0 & 0 & q^2 & 0 & q & 1 \end{bmatrix}$$

and its inverse is

$$\frac{1}{(1-q^2)(1-q^6)}\begin{bmatrix} 1 & -q(1+q^4) & -q^3 & q^6 & q^2(1+q^2) & -q^3 \\ -q(1+q^4) & (1+q^2)(1+q^4) & q^2(1+q^2) & -q(1+q^4) & -q(1+q^2)^2 & q^2(1+q^2) \\ -q^3 & q^2(1+q^2) & 1 & -q^3 & -q(1+q^4) & q^6 \\ q^6 & -q(1+q^4) & -q^3 & 1 & q^2(1+q^2) & -q^3 \\ q^2(1+q^2) & -q(1+q^2)^2 & -q(1+q^4) & q^2(1+q^2) & (1+q^2)(1+q^4) & -q(1+q^4) \\ -q^3 & q^2(1+q^2) & q^6 & -q^3 & -q(1+q^4) & 1 \end{bmatrix}$$

By taking the Taylor expansions in the first row, one can read the minimal projective resolution of $\mathbf{S}_{31}$. The complex is naturally encoded by the noncommutative symmetric function $\mathcal{P}_q(\mathbf{S}_{31})$:

$$(1-q^2)^{-1}(1-q^6)^{-1}(R_{31}+q(1+q^4)R_{22}+q^3R_{211}+q^6R_{13}+q^2(1+q^2)R_{121}+q^3R_{112})$$

$$= R_{31}+qR_{22}+q^2(R_{31}+R_{121})+q^3(R_{22}+R_{211}+R_{112})+q^4(R_{31}+2R_{121})+O(q^5)$$

so that the beginning of the resolution is

$$0 \leftarrow \mathbf{S}_{31} \leftarrow \mathbf{P}_{31} \leftarrow \mathbf{P}_{22} \leftarrow \mathbf{P}_{31}\oplus\mathbf{P}_{121} \leftarrow \mathbf{P}_{22}\oplus\mathbf{P}_{211}\oplus\mathbf{P}_{112} \leftarrow \mathbf{P}_{31}\oplus 2\mathbf{P}_{121} \leftarrow \cdots$$

and the q-characteristics of the successive syzygy modules are $\mathrm{ch}_q(\Omega\mathbf{S}_{31}) = F_{22}+qF_{13}$, $\mathrm{ch}_q(\Omega^2\mathbf{S}_{31}) = F_{31}+F_{121}+qF_{22}$, $\mathrm{ch}_q(\Omega^3\mathbf{S}_{31}) = F_{22}+F_{211}+F_{112}+q(F_{13}+F_{121})$, $\mathrm{ch}_q(\Omega^4\mathbf{S}_{31}) = F_{31}+2F_{121}+q(F_{22}+F_{211}+F_{112})$, and so on. The dimensions of these modules are given by the generating function

$$\sum_{k\ge 0} q^k \dim \Omega^k\mathbf{S}_{31} = \frac{(1+q)(1+q^2)}{(1-q)(1-q^3)} = 1+2q+3q^2+5q^3+6q^4+7q^5+\cdots$$

Example 4.10 The minimal projective resolution of the Specht module $V_{22}(0)$ is encoded by the noncommutative symmetric function

$$\mathcal{P}_q(V_{22}(0)) = (1-q^6)^{-1}(R_{121}+qR_{112}+qR_{211}+q^2R_{121}+q^3R_{22}+q^4R_{13}+q^4R_{31}+q^5R_{22})$$

which has period 6, and whose commutative image tends to s_{22} for $q\to -1$. The Poincaré series of the Ext (Yoneda) algebra of $V_{22}(0)$ is

$$\chi_q(V_{22},V_{22}) = \frac{1+q^5}{1-q^6}\,.$$

5 Matrix quasi-symmetric functions

5.1 Definition

To define our next generalization, we start from a totally ordered set of commutative variables $X=\{x_1<\cdots<x_n\}$ and consider the ideal $\mathbb{C}[X]^+$ of polynomials without constant term. We denote by $\mathbb{C}\{X\} = T(\mathbb{C}[X]^+)$ its tensor algebra. The product of this algebra will be denoted by μ.

In the sequel, we will consider tensor products of elements of this algebra. To avoid confusion, we denote by "$\cdot$" the tensor product of the tensor algebra and call it the dot product. We reserve the notation $\otimes$ for the external tensor product. The reader should keep in mind that, in an expression of the form $\mathbf{m} = m_1 \cdot m_2 \cdots m_k$, none of the m_i are constant monomials. Such a product is said to be in *normal form*. Otherwise we rather write $\mathbf{m} = \mu(m_1, m_2, \cdots, m_k)$.

A natural basis of $\mathbb{C}\{X\}$ is formed by dot products of monomials (called *multiwords* in the sequel), which can be represented by nonnegative integer matrices $M = (m_{ij})$, where m_{ij} is the exponent of the variable x_i in the jth factor of the tensor product. Since constant monomials are not allowed, such matrices have no zero column. We say that they are *horizontally packed*. A multiword $\mathbf{m}$ can be conveniently encoded in the following way. Let A be the *support* of $\mathbf{m}$, that is, the set of those variables x_i such that the ith row of M is non zero, and let P be the matrix obtained form M by removing the null rows. We set $\mathbf{m} = A^P$. A matrix such as P, without zero rows or columns, is said to be *packed*.

For example the multiword $\mathbf{m} = a \cdot ab^3e^5 \cdot a^2d$ is encoded by $\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array}\begin{bmatrix} 1&1&2 \\ 0&3&0 \\ 0&0&0 \\ 0&0&1 \\ 0&5&0 \end{bmatrix}$. Its support is the set $\{a, b, d, e\}$, and the associated packed matrix is $\begin{bmatrix} 1&1&2 \\ 0&3&0 \\ 0&0&1 \\ 0&5&0 \end{bmatrix}$.

Let $\mathbf{MQSym}(X)$ be the linear subspace of $\mathbb{C}\{X\}$ spanned by the elements

$$\mathbf{MS}_M = \sum_{A \in \mathcal{P}_k(X)} A^M \tag{104}$$

where $\mathcal{P}_k(X)$ is the set of k-element subsets of X, and M runs over packed matrices of height $h(m) < n$.

For example, on the alphabet $\{a < b < c < d\}$

$$\mathbf{MS}_{\begin{bmatrix} 1&1&2 \\ 0&3&0 \\ 0&0&1 \end{bmatrix}} = \begin{array}{c} a \\ b \\ c \\ d \end{array}\begin{bmatrix} 1&1&2 \\ 0&3&0 \\ 0&0&1 \\ 0&0&0 \end{bmatrix} + \begin{array}{c} a \\ b \\ c \\ d \end{array}\begin{bmatrix} 1&1&2 \\ 0&3&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix} + \begin{array}{c} a \\ b \\ c \\ d \end{array}\begin{bmatrix} 1&1&2 \\ 0&0&0 \\ 0&3&0 \\ 0&0&1 \end{bmatrix} + \begin{array}{c} a \\ b \\ c \\ d \end{array}\begin{bmatrix} 0&0&0 \\ 1&1&2 \\ 0&3&0 \\ 0&0&1 \end{bmatrix}$$

Proposition 5.1 $\mathbf{MQSym}$ *is a subalgebra of* $\mathbb{C}\{X\}$. *Actually,*

$$\mathbf{MS}_P \mathbf{MS}_Q = \sum_{R \in \sqcup\!\sqcup(P,Q)} \mathbf{MS}_R$$

where the augmented shuffle *of P and Q, $\sqcup\!\sqcup(P, Q)$ is defined as follows: let r be an integer between* $\max(p, q)$ *and* $p+q$, *where $p = h(P)$ and $q = h(Q)$. Insert null rows in the matrices P and Q so as to form matrices $\tilde{P}$ and $\tilde{Q}$ of height r. Let R be the matrix $(\tilde{P}, \tilde{Q})$. The set $\sqcup\!\sqcup(P, Q)$ is formed by all the matrices without null rows obtained in this way.*

For example :

$$\mathbf{MS}_{\begin{bmatrix} 2&1 \\ 1&0 \end{bmatrix}} \mathbf{MS}_{\begin{bmatrix} 3&1 \end{bmatrix}} =$$

$$\mathbf{MS}_{\begin{bmatrix} 2&1&0&0 \\ 1&0&0&0 \\ 0&0&3&1 \end{bmatrix}} + \mathbf{MS}_{\begin{bmatrix} 2&1&0&0 \\ 1&0&3&1 \end{bmatrix}} + \mathbf{MS}_{\begin{bmatrix} 2&1&0&0 \\ 0&0&3&1 \\ 1&0&0&0 \end{bmatrix}} + \mathbf{MS}_{\begin{bmatrix} 2&1&3&1 \\ 1&0&0&0 \end{bmatrix}} + \mathbf{MS}_{\begin{bmatrix} 0&0&3&1 \\ 2&1&0&0 \\ 1&0&0&0 \end{bmatrix}}$$

Let us endow $\mathbf{MQSym}$ with a Hopf algebra structure. Let $Y = \{y_1 < \cdots < y_n\}$ be a second totally ordered set of variables, of the same cardinality as X. We identify the tensor product $\mathbf{MQSym}(X) \otimes \mathbf{MQSym}(X)$ with $\mathbf{MQSym}(X \oplus Y)$, where $X \oplus Y$ denotes the ordered sum of X and Y. The natural embedding

$$\Delta \,:\, \mathbf{MQSym}(X) \longrightarrow \mathbf{MQSym}(X \oplus Y) \simeq \mathbf{MQSym}(X) \otimes \mathbf{MQSym}(X) \qquad (105)$$

defined by $\Delta(\mathbf{MS}_M(X)) = \mathbf{MS}_M(X \oplus Y)$ can be interpreted as a comultiplication.

For example

$$\Delta\left(\mathbf{MS}\begin{bmatrix}1&0&3\\0&2&1\\0&0&3\\1&0&2\end{bmatrix}\right) = 1 \otimes \mathbf{MS}\begin{bmatrix}1&0&3\\0&2&1\\0&0&3\\1&0&2\end{bmatrix} + \mathbf{MS}_{[1\ 3]} \otimes \mathbf{MS}\begin{bmatrix}0&2&1\\0&0&3\\1&0&2\end{bmatrix} + \mathbf{MS}\begin{bmatrix}1&0&3\\0&2&1\end{bmatrix} \otimes \mathbf{MS}\begin{bmatrix}0&3\\1&2\end{bmatrix}$$
$$+ \mathbf{MS}\begin{bmatrix}1&0&3\\0&2&1\\0&0&3\end{bmatrix} \otimes \mathbf{MS}_{[1\ 2]} + \mathbf{MS}\begin{bmatrix}1&0&3\\0&2&1\\0&0&3\\1&0&2\end{bmatrix} \otimes 1$$

From now on, unless otherwise stated, we suppose that X is infinite.

Let $\mu \;:\; f \otimes g \mapsto fg$ be the multiplication of $\mathbf{MQSym}$ (induced by the multiplication of the tensor algebra), and let e be the restriction to $\mathbf{MQSym}$ of the augmentation of $T(\mathbb{C}[X]^+)$. Introduce a grading by setting $\deg(\mathbf{MS}_M) = \sum m_{ij}$ and denote by $\mathbf{MQSym}_d$ the homogeneous component of degree d.

Proposition 5.2 $(\mathbf{MQSym}, \mu, 1, \Delta, e)$ *is a self dual graded bialgebra, the duality pairing being given by* $\langle \mathbf{MS}_P, \mathbf{MS}_Q \rangle = \delta_{P,{}^tQ}$.

The Hilbert series of $\mathbf{MQSym}$ can be expressed directly or in terms of scalar products of ribbon Schur functions. One has

$$\dim(\mathbf{MQSym}_d) = \sum_{l>0,\ h>0} \binom{d+lh-1}{lh-1} \left(\frac{1}{2}\right)^{l+h+2} = \sum_{|I|=|J|=d} 2^{2d-\ell(I)-\ell(J)} \langle r_I, r_J \rangle \qquad (106)$$

which yields

$$\sum_{d\geq 0} \dim(\mathbf{MQSym}_d)\, t^d$$
$$= 1 + t + 5t^2 + 33t^3 + 281t^4 + 2961t^5 + 37277t^6 + 546193t^7 + 9132865t^8 + \cdots$$

To a packed matrix P, we can associate two compositions $I = \mathrm{Row}(P)$ and $J = \mathrm{Col}(P)$ formed by the row-sums and column-sums of P. The Hilbert series in an easy consequence of the classical fact that packed matrices of degree d are in bijection with double cosets of $\mathfrak{S}_J \backslash \mathfrak{S}_d / \mathfrak{S}_I$ where $\mathfrak{S}_{(i_1,i_2,\ldots,i_k)}$ is the Young subgroup $\mathfrak{S}_{i_1} \times \mathfrak{S}_{i_2} \times \cdots \times \mathfrak{S}_{i_k}$. It is well known (*cf.* [12]) that their number is $2^{2d-\ell(I)-\ell(J)} \langle r_I, r_J \rangle$.

Let Ev be the linear map defined by

$$\begin{array}{rccl} \mathrm{Ev} \;:\; & \mathbf{MQSym} & \longrightarrow & QSym \\ & \mathbf{MS}_P & \longmapsto & \mathrm{Ev}(\mathbf{MS}_P) = M_{\mathrm{Row}(P)} \end{array} \qquad (107)$$

Proposition 5.3 Ev *is an epimorphism of bialgebras. Dually, the transposed map*

$$\begin{array}{rccl} {}^t\mathrm{Ev} : & \mathbf{Sym} & \to & \mathbf{MQSym} \\ & \mathbf{MS}_P & \longmapsto & {}^t\mathrm{Ev}(S^I) = \sum_{\mathrm{Col}(P)=I} \mathbf{MS}_P \end{array} \tag{108}$$

is a monomorphism of bialgebras.

Therefore, $\mathbf{MQSym}$ admits $QSym$ as a quotient and $\mathbf{Sym}$ as a subalgebra. The basis $\mathbf{MS}_P$ can be regarded as a simultaneous generalization of the dual bases M_I and S^I. Moreover, $\mathbb{C}\langle X\rangle$ is naturally a subalgebra of $\mathbb{C}\{X\}$, words being identified with multiwords with exponent matrix having only one 1 in each column. It is clear that this embedding maps $\mathbf{FQSym}$ to a subspace of $\mathbf{MQSym}$.

5.2 Algebraic structure

We now elucidate the structure of $\mathbf{MQSym}$ as an algebra. To describe a generating family, we need the following definitions. Let P be a packed matrix of height h. To a composition $K = (k_1, \ldots, k_p)$ of h, we associate the matrix $P\rangle K$ defined as follows. Let $R_1, R_2, \ldots, R_h$ be the rows of P. The first row of $P\rangle K$ is the sum of the first k_1 rows of P, the second row is the sum of the next k_2 rows of P, and so on. We end therefore with a matrix of height p. For example,

$$\left.\begin{bmatrix} 1&2&0&2\\ 0&1&2&1\\ 1&2&0&0\\ 0&3&1&5\\ 1&3&1&0 \end{bmatrix}\right\rangle (3,2) = \begin{bmatrix} 2&5&2&3\\ 1&6&2&5 \end{bmatrix}.$$

Generalizing the idea of [23], we set

$$\phi S_P = \sum_{|K|=h} \frac{1}{K!} \mathbf{MS}_{P\,K} \tag{109}$$

where $K! = k_1!k_2!\cdots k_p!$. The family $\{\phi S_P\}$, where P runs over packed matrices, is a homogeneous basis of $\mathbf{MQSym}$.

Let us say that a packed matrix A is *connected* if it cannot be written in block diagonal form

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

where B and C are not necessarily square matrices.

Theorem 5.4 $\mathbf{MQSym}$ *is freely generated by the family* $\{\phi S_A\}$*, where* A *runs over the set of connected packed matrices.*

5.3 Convolution

The goal of this subsection is to find an interpretation of $\mathbf{MQSym}$ in terms of invariant theory. The first part of the construction applies to any Hopf algebra.

Hopf Algebra background

First, let $(\mathcal{H}, 1, \mu, \delta, \epsilon, \alpha)$ be a graded Hopf algebra. One can define a bialgebra structure on the augmentation ideal $T(\mathcal{H}^+)$. The coproduct is $\mathbf{c}$ is defined as follows. Let $\mathbf{m} = m_1 \cdot m_2 \cdots m_p$ be a normal form dot product ($m_i \in \mathcal{H}^+$). Let $\delta(m_i) = \sum m_i' \otimes m_i''$. Then, one sets

$$\mathbf{c}(\mathbf{m}) = \sum \mu\left(m_1', m_2', \cdots, m_p'\right) \otimes \mu\left(m_1'', m_2'', \cdots, m_p''\right) . \tag{110}$$

For example, with $\mathcal{H} = \mathbb{C}[x, y]$, one has

$$\mathbf{c}(x^2 \cdot y) = x^2 \cdot y \otimes 1 + 2x \cdot y \otimes x + y \otimes x^2 + x^2 \otimes y + 2x \otimes x \cdot y + 1 \otimes x^2 \cdot y\,.$$

If δ is cocommutative, then $\mathbf{c}$ is obviously so. The co-unit is the coordinate of the empty tensor

$$\mathbf{e}(1) = 1 \qquad \text{and} \qquad \mathbf{e}(m_1 \cdot m_2 \cdots m_p) = 0 \text{ if } p > 0 \text{ and } m_i \in \mathcal{H}^+\,. \tag{111}$$

If $\mathcal{H}$ is graded, one defines a gradation on $T(\mathcal{H}^+)$ by

$$\deg(m_1 \cdot m_2 \cdots m_p) = \deg(m_1) + \deg(m_2) + \cdots + \deg(m_p)\,, \tag{112}$$

Note that this gradation differs from the standard one on tensors, which we will call length

$$\ell(m_1 \cdot m_2 \cdots m_p) = p\,. \tag{113}$$

Now, f and g being two endomorphisms of $\mathcal{H}$, the convolution of f and g is defined by $f * g = \mu \circ (f \otimes g) \circ \delta$. In the sequel, all the endomorphisms will be homogeneous. The convolution of an endomorphism of degree p with an endomorphism of degree q is of degree $p+q$. One denotes by $\mathrm{Convol}(\mathcal{H})$ the *convolution algebra of the homogeneous endomorphisms of* $\mathcal{H}$ and by $\mathrm{End}^n(\mathcal{H})$ the vector space of *homogeneous endomorphisms of degree* n.

Operator associated with a packed matrix

To each packed matrix A of total sum n we associate a canonical endomorphism f_A of the n^{th} homogeneous component of $T(\mathcal{H}^+)$. First of all, let $K = (k_1, \ldots, k_q) \in \mathbf{N}^q$. Let us define

$$\begin{aligned} \delta^{(K)} \ &: \mathcal{H} \longrightarrow \mathcal{H}^{\otimes q} \\ &\qquad m \longmapsto (\pi_{k_1} \otimes \cdots \otimes \pi_{k_q}) \circ \delta^q(m) \end{aligned} \tag{114}$$

where π_d is the projector on the homogeneous component of degree d of $\mathcal{H}$. Thus $\delta^{(K)}$ takes an element of degree $|K|$ and sends it to an element of $\mathcal{H}^{\otimes q}$ of degree $(k_1, \ldots, k_q)$, killing all components of other degrees.

Let $A = (a_{i,j})$ be a packed $p \times q$ matrix of total sum n. The row sum of A is a composition $r = (r_1, \ldots, r_p)$ and the column sum is $c = (c_1, \ldots, c_q)$. Let us denote by $R_1, \ldots, R_p$ the rows of A. Finally suppose that $\mathbf{m} = m_1 \cdot m_2 \cdots m_r$ is an element of $T(\mathcal{H}^+)$. Then we define f_A by

$$f_A(m_1 \cdot m_2 \cdots m_r) = \begin{cases} \mu_q^p(\delta^{L_1}(m_1), \cdots, \delta^{L_p}(m_p)) & \text{if } r = p, \\ 0 & \text{otherwise,} \end{cases} \tag{115}$$

where $\mu_q^p = (\mu^p)^{\otimes q}$ is the product of the p tensor $\delta^{L_i}(m_i)$ of length q. Thus we get an element of $(\mathcal{H}^+)^{\otimes q}$ of degree $(c_1, \ldots, c_q)$. Remark that $f_A(\mathbf{m})$ is null unless m is of degree $l = (l_1, \ldots, l_p)$.

Example 5.5 let $A = \left[\begin{smallmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \end{smallmatrix}\right]$. The associated morphism f_A kills all tensors of degree different from $(3, 5)$. Let $\mathbf{m} = abc \cdot a^4 b$. Then

$$\begin{aligned} \delta^{(2,0,1)}(abc) &= ab \otimes 1 \otimes c + ac \otimes 1 \otimes b + bc \otimes 1 \otimes a, \\ \delta^{(0,2,3)}(a^4 b) &= \binom{4}{2}(1 \otimes a^2 \otimes a^2 b) + \binom{4}{1}(1 \otimes ab \otimes a^3)\,. \end{aligned}$$

Finally,

$$\begin{aligned} F_A(abc \cdot a^4 b) = 6\,(\,&ab \cdot a^2 \cdot a^2 bc + ac \cdot a^2 \cdot a^2 b^2 + bc \cdot a^2 \cdot a^3 b\,) \\ +4\,(\,&ab \cdot ab \cdot a^3 c \,+\, ac \cdot ab \cdot a^3 b \,+\, bc \cdot ab \cdot a^4\,)\,. \end{aligned}$$

The following example, which is some sense generic, is of crucial importance.

Example 5.6 Let $A = (a_{i,j})$ of size $p \times q$ and degree (total sum) n. Let us consider

$$\mathbb{K}\{X\} = T(\mathbb{K}[x_1, x_2, \ldots, x_n]^+)\,. \tag{116}$$

To the composition $r = (r_1, \ldots, r_p)$ of the row sum of A, we associate the generic multiword of degree r denoted by $\mathbf{m}_{(r)}$ and defined as follows: let $d_1 = r_1,\; d_2 = r_1 + r_2,\; \ldots,\; d_i = r_1 + \cdots + r_i,\; \ldots$, and $d_p = n$ the descents of r. Define

$$\mathbf{m}_{(r)} = \left(\prod_{i=1}^{d_1} x_i\right) \cdot \left(\prod_{i=d_1+1}^{d_2} x_i\right) \cdots \left(\prod_{i=d_{p-1}+1}^{d_p} x_i\right)\,, \tag{117}$$

or, equivalently,

$$\mathbf{m}_{(r)} = x_1 \diamond_1 x_2 \diamond_2 x_3 \diamond_3 \cdots \diamond_{n-1} x_n, \tag{118}$$

where $\diamond_i$ is the commutative multiplication if i is not a descent of r, and the dot product otherwise. Let $X_D = \prod_{i \in D} x_i$ where D is a subset of $\{1 \ldots n\}$ and moreover, let D_i denote the integer interval $\{d_{i-1} + 1, \ldots, d_i\}$. Then

$$\mathbf{m}_{(r)} = X_{D_1} \cdot X_{D_2} \cdots X_{D_p}\,. \tag{119}$$

Let us compute the image of $\mathbf{m}_{(r)}$ by f_A. For all $K = (k_1, \ldots, k_p) \in \mathbf{N}^q$ of sum s one has

$$\delta^{(K)}(X_{\{u,\ldots,u+s\}}) = \sum_{I_1,\ldots,I_q} (X_{I_1} \otimes \cdots \otimes X_{I_q})\,, \tag{120}$$

where the sum is over all set-partitions $I_1, \ldots, I_q$ of the integer interval $\{u, \ldots, u+s\}$ such that $\#(I_1) = k_1, \ldots, \#(I_q) = k_q$. It follows that

$$f_A(\mathbf{m}_{(r)}) = \sum_{(I_{i,j})} \left(X_{\bigcup I_{i,1}} \otimes \cdots \otimes X_{\bigcup I_{i,q}}\right)\,, \tag{121}$$

the sum is over all $p \times q$-matrices $(I_{i,j})$ whose entries are subsets of $\{1, \ldots n\}$ and such that

- for all i,j, one has $\#(I_{i,j}) = a_{i,j}$,
- for all i the set $\{I_{i,1},\dots,I_{i,q}\}$ defines a partition of the interval $D_i = [d_{i-1}+1,\dots,d_i]$.

For example, with $A = \left[\begin{smallmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{smallmatrix}\right]$, one has $r = (2,3)$. Then the generic multiword $\mathbf{m}_{(r)}$ reads

$$\mathbf{m}_{(2,3)} = x_1x_2 \cdot x_3x_4x_5\,. \tag{122}$$

Then

$$\begin{array}{rclcrcl} \delta^{(0,1,1)}(x_1x_2) & = & 1\otimes x_1\otimes x_2 & + & 1\otimes x_2\otimes x_1, & & \\ \delta^{(1,0,2)}(x_3x_4x_5) & = & x_3\otimes 1\otimes x_4x_5 & + & x_4\otimes 1\otimes x_3x_5 & + & x_5\otimes 1\otimes x_3x_4\,. \end{array}$$

Finally

$$\begin{array}{rclcrcl} F_A(x_1x_2\cdot x_3x_4x_5) & = & x_3\cdot x_1\cdot x_2x_4x_5 & + & x_4\cdot x_1\cdot x_2x_3x_5 & + & x_5\cdot x_1\cdot x_2x_3x_4 \\ & + & x_3\cdot x_2\cdot x_1x_4x_5 & + & x_4\cdot x_2\cdot x_1x_3x_5 & + & x_5\cdot x_2\cdot x_1x_3x_4\,. \end{array}$$

Theorem 5.7 *The map*

$$\begin{aligned} \mathbf{MQSym} &\longrightarrow \mathrm{Convol}(T(\mathcal{H}^+)) \\ \mathbf{MS}_A &\longmapsto f_A \end{aligned} \tag{123}$$

is a homomorphism of algebras.

Proof – The first step of the proof is to see that the definition of the morphism f_A can be extended to non-packed matrices. If B is an integer $p\times q$ matrix the preceding definition gives a morphism

$$\tilde{f}_B \;:\; \mathcal{H}^p \longrightarrow T(\mathcal{H}^+)\,, \tag{124}$$

With this notation, one has the following easy lemma:

Lemma 5.8 *Let A be a packed matrix of height h. Let*

$$\mathbf{m} = \mu(m_1, m_2, \dots, m_p)$$

a tensor, not necessarily in normal form (m_i can be constant). Then,

$$f_A(\mathbf{m}) = \sum_{B\in(A\updownarrow_p)} \tilde{f}_B(m_1, m_2, \dots, m_p)\,, \tag{125}$$

where B runs over the set $(A\updownarrow_p)$ of matrices of height p obtained by inserting 0 rows in the matrix A.

Now, let $\mathbf{m} = m_1\cdot m_2\cdots m_p$. Suppose $\delta(m_i) = \sum m_i'\otimes m_i''$. By definition

$$\mathbf{c}(\mathbf{m}) = \sum \mu\left(m_1', m_2', \cdots, m_p'\right)\otimes\mu\left(m_1'', m_2'', \cdots, m_p''\right)\,. \tag{126}$$

And therefore, if A and A' are two packed matrices,

$$(f_A\otimes f_{A'})\circ\mathbf{c}(\mathbf{m}) = \sum_{B\in(A\updownarrow_p),\ B'\in(A'\updownarrow_p)} \tilde{f}_B\left(m_1',\dots,m_p'\right)\otimes\tilde{f}_{B'}\left(m_1'',\dots,m_p''\right)\,, \tag{127}$$

which gives

$$\mu \circ (f_A \otimes f_{A'}) \circ \mathbf{c}(\mathbf{m}) = \sum_{B \in (A \updownarrow_p),\ B' \in (A' \updownarrow_p)} \tilde{f}_{BB'}\,(m_1, \ldots, m_p)\ , \tag{128}$$

where BB' is the concatenation of B and B'. This is exactly the set of unpackings $(C \updownarrow_p)$ of the matrices C appearing in the product $\mathbf{MS}_A\mathbf{MS}_{A'}$ which acts non-trivialy on $\mathbf{m}$. ■

Note that the theorem is true even if $\mathcal{H}$ is not cocommutative.

Interpretation

First, we reformulate the definition of $\mathbb{C}\{X\}$ (with $X = \{x_1, \ldots, x_n\}$) in a slightly more abstract way. Let V be an n-dimensional vector space with basis X. The polynomials in X can be seen as the symmetric algebra of V. The graded bialgebra structure on $\mathbb{C}\{X\} = T(\mathbb{C}[X]^+)$ gives a structure on $T(S^+(V))$. Moreover, since the definition of the operations does not depend on the basis, this structure is canonical. Let ρ be the natural representation of $GL(V)$ in $\mathrm{End}(T(S^+(V)))$.

Theorem 5.9 *There exists a canonical homomorphism*

$$\phi\ :\ \mathbf{MQSym} \longrightarrow \mathrm{End}(T(S^+(V)))$$

from $\mathbf{MQSym}$ *to* $\mathrm{End}(T(S^+(V)))$ *regarded as a convolution algebra, such that for all* d, $\phi(\mathbf{MQSym}_d)$ *is the commutant* $\mathrm{End}_{GL(V)}(T(S^+(V))_d)$ *of* $\rho(GL(V))$ *in the homogeneous component of degree* d *of* $\mathrm{End}(T(S^+(V)))$. *Moreover,* ϕ *is one-to-one for* $d \leq n$.

Proof – The endomorphism f_A associated with a matrix A is defined by means of the product, coproduct, and the homogeneous projector of $T(S^+(V))$. But all these operations commute with the action of $GL(U)$. Then f_A commutes with $GL(U)$.

We will prove the theorem in two steps:

- In the first step, we suppose that the dimension N of V is greater than n. We will prove that ϕ is one-to-one and, by an argument of dimension, we get that $\phi(\mathbf{MQSym}_d)$ is exactly the commutant $\mathrm{End}_{GL(V)}(T(S^+(V))_d)$.
- Then by a restriction argument we will conclude in every case.

Let us choose a basis $X = \{x_1, \ldots x_n\}$ of V. In example 5.6, we have computed the image of the generic multiword $\mathbf{m}_{(r)}$ by f_A where A is a matrix of row sum r. Notice that n is sufficient to express $\mathbf{m}_{(r)}$ since N is bigger than the sum n of A.

Let us recall some notation: $d_1, \ldots, d_p$ denote the descents of r and D_i the integer interval $[d_{i-1}+1, \ldots, d_i]$. Let us suppose that $\mathbf{m}'$ is a multiword of the form

$$\mathbf{m}' = X_{I_1} \cdot X_{I_2} \cdots X_{I_q}. \tag{129}$$

where $I_1, \dots, I_q$ is a partition of the set $\{1, \dots, n\}$. There exists only one matrix A such that $\mathbf{m}'$ appears in the image of $\mathbf{m}_{(r)}$ by f_A:

$$A = \begin{bmatrix} \#(I_1 \cap D_1) & \cdots & \#(I_q \cap D_1) \\ \vdots & \ddots & \vdots \\ \#(I_1 \cap D_p) & \cdots & \#(I_q \cap D_p) \end{bmatrix}. \tag{130}$$

This proves the injectivity of ϕ.

Now we will show that the dimension of $\mathbf{MQSym}^n$ and of the commutant of $GL(U)$ in $\mathrm{End}^n(T(S^+(U)))$ are equal. Let us compute the graded character of the representation $GL(U)$ on $T(S^+(U))$. It is well known that the character of $S^d(U)$ is the Schur function $s_{(d)}$ which is equal to the complete function h_d.

The graded character of $S^+(U)$ is then:

$$\mathrm{ch}_t(S^+(U)) = \sum_{d>0} h_d t^d . \tag{131}$$

Therefore

$$\mathrm{ch}_t(T(S^+(U))) = \sum_I h_I t^{|I|} , \tag{132}$$

where I runs over the set of all compositions. Note that h_I only depends on the partition associated with I. Then one uses the classical identity

$$\sum_I h_I u^{\ell(I)} = \sum_J r_J u^{\ell(J)} (1+u)^{|J|-\ell(J)}. \tag{133}$$

Now, we extract the homogeneous components, with $u = 1$. This gives

$$\mathrm{ch}_t(T(S^+(U))) = \sum_J r_J 2^{d-\ell(J)} t^{|J|} \tag{134}$$

The multiplicity of the irreducible representation χ_λ of GL_N in the homogeneous component of degree d of $T(S^+(U))$ is therefore given by the scalar product

$$\sum_{\lambda \vdash n} \sum_{I \vDash n,\ J \vDash n} 2^{2n-\ell(I)-\ell(J)} \langle r_I, s_\lambda \rangle \langle s_\lambda, r_J \rangle . \tag{135}$$

The sum is extended to partitions all λ of length smaller than the dimension N of U. Thus if $N \geq n$, all the Schur functions s_λ appear. Moreover, since they form an orthonormal basis of Sym

$$\sum_{\lambda \vdash n} \langle r_I, s_\lambda \rangle \langle s_\lambda, r_J \rangle = \langle r_I, r_J \rangle . \tag{136}$$

This proves that the dimension of the commutant of $GL(U)$ in the n-homogeneous space of $T(S^+(U))$ is equal to the dimension of $\mathbf{MQSym}_n$, which implies the first part of the theorem.

Now we are in the case where the dimension N of the vector space U is less than n. Let $V = U \oplus W$ be of dimension n.

Lemma 5.10 *Let $U \subset V$ be two vector spaces. Then the restriction*

$$\begin{aligned} \textit{Rest}_{U\subset V}\ : \mathrm{End}_{GL(V)}(T(S^+(V))_n) &\longrightarrow \mathrm{End}_{GL(U)}(T(S^+(U))_n) \\ f &\longmapsto f_{T(S^+(U))_n} \end{aligned}$$

is surjective.

Using this lemma one has that every element of $\mathrm{End}_{GL(U)}(T(S^+(U))_n)$ is the restriction of some element of $\rho_V(\mathbf{MQSym})$. But it is clear that the endomorphism F_A^U associated with a matrix A on U is the restriction to $T(S^+(U))$ of F_A^V associated with A on V. This concludes the proof of the theorem.

It remains to prove the lemma. First we have to prove that the image of an element of $T(S^+(U))$ by f is still in $T(S^+(U))$.

Let us set $\mathrm{Comm}_U = \mathrm{End}_{GL(U)}(T(S^+(U)))$. Let $\mathbf{m}$ be an element of $T(S^+(V))$. Let $\mathrm{Vect}(\mathbf{m})$ be the smallest subspace $W \subset V$ such that $\mathbf{m} \in T(S^+(W))$. Then clearly if $g \in GL(V)$ then $\mathrm{Vect}(g(\mathbf{m})) = g(\mathrm{Vect}(\mathbf{m}))$. But if f commutes with GL_V, it also commutes with the projectors on $\mathrm{Vect}(\mathbf{m})$, so that

$$\text{for all } f \in \mathrm{Comm}_V, \quad f(\mathbf{m}) \in T(S^+(\mathrm{Vect}(\mathbf{m})))\,. \tag{137}$$

Now, let us prove the surjectivity. Let $g \in \mathrm{Comm}_U$. Let $\mathbf{m} \in T(S^+(V))$. Under the assumption $\dim(\mathrm{Vect}(\mathbf{m})) \geq \dim(u)$, one can define the image $f(\mathbf{m})$ by conjugation as follows: choose an injective morphism $h_{\mathbf{m}} : \mathrm{Vect}(\mathbf{m}) \mapsto U$. Then obviously $h_{\mathbf{m}}(\mathrm{Vect}(\mathbf{m})) = \mathrm{Vect}(h_{\mathbf{m}}(\mathbf{m}))$ and one can set

$$f(\mathbf{m}) = \begin{cases} h_{\mathbf{m}}^{-1} \circ g \circ h_{\mathbf{m}}(\mathbf{m}) & \text{if } \dim(V(\mathbf{m})) \leq u, \\ 0 & \text{otherwise.} \end{cases} \tag{138}$$

Since g commutes with $GL(U)$, the vector $f(\mathbf{m}) = h_{\mathbf{m}}^{-1} \circ g \circ h_{\mathbf{m}}(\mathbf{m})$ does not depend on the choice of $h_{\mathbf{m}}$. Hence if $\mathbf{m} \in T(S^+(U))$, one can take $h_{\mathbf{m}} = id_u$ and thus $f_U = g$. Moreover it is easy to see that f commutes with $GL(V)$. ∎

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