The Algebra of Binary Search Trees

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Abstract

We introduce a monoid structure on the set of binary search trees, by a process very similar to the construction of the plactic monoid, the Robinson-Schensted insertion being replaced by the binary search tree insertion. This leads to a new construction of the algebra of Planar Binary Trees of Loday-Ronco, defining it in the same way as Non-Commutative Symmetric Functions and Free Symmetric Functions. We briefly explain how the main known properties of the Loday-Ronco algebra can be described and proved with this combinatorial point of view, and then discuss it from a representation theoretical point of view, which in turns leads to new combinatorial properties of binary trees.

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1 Introduction

There are certain analogies between the combinatorics of binary trees and that of Young tableaux. For example, the decreasing labelings of a given binary tree and the standard Young tableaux of a given shape are both counted by a "hook-length formula" (see [9,34,16,27]), both admitting natural q-analogs (see [34,3]). It is also known that one can construct from the Robinson-Schensted correspondence a Hopf algebra **FSym** whose basis is given by standard Young tableaux [29]. Moreover, a natural realization of this algebra by means of non-commutative polynomials gives a enlightening proof of the Littlewood-Richardson rule (see [20,5,6]). In this realization, each tableau t of shape λ is interpreted as a homogeneous polynomial of degree $n = |\lambda|$, whose commutative image is the Schur function s_λ.

Recently, Loday and Ronco have introduced a Hopf algebra whose basis is the set of planar binary trees (see [23,24]). This algebra, denoted by **PBT**, can as the previous one be realized in the free associative algebra (see [5,6,12,13]). Each complete binary tree with n internal nodes (or, equivalently, each binary tree with n nodes) is represented by a homogeneous polynomial of degree n in some non-commutative indeterminates. Actually, both algebras, tableaux and binary trees, were originally defined as Hopf sub-bialgebras of the bialgebra of permutations of [25], which has been realized in [5,6] as the algebra of free quasi-symmetric functions **FQSym**.

This realization suggests that the algebra of Young tableaux **FSym** and the algebra of planar binary trees **PBT** might be two particular cases of the same construction, both based on the existence of a Robinson-Schensted-like correspondence and a plactic-like monoid. We exhibit such a construction, the sylvester monoid. All this process seems even more important since there is a third example which fits in this setting: the pair of mutually dual Hopf algebras (**Sym**, QSym) (Noncommutative Symmetric Functions and Quasi-Symmetric Functions), which corresponds to the hypoplactic monoid (see [18,26]) and the Krob-Thibon correspondence.

Since our realization does yield a Hopf subalgebra of **FQSym**, the basic properties of **PBT** can be derived in a very natural and straigthforward way.

The algebra of permutations **FQSym** is naturally equipped with a scalar product (see [5,6]) and it is known that the integers

$$c_{I,J} = \langle R_I, R_J \rangle \quad |I| = |J| = n\,, \tag{1}$$

where R_I stands for the noncommutative ribbon Schur function of shape I, can be interpreted as Cartan invariants of the 0-Hecke algebra: the coefficient

$c_{I,J}$ is equal to the multiplicity of the simple module S_I in the indecomposable projective module P_J (see [18]).

The analogy between ribbon-Schur functions and the natural basis $\mathbf{P}_T$ of $\mathbf{PBT}$, as presented in the sequel, allows one to wonder whether it is possible to interpret in the same way the integers

$$c_{T,U} = \langle \mathbf{P}_T, \mathbf{P}_U \rangle \tag{2}$$

This question leads us to conjecture the existence of a tower of algebras over $\mathbb{C}$: $A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots$ with $\dim_{\mathbb{C}}(A_n) = n!$, whose simple modules S_T (and, accordingly, the indecomposable projective modules P_T) would be indexed by the binary trees of size n. To get the complete analogy with the classical case, one should also describe the embeddings $A_p \otimes A_q \hookrightarrow A_{p+q}$ and identify $\mathbf{PBT}$ and $\mathbf{PBT}^*$ as the direct sums of the Grothendieck groups

$$\mathcal{G} = \bigoplus_{n\geq 0} G_0(A_n)\,, \quad \mathcal{K} = \bigoplus_{n\geq 0} K_0(A_n) \tag{3}$$

in such a way that the products of $\mathbf{PBT}$ and $\mathbf{PBT}^*$ written in the appropriate basis would correspond to the induction process from $A_p \otimes A_q$ to A_{p+q}.

Up to a simple reodering of the matrix, we propose a good candidate for the matrices of Cartan invariants of a tower of algebras associated with $\mathbf{PBT}$ and present a precise conjecture on the induction process, suggested by some new combinatorial properties of binary trees arising in the analysis of these matrices.

This paper is organized as follows: the second Section recalls the basic definitions and some classical combinatorial algorithms. In Section 3, we present the algebra of planar binary trees and explain how it leads to the definition of the sylvester monoid. We then present how one can recover the basic properties of $\mathbf{PBT}$ in our setting. In Section 4, we build new bases and derive further properties of $\mathbf{PBT}$. Section 5 details the outcome of our representation theoretical investigation of $\mathbf{PBT}$. Finally, Section 7 gives some transition matrices between different bases of $\mathbf{PBT}$.

The main results of this paper have been announced in [12,13].

Acknowledgments

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2 Basic definitions and notations

2.1 Alphabets, words, and products

In all the paper, we will assume that we are given a totally ordered infinite alphabet A, represented either by $\{a, b, c, \ldots\}$ or by $\{1, 2, 3, \ldots\}$. Nevertheless, in some examples or for some constructions, a finite alphabet is needed. This does not change the formulas, the only difference being that some terms vanish in an obvious way.

The free associative algebra over an alphabet A, *i.e.*, the algebra spanned by words, the product being the concatenation, is denoted by $\mathbb{K}\langle A \rangle$, and its unity is denoted by ϵ. Here, $\mathbb{K}$ is some field of characteristic zero.

A *permutation* is a word without repetition on an initial interval of the alphabet. We shall also make use of a modification of the concatenation product, so that, starting from two words that are permutations, one gets a permutation. For a word $w = x_1 x_2 \cdots x_n$ on the alphabet $\{1, 2, \ldots\}$ and an integer k, denote by $w[k]$ the word $(x_1 + k)(x_2 + k) \cdots (x_n + k)$, as *e.g.*, $312[4] = 756$. The *shifted concatenation* of two words u and v is defined as

$$u \bullet v = u \cdot (v[k]) \tag{4}$$

where k is the length of u.

There is another algebraic structure on $\mathbb{K}\langle A \rangle$ known as the *shuffle product*. Let w_1 and w_2 be two words. Then the shuffle $w_1 ⧢ w_2$ is recursively defined by

- $w_1 ⧢ \epsilon = w_1, \quad \epsilon ⧢ w_2 = w_2,$
- $au ⧢ bv = a(u ⧢ bv) + b(au ⧢ v),$

where a, b are letters, and u, v words.

For example,

$$12 ⧢ 43 = 1243 + 1423 + 1432 + 4123 + 4132 + 4312\,. \tag{5}$$

Note that if, as in the previous example, one shuffles a permutation and another one, shifted by the size of the first one, one obtains a sum of permutations. This process is called the *shifted shuffle*.

2.2 Standardization

The symmetric group on n letters will be denoted by $\mathfrak{S}_n$, and its algebra by $\mathbb{K}[\mathfrak{S}_n]$.

Let us recall the standardization process sending a word to a permutation.

Let $A = \{a < b < \cdots\}$ be a totally ordered infinite alphabet. With each word w of A^* of length n, we associate a permutation $\mathrm{Std}(w) \in \mathfrak{S}_n$ called the *standardized* of w defined as the permutation obtained by iteratively scanning w from left to right, and labelling $1, 2, \ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. Alternatively, $\mathrm{Std}(w)$ is the permutation having the same inversions as w.

For example, $\mathrm{Std}(abcadbcaa) = 157296834$:

$$\begin{array}{ccccccccc} a & b & c & a & d & b & c & a & a \\ a_1 & b_5 & c_7 & a_2 & d_9 & b_6 & c_8 & a_3 & a_4 \\ 1 & 5 & 7 & 2 & 9 & 6 & 8 & 3 & 4 \end{array} \tag{6}$$

2.3 The weak order

The *weak order* (also called *right permutohedron order*) is the order on permutations obtained by defining the successors of a permutation σ as the permutations $\sigma \cdot s_i$ if this permutation has more inversions than σ, where $s_i = (i, i+1)$ exchanges the numbers at places i and $i+1$ of σ (see Figure 1).

2.4 Permutations and saillances

The *saillances* of a permutation σ of size n are the $i \leq n$ such that all the elements to the right of i in σ are smaller than i. For example, the saillances of 893175624 are, read from right to left, 4, 6, 7, and 9.

For technical reasons, we build the *saillance sequence* associated with the

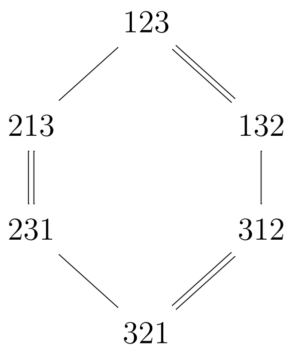


Fig. 1. The weak order of $\mathfrak{S}_3$.

saillances of a permutation by recording the positions of the saillances in *decreasing* order. So the saillance sequence of 893175624 is $(9, 7, 5, 2)$.

2.5 Transition matrices

As we shall need many different bases of **PBT** and of various algebras, we introduce a notation for the matrices expressing one basis into another. The matrix $M_{A,B}$ is the matrix whose ith column expresses the ith element of the basis A as linear combination of elements of the basis B.

3 The Algebra of Planar Binary Trees

In all the paper, a *planar binary tree* (or binary tree) is an incomplete planar rooted binary tree: a binary tree is either void ($\emptyset$) or a pair of possibly void binary trees grafted on an *internal node*. The size of a tree is the number of its nodes. The number of planar binary trees of size n is the Catalan number

$$C_n := \frac{\binom{2n}{n}}{n+1} . \tag{7}$$

A *labeled tree* is a tree with a label attached to each node, the label being taken either in the alphabet A or in $\mathbb{N}$.

A *right comb tree* (resp. *left comb tree*) is a tree having a sequence of right (resp. left) edges starting from the root and trees with only left (resp. right) edges attached to the previous nodes.

3.1 The Loday-Ronco Algebra

In [23], Loday and Ronco introduced a Hopf Algebra of planar binary trees, arising in their study of dendriform algebras. Actually, this algebra is the free dendriform algebra over one generator. In the same paper, they proved that it is a subalgebra of the convolution algebra of permutations, studied by Reutenauer [30], Malvenuto-Reutenauer [25], and Poirier-Reutenauer [29]. We will first present this algebra in our setting and then get back to our construction of the algebra of planar binary trees. It will be denoted by **PBT**, standing for *Planar Binary Trees*.

3.2 Free quasi-symmetric functions

In [25], Malvenuto and Reutenauer made a combinatorial study of the *convolution* of permutations, defined from the interpretation of permutations as elements of the endomorphism algebra of the bialgebra $\mathbb{K}[A^*]$. In particular, they explicited the coproduct which endows the space $\mathbb{K}[\mathfrak{S}] := \oplus_{n\geq 0}\mathbb{K}[\mathfrak{S}_n]$ (where $\mathfrak{S}_n$ denotes the symmetric group) with a Hopf algebra structure.

Their theory can be significantly simplified if one embeds $\mathbb{K}[\mathfrak{S}]$ in $\mathbb{K}\langle A \rangle$ as in [6]. This also sheds some light on the connection between this algebra and the algebra of Quasi-Symmetric Functions defined by Gessel in [11].

The image of $\mathbb{K}[\mathfrak{S}]$ under this embedding is called the algebra of *Free Quasi-Symmetric functions* over A and denoted by $\mathbf{FQSym}(A)$ or simply by **FQSym** if there is no ambiguity. Its natural basis $\mathbf{F}_\sigma$, where σ runs over all permutations, is given by the following construction.

Definition 1 *Let σ be a permutation. The* Free Quasi-Ribbon $\mathbf{F}_\sigma$ *is the noncommutative polynomial*

$$\mathbf{F}_\sigma := \sum_{w;\ \mathrm{Std}\,(w)=\sigma^{-1}} w\,, \tag{8}$$

where $\mathrm{Std}\,(w)$ *denotes the standardized word of w and w runs over the words on A.*

For example, on the alphabet $\{1,2,3\}$,

$$\mathbf{F}_{123} = 111+112+113+122+123+133+222+223+233+333\,, \tag{9}$$

$$\mathbf{F}_{213} = 212+213+313+323\,, \tag{10}$$

$$\mathbf{F}_{312} = 221+231+331+332\,. \tag{11}$$

With the help of the shifted shuffle, one easily describes the product of free quasi-ribbon functions.

Proposition 2 *The free quasi-ribbons span a $\mathbb{Z}$-subalgebra of the free associative algebra. Their product is given by the following formula. Let $\alpha \in \mathfrak{S}_k$ and $\beta \in \mathfrak{S}_l$. Then*

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{\sigma \in \alpha \,\text{ш}\, (\beta[k])} \mathbf{F}_\sigma \,. \tag{12}$$

This algebra is in fact a Hopf algebra, the coproduct being defined as follows. Let A' and A'' be two mutually commuting ordered alphabets. Identifying $F \otimes G$ with $F(A')G(A'')$, we set $\Delta(F) = F(A' \oplus A'')$, where $\oplus$ denotes the ordered sum. Clearly, this is an algebra homomorphism and thus defines a coproduct compatible with the product.

Proposition 3 *The coproduct of $\mathbf{F}_\sigma$ is given by*

$$\Delta \mathbf{F}_\sigma = \sum_{u \cdot v = \sigma} \mathbf{F}_{\mathrm{Std}\,(u)} \otimes \mathbf{F}_{\mathrm{Std}\,(v)} \,. \tag{13}$$

Moreover, **FQSym** is a self-dual Hopf algebra. One can see it by setting $\mathbf{G}_\sigma = \mathbf{F}_{\sigma^{-1}}$ as a basis of **FQSym*** and defining the scalar product by

$$\langle\, \mathbf{F}_\sigma, \mathbf{G}_\tau \,\rangle = \delta_{\sigma,\tau} \,. \tag{14}$$

Let us recall that the convolution algebra of permutations is the graded dual **FQSym***: the product of **G** functions is the so-called *convolution* of permutations. It consists in taking the inverses of both permutations, make their shifted shuffle and invert the resulting permutations.

3.3 The sylvester monoid

Let us now get back to the Loday-Ronco algebra. In [23], Loday and Ronco proved that **PBT** is a subalgebra of the convolution algebra and gave an explicit embedding via the construction of the decreasing tree of a permutation as described, for example, in [16,35].

Definition 4 *A* decreasing tree *T is a labeled tree such that the label of each internal node is greater than the label of its children. If the labels are the integers from 1 to the number n of nodes of T, we say that T is a* standard decreasing tree*.*

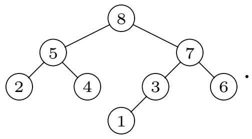
Definition 5 *Let w be a word without repetition. Its* decreasing tree *$\mathcal{T}(w)$ is obtained as follows: the root is labeled by the greatest letter, n of w, and if*

$w = unv$*, where* u *and* v *are words without repetition, the left subtree is* $\mathcal{T}(u)$ *and the right subtree is* $\mathcal{T}(v)$*.*

Note 1 The left infix reading (recursively read the left subtree, the root and the right subtree) of $\mathcal{T}(w)$ is w.

For example, the decreasing tree of 25481376 is

$$\mathcal{T}(25481376) = \tag{15}$$



Loday and Ronco defined an embedding of **PBT** in the Malvenuto-Reutenauer Hopf algebra by expressing a basis of **PBT** that we shall denote by $\mathbf{P}_T$, as

$$\mathbf{P}_T := \sum_{\sigma;\ \mathrm{shape}\,(\mathcal{T}(\sigma))=T} \sigma\,. \tag{16}$$

where T is a non-labeled tree and $\mathrm{shape}\,(T)$ is the shape of the tree T.

This gives an embedding in **FQSym**, which reads

$$\mathbf{P}_T = \sum_{w;\ \mathrm{shape}\,(\mathcal{T}(w))=T} w = \sum_{\sigma;\mathrm{shape}\,(\mathcal{P}(\sigma))=T} \mathbf{F}_\sigma\,, \tag{17}$$

where $\mathcal{P}$ is a simple algorithm: it is the well-known *binary search tree insertion*, such as presented, for example, by Knuth in [16]. Lemma 11 will show that this definition is consistent.

Definition 6 *A* right strict binary search tree T *is a labeled binary tree such that for each internal node* n*, its label is greater than or equal to the labels of its left subtree and strictly smaller than the labels of its right subtree.*

Definition 7 *Let* w *be a word. Its* binary search tree $\mathcal{P}(w)$ *is obtained as follows: reading* w *from* right to left*, one inserts each letter in a binary search tree in the following way: if the tree is empty, one creates a node labeled by the letter ; otherwise, this letter is recursively inserted in the left (resp. right) subtree if it is smaller than or equal to (resp. strictly greater than) the root.*

Note 2 The left infix reading of $\mathcal{P}(w)$ is the non-decreasing permutation of w. This is the well-known algorithm of binary search tree sorting.

Figure 2 shows the binary search tree of *cadbaedb*.

Some known examples of Hopf subalgebras of **FQSym** (Free Symmetric Functions and Noncommutative Symmetric Functions) suggest to look for a monoid structure on words to explain the previous construction.

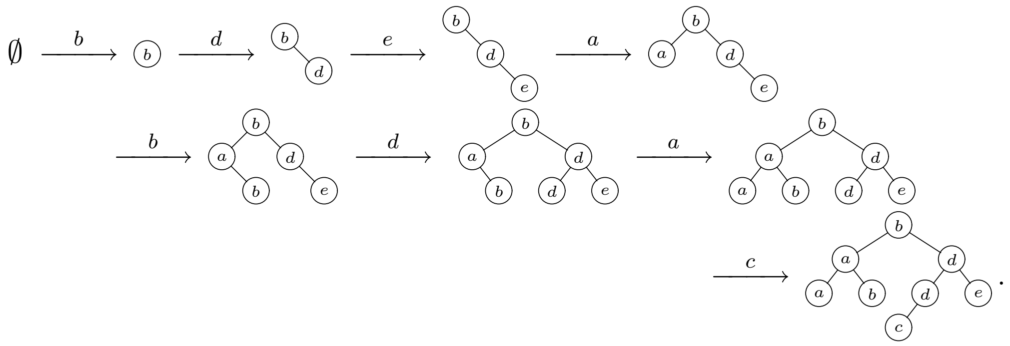


Fig. 2. The Binary Search Tree of *cadbaedb*.

Definition 8 *Let w_1 and w_2 be two words. One says that they are* sylvester-adjacent *if there exists three words u, v, w and three letters $a \leq b < c$ such that*

$$w_1 = u\, ac\, v\, b\, w \qquad \textit{and} \qquad w_2 = u\, ca\, v\, b\, w\,. \tag{18}$$

The sylvester congruence *is the transitive closure of the relation of sylvester adjacence. That is, two words u, v are sylvester-congruent if there exists a chain of words*

$$u = w_1, w_2, \ldots, w_k = v\,, \tag{19}$$

such that w_i and w_{i+1} are sylvester-adjacent for a all i. In this case, we write $u \equiv_{sylv} v$.

It is plain that this is actually a congruence on A^*.

Definition 9 *The* sylvester monoid $\mathrm{Sylv}(A)$ *is the quotient of the free monoid A^* by the sylvester congruence:* $\mathrm{Sylv}(A) := A^* / \equiv_{sylv}$.

For example, the classes of $w = 21354$ and $w = 614723$ respectively are the sets

$$\mathrm{Sylv}(21354) = \{21354, 21534, 25134, 52134\}\,, \tag{20}$$

$$\begin{aligned}\mathrm{Sylv}(614723) = \{&126473, 162473, 164273, 164723, 612473,\\ &614273, 614723, 641273, 641723, 647123\}\,.\end{aligned} \tag{21}$$

Recall that the *right to left postfix reading* of a tree T is the word w_T obtained by reading the right subtree, then the left and finally the root. Notice that the insertion of w_T with the binary search tree insertion algorithm gives T back. A word which is the postfix reading of a tree is called the *canonical word* of its sylvester class.

Note 3 Thanks to Note 2, one can easily see that there only is one binary search tree of a given shape labeled by a permutation. In the sequel, this element is called the *standard canonical element of the tree.* It now makes sense to define the canonical element of an unlabeled tree as the canonical element of its unique binary search tree labeled by a permutation.

Theorem 10 *Let w_1 be a word. Then $\mathcal{P}(w_1) = T$ if and only if w_1 and w_T are sylvester-congruent.*

PROOF. If w_1 and w_T are sylvester-congruent, they give the same result by the binary search tree algorithm since it is always the case for two words w_1 and w_2 which are sylvester-adjacent.

Conversely, any word w is sylvester-congruent to a word w_T. Indeed, by induction, one can assume that $w = a \cdot w_1$ where a is a letter and w_1 a canonical word. It is then easy to see that if a is smaller than or equal to the last letter of w_1, it can move to the left sub-tree of w_T. The result follows by induction. □

To get our version of the Loday-Ronco algebra, there only remains to prove that Formula (17) holds. Indeed, if we prove

$$\mathbf{P}_T = \sum_{\sigma;\ \mathrm{shape}\,(\mathcal{P}(\sigma))=T} \mathbf{G}_{\sigma^{-1}}\,, \tag{22}$$

we are done since **FQSym** is isomorphic to its dual, through the identification $\mathbf{F}_\sigma = \mathbf{G}_{\sigma^{-1}}$. This amounts to to the following Lemma.

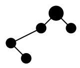
Lemma 11 *Let w be a word and σ its standardized word. Then $\mathcal{P}(w)$ has the same shape as $\mathcal{P}(\sigma)$ and $\mathcal{T}(\sigma^{-1})$.*

PROOF. It is obvious that $\mathcal{P}(w)$ and $\mathcal{P}(\sigma)$ have the same shape. Let us now prove the second part of the lemma. Let n be the size of σ. Contemplating the outputs of both insertion algorithms, the first observation is that the size of the left subtree of $\mathcal{P}(\sigma)$ is the same as the size of the left subtree of $\mathcal{T}(\sigma^{-1})$: it is respectively the number of elements smaller than $\sigma(n)$ and the number of elements to the left of n in σ^{-1}. Now, the proof follows by induction, since the inverse of the standardized word of the restriction of σ to elements smaller than $\sigma(n)$ is the standardized word of the elements to the left of n in σ^{-1}. □

We have now proved that the Loday-Ronco algebra of planar binary trees is isomorphic to the algebra of sums of free quasi-ribbons over sylvester classes. Within our description, one has

$$\mathbf{P}_T = \sum_{\sigma;\ \mathrm{shape}\,(\mathcal{P}(\sigma))=T} \mathbf{F}_{\sigma}\,. \tag{23}$$

For example, according to Equation (20), one has:



$$\mathbf{P} = \mathbf{P}_{52134} = \mathbf{F}_{21354} + \mathbf{F}_{21534} + \mathbf{F}_{25134} + \mathbf{F}_{52134}\,. \tag{24}$$

We have therefore realized **PBT** as a Hopf subalgebra of **FQSym** in the same way as other interesting algebras that we will briefly recall, before adaptating the constructions to the sylvester case.

3.4 Analogous constructions

The algebra of Free Symmetric Functions has been designed to give a simple and transparent proof of the Littlewood-Richardson rule, a famous combinatorial rule for computing tensor products of group representations which was stated without proof in 1934, and of which no complete proof had been known until the end of the seventies (one can find a detailed version of all this in [6,20]). The monoid that will play the role of the sylvester monoid is the well-known *plactic monoid* defined by Lascoux and Schützenberger (see [21]) from Knuth's rewriting rules (see [17]).

For later reference, let us recall that the plactic equivalence is the congruence generated by the relations

$$\begin{cases} acb \equiv cab, & \text{for } a \le b < c, \\ bca \equiv bac, & \text{for } a < b \le c. \end{cases} \tag{25}$$

Let t be a standard tableau of shape λ. Let

$$\mathbf{S}_t := \sum_{P(\sigma)=t} \mathbf{F}_\sigma = \sum_{Q(w)=t} w\,, \tag{26}$$

where $w \mapsto (P(w), Q(w))$ is the usual Robinson-Schensted map, sending a word to a pair of Young Tableaux of the same shape, the second one being standard. As pointed out in [20], Schützenberger's version of the multiplication of Schur functions, the *Littlewood-Richardson rule* is equivalent to the following statement, which shows in particular that the free Schur functions span a subalgebra of **FQSym**. It is called the algebra of *Free Symmetric Functions* and denoted by **FSym**. It provides a realization of the algebra of tableaux introduced by Poirier and Reutenauer [29] as a subalgebra of the free associative algebra.

Proposition 12 *Let t', t'' be standard tableaux, and let k be the number of cells of t'. Then,*

$$\mathbf{S}_{t'}\mathbf{S}_{t''} = \sum_{t \in \mathrm{Sh}(t',t'')} \mathbf{S}_t \,, \tag{27}$$

where $\mathrm{Sh}(t',t'')$ *is the set of standard tableaux in the shuffle of t' (regarded as a word via its row reading) with the plactic class of $t''[k]$.*

The proof of this statement is relatively easy. It follows from simple combinatorial properties of independent interest. In fact, the only non trivial element of the proof is the idea of the Robinson-Schensted correspondence. If one is willing to accept it as natural, then, the fact that the commutative image of $\mathbf{S}_t$ is s_λ can be considered as a one-line proof of the Littlewood-Richardson rule.

Notice that one can perform the same construction, replacing the plactic monoid by the *hypoplactic monoid* (see [18,26]). One then obtains the algebra of Noncommutative Symmetric Functions (see [7]).

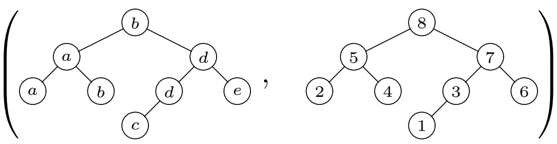
3.5 A Schensted-like algorithm

We have already mentioned the Robinson-Schensted correspondence, a bijection between words and pairs of tableaux computed by the Schensted algorithm (see [32]). The same construction generalizes to pairs composed of ribbons and quasi-ribbons in the case of the hypoplactic monoid (see [18,26]). This construction also generalizes to the sylvester case.

Let w be a word. The *Sylvester Schensted Algorithm SSA* sends it to the pair composed of its binary search tree $\mathcal{P}(w)$ and the decreasing tree of the inverse of its standardized $\mathcal{Q}(w) = \mathcal{T}((\mathrm{Std}\,(w))^{-1})$.

For example, according to Formula (15) and Figure 2, one has

$SSA(cadbaedb) =$



(28)

Consider a pair composed of a binary search tree t_1 and a standard decreasing tree t_2 of the same shape. Algorithm SSB sends this pair to the word obtained by reading the labels of t_1 in the order of the corresponding labels in t_2.

Theorem 13 *The algorithm SSA yields a bijection between the set of words of size n and pairs composed of a binary search tree of size n and a standard decreasing tree of the same shape. The reciprocal bijection is computed by*

Algorithm SSB.

PROOF. First, the output of Algorithm SSA is a pair of trees of the same shape, thanks to Lemma 11. So it maps w to a pair of the right form. Moreover, thanks to Notes 1 and 2 and since w is equal to the word obtained by permuting its non-decreasing word with the infix reading of $\mathcal{T}(\mathrm{Std}\,(w)^{-1})$, one can conclude. $\square$

Note 4 If one regards a tree as a partially ordered set, the leafs being the smallest elements and the root the greatest, the sylvester class associated with a given tree consists in the linear extensions of this partial order.

3.6 Basic properties of the sylvester monoid and of its algebra

3.6.1 Sylvester classes and the permutohedron

First, let us describe the structure of the sylvester equivalence classes seen as parts of the permutohedron. The first two properties are obvious thanks to Note 4.

Proposition 14 *The greatest word for the lexicographic order of the sylvester class of T is w_T.*

Proposition 15 *The smallest word for the lexicographic order of the sylvester class indexed by a tree T is the left to right postfix reading of T.*

The following property will later be useful to simplify the combinatorial description of **PBT**. We first recall the definition of pattern avoidance.

A permutation σ of $\mathfrak{S}_n$ *avoids the pattern* π of $\mathfrak{S}_k$ if and only if there is no subsequence $i_{\pi(1)} < i_{\pi(2)} < \ldots < i_{\pi(k)}$ in $[1, n]$ such that $\sigma(i_1) < \sigma(i_2) < \ldots < \sigma(i_k)$.

Proposition 16 *A permutation is a canonical sylvester permutation if and only if it avoids the pattern* 132*.*

PROOF. If σ is a canonical sylvester permutation, since it is the right to left postfix reading of a binary search tree, it necessarily avoids the pattern 132. Since the number of permutations avoiding 132 of size n is the Catalan number C_n, there are as many permutations of size n avoiding 132 as binary trees with n nodes. $\square$

If σ avoids a given pattern π then σ^{-1} avoids the pattern π^{-1}. Consequently:

Corollary 17 *Let σ be a canonical sylvester permutation. Then σ^{-1} also is a canonical sylvester permutation.*

The next property can be used to prove that the $\mathbf{P}_T$'s span a Hopf subalgebra of **FQSym**. Let us examine the compatibility of the sylvester congruence with restriction to intervals.

Proposition 18 *Assume that u and v are sylvester-congruent. Let I be an interval of the alphabet A, $I = [a_k, \ldots, a_l]$. Let u/I (resp. v/I) be the word obtained by erasing the letters of u (resp. v) that are not in I. Then u/I and v/I are sylvester congruent.*

PROOF. This property is easily checked on the sylvester relations, which implies the result. $\square$

Now, by the very same reasoning as in [6], prop. 3.12, using the fact that the sylvester congruence is compatible with the restriction to intervals and to de-standardization, one deduces that **PBT** is a Hopf subalgebra of **FQSym**. All formulas will be given in the next Section.

Note 5 In [24], the product $\mathbf{P}_{T'}\mathbf{P}_{T''}$ is described by means of an order on the planar binary trees, also known as the *Tamari* order (see Section 4.4 for more details). In our setting, this order is obtained from the weak order of the symmetric group: it is its restriction to canonical words (see Theorem 27). It should be noticed that the results of Björner and Wachs (see [4]) show that the sylvester classes are intervals for the weak order.

Note 6 In the context of tableaux, as a simple consequence of Theorem 1 of [6], one can define an order on tableaux as the quotient of the weak order by Knuth's relations (this order is also considered by Björner and Wachs). The sets $\mathrm{Sh}(t', t'')$ of Proposition 12, where t' and t'' are standard tableaux, are intervals for this order.

3.6.2 Hook-length formula for trees

For sake of completeness, we include Knuth's hook-length formula for trees, and its q-analog, due to Björner and Wachs.

The cardinality of a standard plactic class is equal to the number of standard tableaux of a certain shape which is given by the celebrated hook-length formula (see [9,27]). This formula admits a q-analog which enumerates permutations by their major index (see [34]). In the same way, the enumeration of the sylvester class associated with a tree T of size n is given by the specialization $(q)_n\mathbf{P}_T(1,q,q^2,\cdots)$ which is equal (see [3]) to

$$\sum_{\mathcal{P}(\sigma)=T} q^{\mathrm{maj}\,(\sigma)} = (q)_n\mathbf{P}_T(1,q,q^2,\cdots) = \frac{[n]_q!}{\prod_{\circ\in T} q^{-\delta_\circ}[h_\circ]_q}\,, \tag{29}$$

where for a node $\circ$ of T, the coefficient $h_\circ$ is the size of the subtree rooted at $\circ$ and $\delta_\circ$ the size of its right subtree. Here we use the standard notations of the q-calculus, that is, for any integer n, the q-integer $[n]_q$ is equal to $1+q+\cdots+q^{n-1}$, the q-factorial $[n]!_q$ is the product of the corresponding q-integers: $[n]!_q := [1]_q[2]_1\dots[n]_q$, and finally

$$(q)_n := (1-q)^n[n]!_q = (1-q)(1-q^2)\dots(1-q^n)\,. \tag{30}$$

3.7 A sylvester description of PBT

We are now in position to describe the structure of **PBT** in a similar way as in [6,28] for other combinatorial Hopf algebras (in the sense of [2]). In particular we will give alternative formulas to compute the product, the coproduct and the antipode of **PBT** using only combinatorial properties of the sylvester classes.

Theorem 19 *Let T' and T'' be two binary trees. Then*

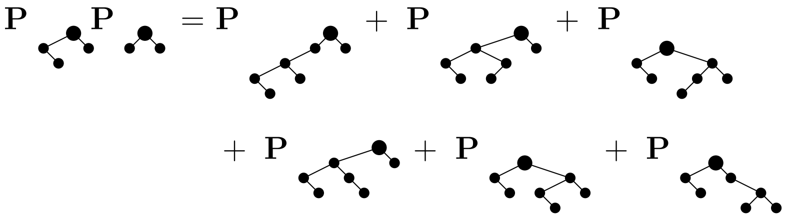
$$\mathbf{P}_{T'}\mathbf{P}_{T''} = \sum_{T\in\mathrm{Sh}(T',T'')} \mathbf{P}_T\,, \tag{31}$$

where $\mathrm{Sh}(T',T'')$ is the set of trees T such that w_T occur in the shuffle product $u\sqcup\!\sqcup v$, $u = w_{T'}$ and $v = w_{T''}[k]$ is the canonical word of T'' shifted by the size of T'.

PROOF. The product $\mathbf{P}_{T'}\mathbf{P}_{T''}$ can be expanded into $\mathbf{F}_\sigma$'s using the shifted shuffle on permutations (see Formulas 17 and 12). It can be factored as a sum of $\mathbf{P}_T$'s thanks to the compatibility of the sylvester relation with restriction to intervals. Finally, $\mathbf{P}_T$ arises in the product $\mathbf{P}_{T'}\mathbf{P}_{T''}$ if and only if its canonical word w_T appears in the shuffle of the sylvester classes associated with T' and T''. The last part of the theorem then comes from the fact that if w and w' are not both canonical words, there is no canonical word in their shifted shuffle. $\square$

For example,

$$\mathbf{P}_{4213}\mathbf{P}_{312} = \mathbf{P}_{7421356} + \mathbf{P}_{7452136} + \mathbf{P}_{7456213} + \mathbf{P}_{7542136} + \mathbf{P}_{7546213} + \mathbf{P}_{7564213}\,, \quad (32)$$



(33)

Let us now compute the coproduct of a $\mathbf{P}_T$ in a similar way.

Theorem 20 *Let T be a tree. The coproduct of $\mathbf{P}_T$ is given by*

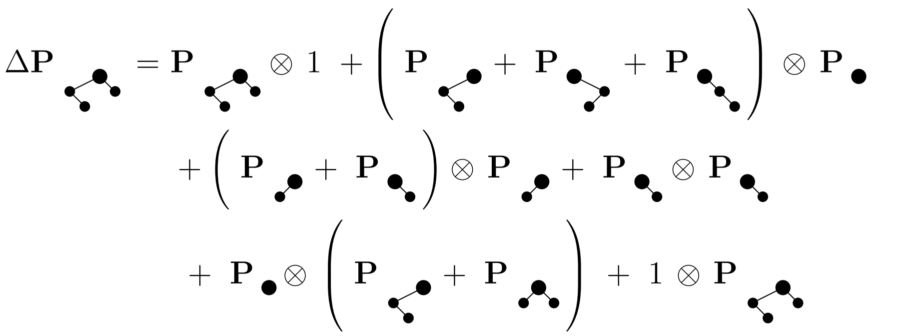
$$\Delta \mathbf{P}_T = \sum_{(T',T'')\in \mathrm{Dec}(T)} \mathbf{P}_{T'} \otimes \mathbf{P}_{T''}\,, \quad (34)$$

where $\mathrm{Dec}(T)$ *is the set of pairs of trees (T', T'') such that $w_{T'}$ (resp. $w_{T''}$) are the standardized words of elements w_1 (resp. w_2) where $w_1 \cdot w_2$ is in the sylvester class of T.*

PROOF. The coproduct of a $\mathbf{P}_T$ can be expanded into $\mathbf{F}_\sigma$'s, then described by using the deconcatenation of permutations (see Formulas (17) and (13)). It can be factored into a sum of $\mathbf{P}_T$'s thanks to the compatibility of the sylvester relation with de-standardization. Finally, a pair $(\mathbf{P}_{T'}, \mathbf{P}_{T''})$ arises in the coproduct of $\mathbf{P}_T$ if and only if both canonical words $w_{T'}$ and $w_{T''}$ appear in the deconcatenation of an element of the sylvester class of T. $\square$

For example,

$$\begin{aligned}\Delta \mathbf{P}_{4213} = {} & \mathbf{P}_{4213} \otimes 1 + (\mathbf{P}_{213} + \mathbf{P}_{231} + \mathbf{P}_{321}) \otimes \mathbf{P}_1 + (\mathbf{P}_{12} + \mathbf{P}_{21}) \otimes \mathbf{P}_{12} \\ & + \mathbf{P}_{21} \otimes \mathbf{P}_{21} + \mathbf{P}_1 \otimes (\mathbf{P}_{213} + \mathbf{P}_{312}) + 1 \otimes \mathbf{P}_{4213}\,,\end{aligned} \quad (35)$$



(36)

Since **PBT** is a connected graded bialgebra of finite dimension in each component, one can define the antipode ν of **PBT** without ambiguity. It then endows **PBT** with the structure of a graded Hopf algebra. We will provide its formula on the dual basis of the $\mathbf{P}_T$ functions (see Equation (44)).

4 Properties of PBT

4.1 Duality

The sylvester congruence also gives a nice characterization of the dual algebra of **PBT**:

Theorem 21 *The (graded) dual* $\mathbf{PBT}^*$ *of the algebra of planar binary trees* **PBT** *is isomorphic to the image of* $\mathbf{FQSym}^*$ *under the canonical projection*

$$\pi : \ \mathbb{C}\langle A \rangle \longrightarrow \mathbb{C}[\mathrm{Sylv}(A)] \simeq \mathbb{C}\langle A \rangle / \equiv_{sylv} \ . \tag{37}$$

The dual basis $\mathbf{Q}_T$ *of* $\mathbf{P}_T$ *is expressed as* $\mathbf{Q}_T = \pi(\mathbf{G}_\sigma)$*, where* σ *is any permutation of the sylvester class associated with* T*.*

Notice that all this works within the realization of $\mathbf{FQSym}^*$ since the sylvester monoid is compatible with the de-standardization process. Let us now see how to compute the product and coproduct of $\mathbf{Q}_T$ functions by means of our formalism.

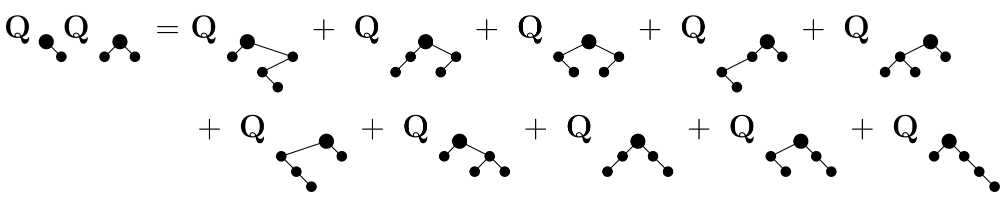
Theorem 22 *Let* T' *and* T'' *be two trees. Then,*

$$\mathbf{Q}_{T'}\mathbf{Q}_{T''} = \sum_{T \in \mathrm{Conv}(T',T'')} \mathbf{Q}_T \,, \tag{38}$$

where $\mathrm{Conv}(T',T'')$ *is the set of trees that are the binary search trees of an element of the convolution product of* $w_{T'}$ *by* $w_{T''}$*.*

For example,

$$\begin{aligned} \mathbf{Q}_{21}\mathbf{Q}_{312} = \ & \mathbf{Q}_{43512} + \mathbf{Q}_{45123} + \mathbf{Q}_{45213} + \mathbf{Q}_{52134} + \mathbf{Q}_{53124} \\ & + \mathbf{Q}_{53214} + \mathbf{Q}_{53412} + \mathbf{Q}_{54123} + \mathbf{Q}_{54213} + \mathbf{Q}_{54312} \,. \end{aligned} \tag{39}$$



(40)

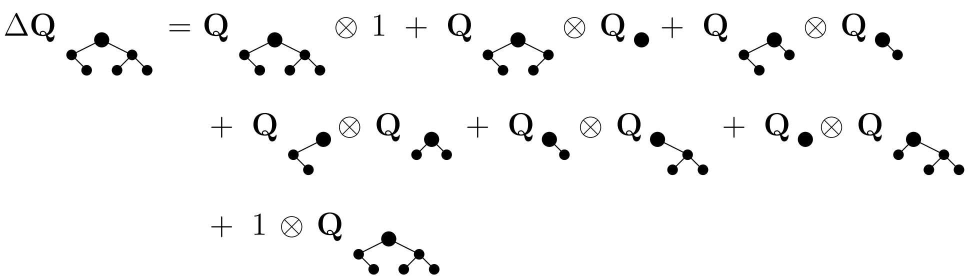
Theorem 23 *Let T be a tree. Then,*

$$\Delta \mathbf{Q}_T = \sum_{(T',T'')\in \mathrm{DeSh}(T)} \mathbf{Q}_{T'} \otimes \mathbf{Q}_{T''}\,, \tag{41}$$

where $\mathrm{DeSh}(T)$ *is the set of pair of trees* (T',T'') *such that their canonical elements are the standardized of the restrictions of the canonical word of* T *to all pairs of intervals* $[1,i]$ *and* $[i+1,n]$ *for* $i \in \{0,\ldots,n\}$.

For example,

$$\begin{aligned}\Delta \mathbf{Q}_{645213} = \ & \mathbf{Q}_{645213} \otimes 1 + \mathbf{Q}_{45213} \otimes \mathbf{Q}_{1} + \mathbf{Q}_{4213} \otimes \mathbf{Q}_{21} + \mathbf{Q}_{213} \otimes \mathbf{Q}_{312} \\ & + \mathbf{Q}_{21} \otimes \mathbf{Q}_{4231} + \mathbf{Q}_{1} \otimes \mathbf{Q}_{53412} + 1 \otimes \mathbf{Q}_{645213}\,.\end{aligned} \tag{42}$$


(43)

4.2 Antipode of **PBT**

We have already seen that since **PBT** is a connected graded bialgebra of finite dimension in each degree, one can define the antipode ν of **PBT** without ambiguity. It then endows **PBT** with a structure of graded Hopf algebra. Its formula on the basis $\mathbf{Q}_T$ is:

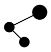
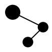
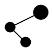
$$\nu(\mathbf{Q}_T) = \sum_{I \vDash n} (-1)^k \mathbf{Q}_{w_T(I,0)} \cdots \mathbf{Q}_{w_T(I,k-1)}\,, \tag{44}$$

where k is the length of I and $w(I,j)$ is the restriction of the word w to the alphabet interval $[i_1 + \cdots + i_j + 1, i_1 + \cdots + i_{j+1}]$.

Note 7 In Section 4.5, we define analogs of the complete homogeneous symmetric functions denoted by $\mathbf{H}_T$ and elementary symmetric functions denoted by $\mathbf{E}_T$ in **PBT**. Then the image of $\mathbf{H}_T$ by the antipode, where T is a right comb tree, is equal up to sign to T reversed, on the $\mathbf{E}$ basis.

One proves this property by remarking that the $\mathbf{H}_T$ correspond to products of complete functions of **Sym** whereas the $\mathbf{E}_T$ corresponding to left comb trees are products of elementary functions of **Sym**.

Notice that the antipode is not an involution as one can see:

$\nu(\nu(\mathbf{H}$  $)) = 2\mathbf{H}$  $- 2\mathbf{H}$  $+ \mathbf{H}$  . (45)

4.3 Pairs of Fomin graphs

One can build a pair of graded graphs (Γ, Γ^*) in duality as in Fomin's setting [8], whose vertices of degree n are the binary trees of size n. In Γ, there is an edge between T and T' if $\mathbf{P}_{T'}$ appears in the product $\mathbf{P}_T\mathbf{P}_\bullet$ (the dot $\bullet$ is the tree of size 1). In Γ^*, this edge appears if $\mathbf{Q}_{T'}$ appears in $\mathbf{Q}_T\mathbf{Q}_\bullet$. The sylvester correspondence is the Fomin correspondence associated to this pair of graphs.

4.4 The Tamari order and equivalent orders

In [24], Loday and Ronco describe the product $\mathbf{P}_{T'}\mathbf{P}_{T''}$ as an interval of the so-called *Tamari* order. The situation is the following: given the structure of all sylvester classes inside the permutohedron, we can easily prove in our setting that the product of two $\mathbf{P}_T$ functions is an interval of the permutohedron (see Note 9). It happens that the restriction of the weak order to sylvester classes is the same as the Tamari order (see Theorem 24), which yields in particular a simple proof of the result of [24].

Let us first give some definitions. Following Stanley in [36] (ex. 6.32.a p. 234), we define the Tamari order O_n as the poset of all integer vectors $(a_1, \ldots, a_n)$ such that $i \leq a_i \leq n$ and such that, if $i \leq j \leq a_i$ then $a_j \leq a_i$, ordered coordinatewise (see Figures 4 and 5).

Let O'_n be the *sylvestrohedron order* defined on sylvester classes as follows: a sylvester class S is smaller than S' if there exist $\sigma \in S$ and $\sigma' \in S'$ such that $\sigma < \sigma'$ for the weak order.

Theorem 24 *The sylvestrohedron order coincides with the Tamari order.*

To prove this property, we will go through a third equivalent order. Let O''_n be the *sylvester order* defined on canonical sylvester words as the restriction of the weak order to those elements.

PROOF. The proof of Theorem 24 results from the following three lemmas:

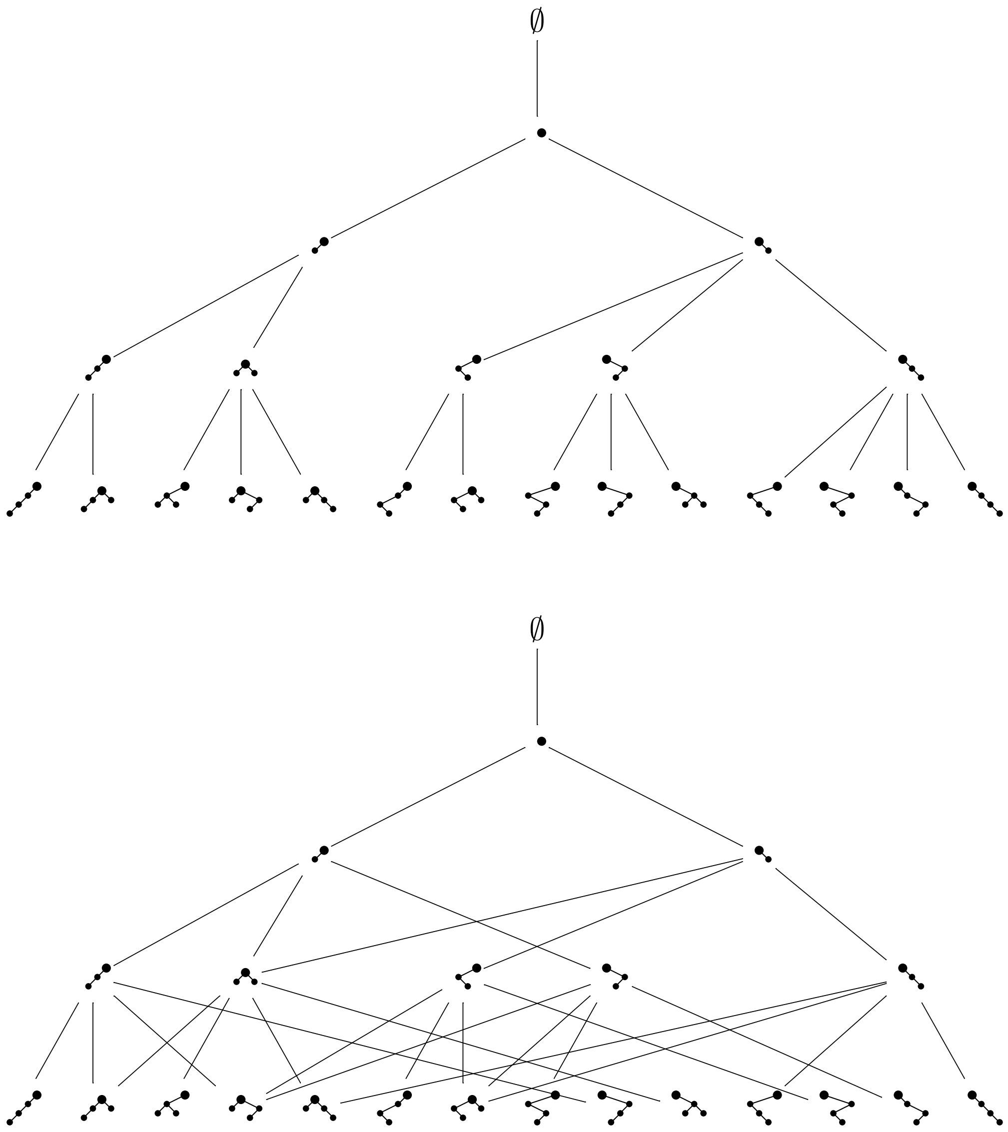


Fig. 3. The two graded graphs in duality.

Lemma 25 *Let σ and σ' be two permutations such that σ is smaller than σ' for the weak order. Then the canonical word corresponding to σ is smaller than or equal to the canonical word of σ'.*

Lemma 26 *The sylvestrohedron order coincides with the sylvester order.*

Lemma 27 *The Tamari order coincides with the sylvester order.*

Let us first prove Lemma 25. We only need to prove it for an elementary transposition. Assume that an elementary transposition s_i sends a permutation σ to σ', belonging to another sylvester class. Let us prove that there is an elementary transposition that sends the canonical word associated with σ

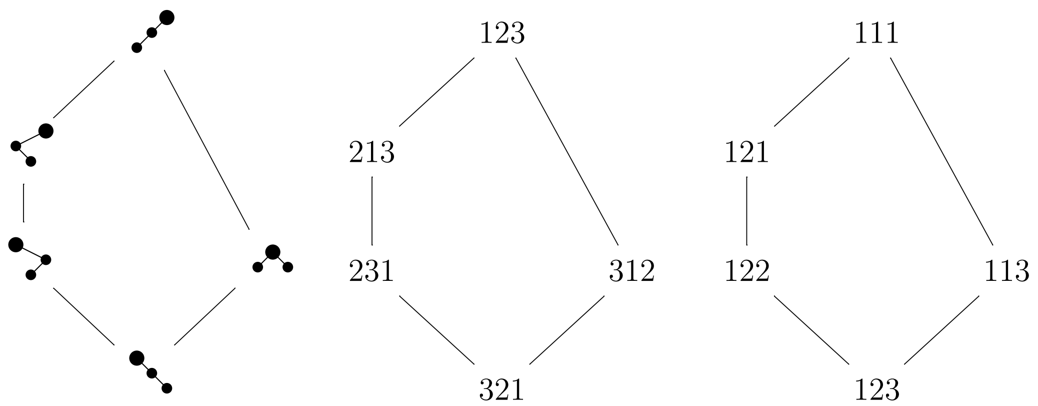


Fig. 4. The same order on trees, canonical words, and Tamari elements of size 3.

inside the class of σ'.

If σ is the canonical word of its class, then it is smaller than the canonical word of the class of σ' by transitivity. Assume now that σ is not a canonical word. Then there is another elementary transposition s_j that sends σ to σ_0, belonging to the same sylvester class (let us recall that thanks to [4], the sylvester classes are intervals of the permutohedron). If $j \neq i-1$ and $j \neq i+1$, then s_i sends σ_0 to an element of the same sylvester class as σ'. Otherwise, let us assume that $j = i-1$. Apply s_i and then s_{i-1} to σ_0. The resulting element is in the same sylvester class as σ'. Applying this property until σ_0 is a canonical word proves that there exists an elementary transposition that sends the canonical word associated with σ to the class of σ'. $\square$

Let us now prove Lemma 26. The isomorphism between both orders is trivial: a sylvester class is sent to its canonical word. Now, by definition, if $\sigma < \sigma'$ for the sylvester order, then $\sigma < \sigma'$ for the sylvestrohedron order. Lemma 25 proves the converse. $\square$

Let us finally prove Lemma 27. Notice that there exists a well-known simple bijection between the elements of the Tamari poset as defined before and the canonical sylvester words. Indeed, send each permutation to the sequence defined as follows: for each i, compute the number of elements smaller than i and to the right of i. Then add 1 to each component of the resulting vector. It is a Tamari element. Conversely, subtract 1 to each component of a Tamari element and rebuild the permutation which has these numbers of inversions. For example, if $\sigma = 435216$, one finds the sequence 012320 and the Tamari element $t = 123431$.

Now, given the bijection, it is immediate to see that if two permutations are comparable for the weak order, then so are the corresponding Tamari elements for the Tamari order. And conversely, if two Tamari elements are comparable for the Tamari order, then so are the corresponding permutations. $\square$

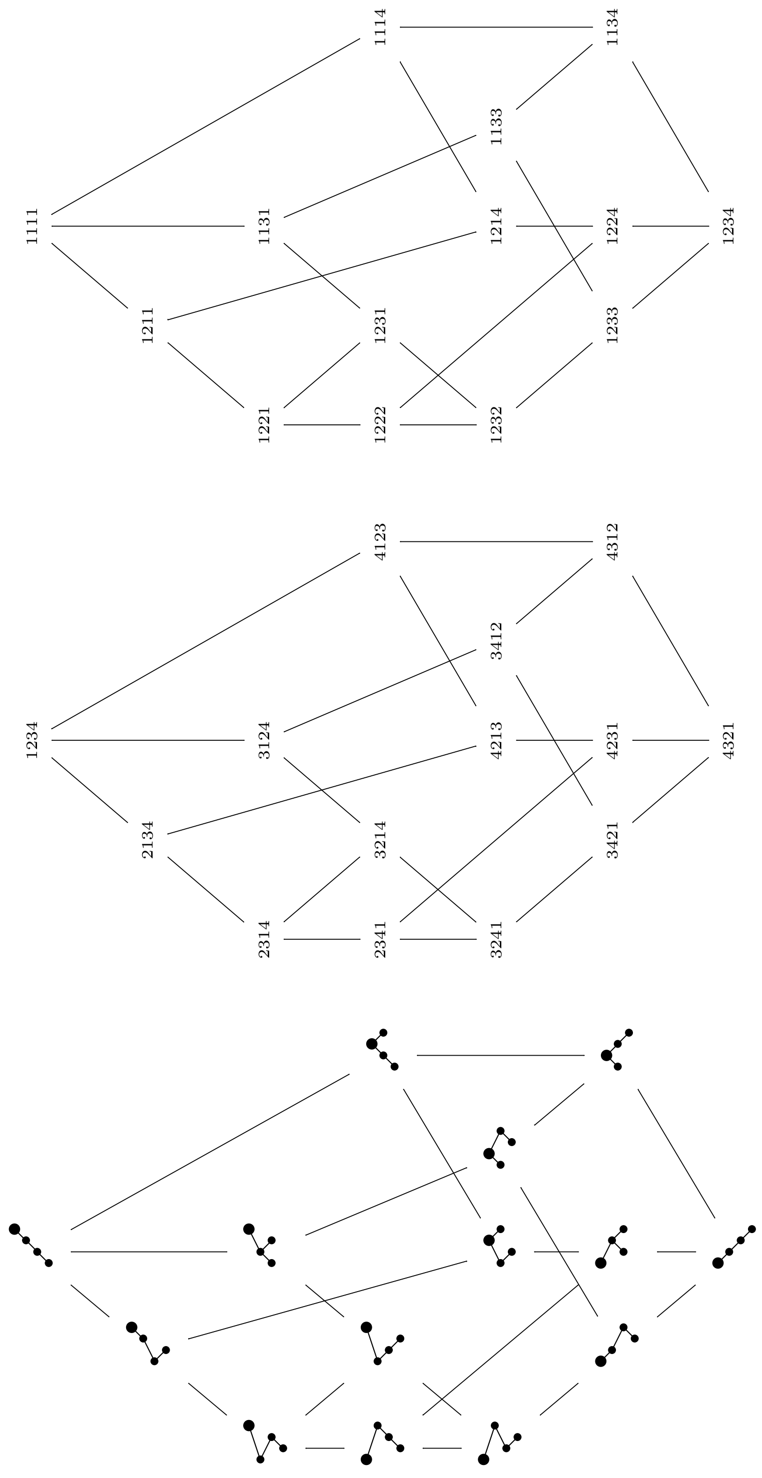


Fig. 5. The same order on trees, canonical words, and Tamari elements of size 4.

Lemmas 26 and 27 together imply the theorem. $\square$

Note 8 The construction of the sylvestrohedron order and the fact that it is the same as the restriction of the weak order to canonical words does not work for the other known interesting example, that is the algebra of Free Symmetric Functions **FSym**: in this algebra, if one says that a plactic class is smaller than another one if there is an element of the first one smaller than an element of the second one for the weak order, this relation is not transitive.

Note 9 It is well known that the set of permutations arising in the shifted shuffle of two permutohedron intervals is a permutohedron interval. Since each sylvester class is an interval of the permutohedron and since the product of quasi-ribbon functions is given by the shifted shuffle, it immediately comes that the product $\mathbf{P}_{T'}\mathbf{P}_{T''}$ expressed on the $\mathbf{F}_\sigma$'s is an interval of the permutohedron, and so, is an interval of the sylvestrohedron. This property holds for other quotients of $\mathbf{FQSym}^*$ as soon as the congruence is compatible with de-standardization.

Note 10 As already proved, the Tamari order is the same as the restriction of the weak order to canonical words. One can translate on canonical words the covering relation built by Loday and Ronco: given a canonical word σ, for any rise $\sigma(i) < \sigma(i+1)$, one builds the permutation obtained from σ by exchanging $\sigma(i+1)$ with the element to its left as long as it is smaller than or equal to $\sigma(i)$. This last permutation is a canonical word.

For example, if one chooses $\sigma = (12, 10, 8, 9, 6, 7, 4, 2, 1, 3, 5, 11)$ and $i = 11$, one obtains $(12, 10, 8, 9, 6, 7, 11, 4, 2, 1, 3, 5)$, which is canonical. Doing this on all consecutive elements such that $\sigma(i) < \sigma(i+1)$, one recovers all covering relations of the Tamari order.

Lemma 25 also proves that

Corollary 28 *The intervals of the permutohedron starting at the identity permutation and finishing at a canonical sylvester permutation are unions of sylvester classes.*

This property will be the main tool for constructing multiplicative bases.

4.5 Multiplicative bases

In their paper [23], Loday and Ronco build a multiplicative basis of **PBT** by associating with a tree, a function obtained by multiplying the $\mathbf{P}_T$'s obtained by cutting the right subtrees connected by right edges to the root of T.

However, in our setting, there is a more natural and general way to build multiplicative bases. Let us first fix the notation: a tree T is said to be smaller than a tree T', and we write $T < T'$ if $w_T < w_{T'}$ for the sylvester order.

Let T be a tree. The *complete* ($\mathbf{H}$) and *elementary* ($\mathbf{E}$) functions of **PBT** are respectively defined by

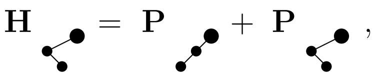
$$\mathbf{H}_T := \sum_{T' \leq T} \mathbf{P}_{T'}\,, \tag{46}$$

$$\mathbf{E}_T := \sum_{T' \geq T} \mathbf{P}_{T'}\,. \tag{47}$$

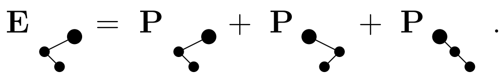
The names complete and elementary functions have been chosen on purpose: as we will see later (see Section 4.8), these are analogs of the homogeneous complete and elementary symmetric functions.

For example,

$$\mathbf{H}_{213} = \mathbf{P}_{123} + \mathbf{P}_{213} : \tag{48}$$



$$\mathbf{E}_{213} := \mathbf{P}_{213} + \mathbf{P}_{231} + \mathbf{P}_{321} : \tag{49}$$



More examples are given at the end of the paper. For the matrices $M_{\mathbf{H},\mathbf{P}}$, see Figures 14 and 15. For $M_{\mathbf{E},\mathbf{P}}$, see Figures 16 and 17. For $M_{\mathbf{E},\mathbf{H}}$, see Figures 18 and 19.

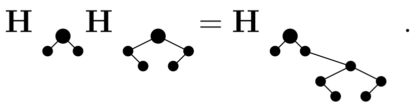
Theorem 29 *The basis of* $\mathbf{H}$ *functions is a multiplicative basis of* **PBT**, *whose product is given by*

$$\mathbf{H}_{T'}\mathbf{H}_{T''} = \mathbf{H}_T\,, \tag{50}$$

where the canonical word w_T *of* T *is obtained by concatenating* $w_{T''}[k]$ *with* $w_{T'}$ *where* k *is the size of* T'.

For example,

$$\mathbf{H}_{312}\mathbf{H}_{45213} = \mathbf{H}_{78564312} : \tag{51}$$



Notice that this operation coincides with the *over* operation defined by Loday-Ronco in [23]. It consists in grafting the tree T' on the right of the rightmost element of T''.

PROOF. As already pointed out in Note 28, the product in **FQSym** of an initial interval of $\mathfrak{S}_i$ by an initial interval of $\mathfrak{S}_{n-i}$ gives rise to an initial interval of $\mathfrak{S}_n$. This proves that the $\mathbf{H}$'s form a multiplicative basis. Now, the greatest possible element of this product is the word w_T defined in the theorem. $\square$

The same theorem holds for the $\mathbf{E}$'s, and the proof needs one small change: the smallest possible element of a given product is not a canonical word but can be easily rewritten as the one described in the next theorem. One can use the fact that sylvester classes are invariant through reversion of the alphabet, considering $(A,>)$ instead of $(A,<)$. Indeed, if one rewrites everything in terms of the smallest element of each sylvester class instead of the greatest one, the product of $\mathbf{E}_{T'}$ by $\mathbf{E}_{T''}$ is given by $\mathbf{E}_T$ where $w_T = w_{T'} \cdot w_{T''}[k]$.

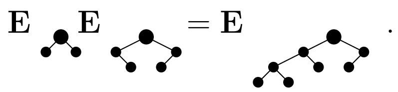
Theorem 30 *The basis of* $\mathbf{E}$ *functions is a multiplicative basis of* $\mathbf{PBT}$*, whose product is given by*

$$\mathbf{E}_{T'}\mathbf{E}_{T''} = \mathbf{E}_T \,, \tag{52}$$

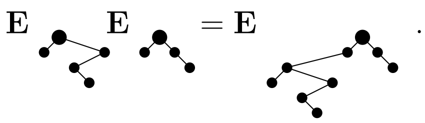
where T is obtained by connecting T' on the left of the left-most element of T''.

For example,

$$\mathbf{E}_{312}\mathbf{E}_{45213} = \mathbf{E}_{78531246} : \tag{53}$$



$$\mathbf{E}_{43512}\mathbf{E}_{4312} = \mathbf{E}_{984351267} : \tag{54}$$



Notice that this operation coincides with the *under* operation defined by Loday-Ronco in [23].

Note 11 The basis change from $\mathbf{E}$ to $\mathbf{H}$ has interesting properties: it sends the $\mathbf{E}$ trees with only left edges to their reversed tree on $\mathbf{H}$. Moreover, any $\mathbf{E}_T$ is an alternating sum of 2^k $\mathbf{H}$ functions, where k is the size of T minus the length of the left edge starting from the root of T.

4.6 $\mathbf{PBT}$ *and* $\mathbf{PBT}^*$ *as free algebras and isomorphic Hopf algebras*

As a consequence of the existence of multiplicative bases on $\mathbf{PBT}$ with a very simple product, $\mathbf{PBT}$ is free as an algebra (it is the algebra of a free monoid).

Before stating and proving this result, let us recall that a permutation σ is *connected* if it cannot be written as a shifted concatenation $\sigma = u \bullet v$, and *anticonnected* if its mirror image $\overline{\sigma}$ is connected.

Theorem 31 *The algebra* **PBT** *is free over the* $\mathbf{P}_T$*'s (or the* $\mathbf{H}_T$*'s) where* T *runs over trees whose root has no right son. In other words,* **PBT** *is free over the* $\mathbf{P}_T$*'s (or the* $\mathbf{H}_T$*'s) where* T *runs over trees whose canonical words are anticonnected permutations.*

PROOF. The two statements of the theorem are equivalent: by definition, a tree whose root has no right son has an anticonnected canonical word, and conversely. Now, thanks to Section 4.5, we know that the matrix that expresses $\mathbf{P}_T$ on the basis $\mathbf{H}_T$ is triangular with 1 on the main diagonal. Moreover, the statement is obvious on the $\mathbf{H}$'s thanks to their product formula: if σ is anticonnected, $\mathbf{H}_\sigma$ cannot be obtained by multiplication of smaller $\mathbf{H}$ elements. Conversely, if σ is not anticonnected, it can be written as $\sigma = u[k] \cdot v$ where u and v are canonical words, since canonical words are the words avoiding the pattern 132. □

Note 12 The same theorem is true with the $\mathbf{E}$'s instead of the $\mathbf{H}$'s. As already said, the product of the $\mathbf{E}$'s is the shifted concatenation of u and v written as $u \cdot v[k]$. This proves that **PBT** is free over the $\mathbf{P}_T$ where T runs over trees whose root has no left son.

Let us now move to $\mathbf{PBT}^*$. A few checks on small examples suggest that $\mathbf{PBT}^*$ also is free on anticonnected canonical words. We could apply the same techniques to prove that it is indeed the case but we will proceed in another way. If **PBT** and $\mathbf{PBT}^*$ are both free, they are isomorphic as algebras. We are going to prove that they are not only isomorphic as algebras but also as Hopf algebras that will, in particular, prove that $\mathbf{PBT}^*$ is a free algebra.

Let us first define a new basis in $\mathbf{PBT}^*$. Let T be a tree. $\mathbf{Q}'_T$ is defined as

$$\mathbf{Q}'_T := \sum_{\sigma;\ \mathcal{P}(\sigma)=T} \mathbf{Q}_{\mathcal{P}(\sigma^{-1})} . \tag{55}$$

For example, representing the trees by their canonical words:

$$\mathbf{Q}'_{231} = \mathbf{Q}_{312} \quad ; \quad \mathbf{Q}'_{312} = \mathbf{Q}_{231} + \mathbf{Q}_{312} , \tag{56}$$

$$\mathbf{Q}'_{4213} = \mathbf{Q}_{3241} + \mathbf{Q}_{3412} + \mathbf{Q}_{4213} , \tag{57}$$

$$\mathbf{Q}'_{54213} = \mathbf{Q}_{43521} + \mathbf{Q}_{45231} + \mathbf{Q}_{45312} + \mathbf{Q}_{53241} + \mathbf{Q}_{53412} + \mathbf{Q}_{54213} , \tag{58}$$

$$\mathbf{Q}'_{53412} = \mathbf{Q}_{45231} + \mathbf{Q}_{45312} + \mathbf{Q}_{52341} + \mathbf{Q}_{53241} + 2\,\mathbf{Q}_{53412} + \mathbf{Q}_{54123} + \mathbf{Q}_{54213} . \tag{59}$$

As one can observe on these examples, the smallest canonical word in the expression of $\mathbf{Q}'_\sigma$ as a sum of $\mathbf{Q}_\tau$ is σ^{-1}. This result shows that $\mathbf{Q}'_\sigma$ is a basis of $\mathbf{PBT}^*$.

Theorem 32 *The set $\mathbf{Q}'_T$ where T runs over the set of planar binary trees is a basis of $\mathbf{PBT}^*$. Moreover, the matrix $M_{\mathbf{Q}',\mathbf{Q}}$ is triangular for the right order:*

$$\mathbf{Q}'_\sigma = \mathbf{Q}_{\sigma^{-1}} + \sum \mathbf{Q}_\tau \,, \tag{60}$$

where τ runs over some set of canonical words greater than σ^{-1} for the lexicographic order.

Before completing the proof of the theorem, let us mention a simple but useful lemma:

Lemma 33 *Let σ be a permutation and T its decreasing tree. Consider the sequence of right sons starting from the root. The number of such right sons is given by the saillances of σ. Moreover, the length of the left subtrees attached to each son starting from the bottom-most one is given by the saillances sequence of σ.*

PROOF. The statement of the theorem is equivalent to the fact that, for any permutation, the inverse of its canonical word is smaller than or equal to the canonical word of its inverse, with equality iff the permutation is a canonical word. Translating these facts with decreasing trees leads to the following equivalent formulation: for any permutation σ, the canonical word of the unlabeled shape of the decreasing tree of its canonical word is smaller than or equal to the canonical word of the unlabeled shape of its decreasing tree with equality iff σ is a canonical word. Let us prove this result.

First, notice that if σ is of size n and ends with an n, the result is equivalent to the same statement for σ' obtained by removing n from σ. So we can assume that σ does not end with n. Moreover, it is obvious that the saillance sequence of σ is greater than or equal to the saillance sequence of its canonical word. So, if the saillance sequence of σ is different from the one of its canonical word, thanks to Lemma 33, its decreasing tree is strictly greater than the decreasing tree of its canonical word. If it is not the case, one restricts the permutation and its canonical word to each interval between two saillances and iterate. On the trees, this operation consists in computing the left subtrees associated with the right sons of the root starting from the bottom-most one. By induction, this proves the theorem. $\square$

Let us now define a linear map ϕ from **PBT** to **PBT*** by

$$\phi(\mathbf{P}_T) := \mathbf{Q}'_T. \tag{61}$$

Theorem 34 *The map ϕ induces a Hopf algebra isomorphism from* **PBT** *to* **PBT****. In other words, one has:*

$$\phi(\mathbf{P}_{T'}\mathbf{P}_{T''}) = \phi(\mathbf{P}_{T'})\phi(\mathbf{P}_{T''}) = \mathbf{Q}'_{T'}\mathbf{Q}''_{T''}\,, \tag{62}$$

$$(\phi \otimes \phi)(\Delta \mathbf{P}_T) = \Delta\phi(\mathbf{P}_T) = \Delta \mathbf{Q}'_T\,. \tag{63}$$

PROOF. First, ϕ is a bijection, since $\mathbf{Q}'_T$ is a basis of **PBT***. Now, ϕ is a composition of Hopf morphisms: it consists in the embedding of **PBT** in **FQSym** composed with the morphism that sends $\mathbf{F}_\sigma$ to $\mathbf{G}_{\sigma^{-1}}$ then composed with the morphism that sends $\mathbf{G}_\sigma$ to its equivalence sylvester class in **PBT***. So ϕ is a Hopf isomorphism and both Equations (62) and (63) hold. □

As a corollary, **PBT** and **PBT*** are isomorphic as algebras. Since **PBT** is a free algebra, the same is true of **PBT***.

Corollary 35 *The algebra* **PBT*** *is free over the functions $\mathbf{Q}_T$ (and $\mathbf{Q}'_T$) where T runs over trees whose root has no right son. The algebra* **PBT*** *is free over the functions $\mathbf{Q}_T$ (and $\mathbf{Q}'_T$) where T runs over trees whose root has no left son.*

Note 13 One can use the isomorphism to build multiplicative bases of **PBT***: they are the sums of $\mathbf{Q}'_T$ functions over upper or lower intervals of the Tamari lattice.

4.7 Primitive elements

It is well-known that the dual basis of a multiplicative basis restricted to indecomposable elements, is a basis of the Lie algebra of primitive elements of the dual. Since we have two mutiplicative bases on the **PBT** side, we then obtain two different bases of primitive elements on **PBT***. We could have worked out the multiplicative bases on the **PBT*** side but this would have been useless since we have an explicit isomorphism of **PBT** to **PBT***. We obtain in this way a description of the primitive elements which differs from that of [31].

Let us denote by $\mathbf{M}_T$ (resp. $\mathbf{N}_T$) the dual bases of the $\mathbf{H}_T$ (resp. $\mathbf{E}_T$). The basis $\mathbf{M}$ in an analog of the basis of monomial symmetric functions, whereas the basis $\mathbf{N}$ is an analog of the forgotten symmetric functions. The following results hold:

Theorem 36 *The Lie algebra of primitive elements of* $\mathbf{PBT}^*$ *is spanned by the* $\mathbf{M}_T$*'s where* T *runs over trees whose roots have no right son. The Lie algebra of primitive elements of* $\mathbf{PBT}^*$ *is spanned by the* $\mathbf{N}_T$*'s where* T *runs over trees whose roots have no left son.*

The first matrices $M_{\mathbf{M}_T,\mathbf{Q}_T}$ and $M_{\mathbf{N}_T,\mathbf{Q}_T}$ for trees up to 4 is respectively given in Figures 22–25.

Note 14 Since $M_{\mathbf{M}_T,\mathbf{Q}_T}$ is the transpose of $M_{\mathbf{P}_T,\mathbf{H}_T}$, the expression of $\mathbf{M}_T$ on $\mathbf{Q}_T$ is derived from the Möbius inversion of the Tamari lattice.

4.8 Embeddings and quotients

4.8.1 The full diagram of embeddings

In [23], Loday and Ronco defined different morphisms starting from or getting to **PBT**. These morphisms can be naturally understood and realized in our framework since we have non-commutative polynomial realizations of all of those: **FQSym**, **PBT**, and **Sym**. Indeed, the morphisms become trivial: all algebras are included in the same non-commutative polynomial algebra.

Let us first present the general diagram containing all these algebras and a few other ones (see Figure 6).

$$
\begin{tikzcd}
& \mathbf{FQSym} \arrow[d, shift left] & \\
& \mathbf{PBT}, \mathbf{FSym} \arrow[u, hook, shift left] \arrow[dl] & \\
\mathrm{QSym} & & \mathbf{Sym} \arrow[ul, hook] \arrow[dl] \\
& \mathit{sym} \arrow[ul, hook] &
\end{tikzcd}
$$

Fig. 6. Morphisms between known related Hopf algebras.

The algebra *sym* is the usual algebra of commutative Symmetric Functions.

As one can see on Figure 6, all these algebras are subalgebras or quotients of **FQSym**. As this algebra can be realized in the free algebra on an infinite alphabet, it is the same for all the other algebras. Then the up arrows just are inclusions and the down arrows are induced by commutation rules amoung the letters of the alphabet.

Let us describe in more detail two arrows:

- **Sym** to **PBT**: the algebra of non-commutative symmetric functions is generated by the homogeneous symmetric functions S_n which can be realized as the sum of all non-decreasing words of length n. As a polynomial, S_n is equal to the $\mathbf{P}_T = \mathbf{H}_T$ function where T is the tree with $n-1$ left edges. Since S_n and $\mathbf{H}_T$ are both multiplicative bases, it is obvious that one gets a realization of **Sym** inside **PBT** as the subalgebra generated by the $\mathbf{H}_T$ where T runs over the set of right comb trees. Moreover, the basis of elementary non-commutative symmetric functions Λ_n is realized as the sum of all decreasing words of size n that happens to be $\mathbf{E}_T = \mathbf{P}_T$, where T is the tree with $n-1$ right edges. Thus, the linear basis of Λ^I is realized as the $\mathbf{E}_T$ where T runs over the left comb trees.
- $\mathbf{PBT}^*$ to QSym: the algebra $\mathbf{PBT}^*$ is the specialization of $\mathbf{FQSym}^*$ to a "sylvester alphabet", an ordered alphabet satisfying the sylvester relations whereas QSym is the specialization of both $\mathbf{PBT}^*$ and $\mathbf{FQSym}^*$ to a commutative alphabet.

Note 15 Figure 6 is a commutative diagram. Indeed, both compound morphisms **Sym** $\to$ *sym* $\to$ QSym and **Sym** $\to$ **PBT** $\to$ **FQSym** $\to$ $\mathbf{FQSym}^*$ $\to$ $\mathbf{PBT}^*$ $\to$ QSym amount to compute commutative images of polynomials.

Note 16 The pair of Hopf algebras in duality **FSym** and $\mathbf{FSym}^*$ play the same role as **PBT** and $\mathbf{PBT}^*$. However, there exists a difference between these algebras: the compound morphism **PBT** $\to$ **FQSym** $\to$ $\mathbf{PBT}^*$ is a Hopf isomorphim whereas the compound map **FSym** $\to$ **FQSym** $\to$ $\mathbf{FSym}^*$ is not an isomorphism, since it is not even injective.

5 Representation theory

As mentioned in the introduction, it is known that the integers

$$c_{I,J} = \langle\, R_I, R_J \,\rangle \quad |I| = |J| = n\,, \tag{64}$$

where R_I stands for the ribbon-Schur function of shape I, can be interpreted as Cartan invariants of the 0-Hecke algebra: the coefficient $c_{I,J}$ is equal to the multiplicity of the simple module S_I in the indecomposable projective module P_J (see [18]).

The analogy between ribbon Schur functions and the natural basis $\mathbf{P}_T$ of **PBT** allows one to wonder whether one can interpret in the same way the integers

$$c_{T,U} = \langle\, \mathbf{P}_T, \mathbf{P}_U \,\rangle = \mathrm{Card}\{\sigma;\ \mathcal{P}(\sigma) = T,\ \mathcal{P}(\sigma^{-1}) = U\}. \tag{65}$$

Let $M^{(n)}$ be the matrix of the $c_{T,U}$ ordered in rows and columns by the lexicographic order of the canonical words. It suffices to compute $M^{(3)}$ (see Figure 7) to understand that this matrix cannot be a matrix of Cartan invariants: it has a 0 on the diagonal.

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & & 1 & \\ & & 1 & 1 & \\ & & & & 1 \end{pmatrix} \qquad \begin{pmatrix} 123 & & & & \\ & 213 & & & \\ & & & 231 & \\ & & 312 & 132 & \\ & & & & 321 \end{pmatrix} \tag{66}$$

Fig. 7. The matrix $M^{(3)}$ and the corresponding permutations.

Indeed, if one assumes that $\langle\, \mathbf{P}_T, \mathbf{P}_U \,\rangle = \dim \hom_{A_n}(\mathbf{P}_T, \mathbf{P}_U)$, or, equivalently if one assumes that the simple modules are indexed in such a way that $S_T = P_T/\mathrm{Rad}\ P_T$, each diagonal entry is at least 1 since $\hom_{A_n}(\mathbf{P}_T, \mathbf{P}_T)$ contains at least the identity map.

5.1 Combinatorial analysis of the scalar product

5.1.1 The Gram matrices

However, the Gram matrices $M^{(n)}$ have an interesting block structure. This leads to enquire whether there exists a simple transformation building a more interesting sequence of matrices. We already solved this question in Section 4.6 and more precisely in Theorem 32. Indeed, if one orders the rows of $M^{(n)}$ with the lexicographic order of their canonical words and the columns with the lexicographic order of the inverses of the canonical words, the matrix M becomes the matrix expressing the $\mathbf{Q}'_T$ on the $\mathbf{Q}_T$ basis.

We will now present the block structure in order to get the right order on trees and its interpretation in terms of the scalar product inherited from **FQSym**.

Let T be a planar binary tree. The *skeleton* of T is the pair of integers (k, l) defined by

- $k \leq n-1$ is the greatest integer such that $w_T(n) = n, \ldots, w_T(n-k+1) = n-k+1$, say, the number of fixed points at the end of w_T (minus 1 if w_T is the identity permutation),
- l is the number of saillances of σ_T, after one has removed its last k elements.

One can geometrically define the skeleton of a tree T as the part of T composed of the highest sequence of right sons and of vertices greater than this sequence. Figure 8 shows the skeleton of $(8, 9, 7, 5, 4, 6, 1, 2, 3, 10, 11)$. The number of fixed points at the end of the permutation (two in the example) corresponds to the size of the left edge minus 1, and the number of saillances of the remaining permutation (four in the example) corresponds to the size of the right edge.

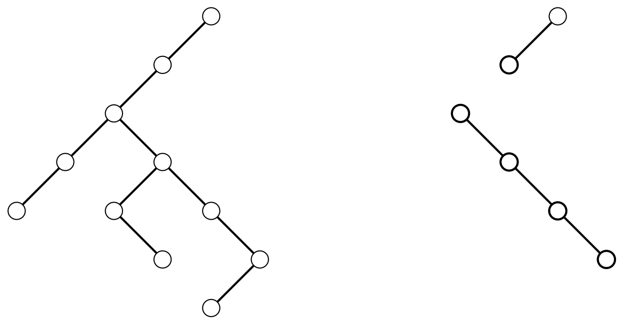


Fig. 8. A tree and its skeleton.

We can now describe the block structure of $M^{(n)}$: let us say that two trees T and U are *in the same block* if there exists a power of $M^{(n)}$ in which the coefficient (T, U) is nonzero. Notice that this relation is symmetric since $M^{(n)}$ is itself symmetric.

Theorem 37 *Two trees are in the same block iff they have the same size and the same skeleton.*

To prove this property, we will need a definition and two simple lemmas:

Definition 38 *Let σ and σ' be two permutations such that there exists three indices $i < j < k$ such that the restriction of both permutations to these indices are two words acb and bca with $a < b < c$. We say that σ and σ' are* co-sylvester adjancent*. This allows us to define the co-sylvester equivalence by transitive closure of the co-sylvester adjacence.*

Lemma 39 *Two permutations σ and σ' are co-sylvester-adjacent (respectively co-sylvester-equivalent) iff σ^{-1} and σ'^{-1} are sylvester-adjacent (resp. sylvester-congruent).*

Lemma 40 *The greatest word for the lexicographic order associated with a* *tree of size n of skeleton (k,l) is given by*

$$w_{k,l,n} := (n-k)\cdots(n-k-l+2)(n-k-l)\cdots 1.(n-k-l+1).(n-k+1)\cdots n. \tag{67}$$

For example, $w_{2,4,9} = 765321489$. There exists another description of w: given the skeleton, build a tree by attaching the tree of the correct size only composed of right edges to the left of the left-most node of the skeleton. Then w is the canonical word associated with this tree. It is a consequence of Lemma 33.

Let us now prove Theorem 37.

PROOF. By definition, two trees are in the same matrix block iff there is a path going from the first one to the second one consisting of pairs of trees, the first one being the binary tree of a permutation and the second one being the binary tree of its inverse. It is obvious that both binary trees have same skeleton since σ and σ^{-1} have the same number of fixed points at the end and the same number of saillances after having removed the previous fixed points.

Conversely, let us prove that all trees with a given skeleton are connected. Thanks to Theorem 32, the permutations corresponding to a given row of $M^{(n)}$ are sylvester-congruent whereas the permutations corresponding to a given column are co-sylvester-equivalent. Consider a skeleton (k,l). Then proving that all trees are connected is equivalent to prove that all permutations of size n having skeleton (k,l) are connected using both sylvester and co-sylvester relations. This will come from the fact that they all are connected to $w_{k,l,n}$.

By induction, we can restrict to the case $k=0$ and to canonical sylvester permutations. Let σ be a canonical permutation of size n and skeleton $(0,l)$. If σ does not begin with n then exchange the two neighboors of n by a co-sylvester rewriting and take the sylvester rewriting on these three elements. This permutation is greater than σ so is its canonical word. Iterating this process, one ends with a permutation beginning with n. If σ begins with n then by induction on n, it is connected to $w_{k,l,n}$. So all permutations of size n and skeleton (k,l) are connected to $w_{k,l,n}$. □

The proof of the next theorem directly follows from Theorem 32.

Theorem 41 *Let ν be the involution on trees defined as $\nu(T) = T'$, where $T' = \mathcal{P}(w_T^{-1})$. Then*

- *the involution ν preserves the blocks: (T and $\nu(T)$ have same skeleton),*
- *the matrix $C^{(n)}$ defined by*

$$C^{(n)}(T,U) = \langle\, \mathbf{P}_T, \mathbf{P}_{\nu(U)}\,\rangle \tag{68}$$

is block lower unitriangular if one orders the trees, first by skeleton, then by lexicographic order on the canonical words of each skeleton class of trees.

Figure 9 contains the first matrices $C^{(n)}$, skipping the zero entries to allow instantaneous reading. The order of the trees in rows and columns corresponds to the lexicographic order on their canonical words:

- 12; 21,
- 123; 213; 231, 312; 321,
- 1234; 2134; 2314, 3124; 3214; 2341, 3241, 3412, 4123, 4213; 3421, 4231, 4312; 4321.

$$
\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}
\quad
\begin{pmatrix}
1 &&&& \\
& 1 &&& \\
&& 1 && \\
&& 1 & 1 & \\
&&&& 1
\end{pmatrix}
\quad
\begin{pmatrix}
1 &&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&& \\
&& 1 &&&&&&&&&&& \\
&& 1 & 1 &&&&&&&&&& \\
&&&& 1 &&&&&&&&& \\
&&&&& 1 &&&&&&&& \\
&&&&&& 1 &&&&&&& \\
&&&&& 1 & 1 & 1 &&&&&& \\
&&&&& 1 && 1 & 1 &&&&& \\
&&&&&& 1 & 1 && 1 &&&& \\
&&&&&&&&&& 1 &&& \\
&&&&&&&&&& 1 & 1 && \\
&&&&&&&&&& 1 & 1 & 1 & \\
&&&&&&&&&&&&& 1
\end{pmatrix}
$$

Fig. 9. The matrices $C^{(n)}$ for $n = 2, 3, 4$.

5.1.2 Combinatorics of the Gram matrices

Let us now study more precisely the block structure of our matrices. We first need a few classical definitions. Define the *Catalan triangle* (see [33]) (resp. the *first kind Stirling triangle*) as the triangular matrix A (resp. B) whose coefficient $a_{i,j}$ (resp. $b_{i,j}$) is the coefficient of $t^i u^j$ in the respective expressions:

$$\sum_{n=1}^{\infty} t^n \sum_{k=0}^{n-1} \frac{n-k}{n+k} \binom{n+k}{k} u^k, \qquad \sum_{n=1}^{\infty} t^n \prod_{k=1}^{n-1} (1 + k\ u). \tag{69}$$

The well-known combinatorial interpretation of these numbers is the following: $a_{i,j}$ is the number of planar binary trees of size i whose number of elements not belonging to the sequence of right sons starting from the root is $j-1$. The coefficient $b_{i,j}$ is the number of permutations of size i whose saillances number is $i-j+1$. Both triangles are represented Figure 10 where we put into parenthesis the trees having no right son at the root, and the permutations having only one saillance. Notice that the same Catalan triangle has been encountered by Aval-Bergeron-Bergeron in [1] while studying the quotient of the algebra of polynomials by the ideal generated by quasi-symmetric polynomials without constant term. The relation between both constructions remains mysterious.

1								1						
1	(1)							1	(1)					
1	2	(2)						1	3	(2)				
1	3	5	(5)					1	6	11	(6)			
1	4	9	14	(14)				1	10	35	50	(24)		
1	5	14	28	42	(42)			1	15	85	225	274	(120)	
1	6	20	48	90	132	(132)		1	21	175	735	1624	1764	(720)

Fig. 10. The Catalan and first kind Stirling triangles.

Theorem 42 *Let us consider a block of skeleton (k,l) of $C^{(n)}$ for a given n.*

- *The number $m_n(k,l)$ of rows of this block is given by the $(n-k-l+1)$-th number of the $(n-k)$-th row of the Catalan triangle.*
- *The sum $d_n(k,l)$ of the entries of this block is given by the $(n-k-l-1)$-th number of the $(n-k)$-th row of the first kind Stirling triangle.*

5.2 A conjectural representation theoretical interpretation

Definition 43 *A* tower of algebras *is a pair $((A_n)_{n\in\mathbb{N}},\ (\rho_{i,j})_{i,j\in\mathbb{N}})$ where the A_n's are algebras, and for all i,j the map $\rho_{i,j}$ is an algebra embedding of $A_i\otimes A_j$ into A_{i+j} such that*

$$\rho_{i+j,k}\circ(\rho_{i,j}\otimes \mathrm{Id}_{A_k}) = \rho_{i,j+k}\circ(\mathrm{Id}_{A_i}\otimes\rho_{j,k})\,. \tag{70}$$

Notice that Equation (70) amounts to require that the direct sum of the maps $(x,y)\mapsto\rho_{i,j}(x\otimes y)$ if $x\in A_i$ and $y\in A_j$ defines an associative product on the direct sum $\bigoplus A_i$ which is compatible with the structure of the A_i.

For any tower of algebras, the induction process with respect to the embeddings $\rho_{i,j}$ defines an algebra structure on the direct sums of the Grothendieck groups

$$\mathcal{G}=\bigoplus_{n\geq 0} G_0(A_n)\,,\quad \mathcal{K}=\bigoplus_{n\geq 0} K_0(A_n). \tag{71}$$

Similarly, the restriction defines a coproduct in such a way that $\mathcal{G}$ and $\mathcal{K}$ are equipped with two mutually dual Hopf algebra structures.

An example of a tower of algebras is the tower of algebras of the symmetric groups together with the linear maps extending the group inclusions $\mathfrak{S}_i \times \mathfrak{S}_j \mapsto \mathfrak{S}_{i+j}$. It is well-known that this leads to the self-dual Hopf algebra of symmetric functions (see [10]). Replacing the symmetric group by its degenerated Hecke algebra leads to the dual pair $(\mathrm{QSym}, \mathbf{Sym})$. And it is likely that the pair $(\mathbf{PBT}, \mathbf{PBT}^*)$ comes from a similar construction.

For any tower of algebras (A_n) having such Gram matrices as Cartan invariants, the skeletons (k, l) correspond to the blocks (the indecomposable subalgebras) $B_n(k, l)$ of A_n. Then, $d_n(k, l)$ is the dimension of $B_n(k, l)$, and $m_n(k, l)$ is the number of its simple modules.

Notice that there might *a priori* exist many non isomorphic towers of algebras such that the $C^{(n)}$ matrices are their Cartan invariants.

We computed the quivers corresponding to each block of the matrix for $n \leq 6$ with the constraint of providing the smallest possible amount of arrows (equivalently the smallest possible amount of relations). The structure of these quivers and their relations seem to have a certain regularity but they unfortunately remain unsufficiently understood to allow us to describe these algebras for any n. Nevertheless, we conjecture the following result:

Conjecture 44 *There exists a tower of algebras (A_n) such that the $C^{(n)}$ are their matrices of Cartan invariants.*

In particular, one should have $\dim A_n = n!$. We can also propose a more precise conjecture.

Conjecture 45 *There exists a tower of algebras A_n, with a basis $(e_\sigma)_{\sigma \in \mathfrak{S}_n}$ such that:*

- *the restriction to canonical words of the morphism $\rho_{m,n}$ is given by the product of the corresponding $\mathbf{P}_T$ functions. In this setting, the indecomposable projective modules of A_n are left ideals $P_T = A_n e_{\sigma_T}$, and therefore are in bijection with the planar binary trees of size n.*
- *If one endows $\mathcal{K}$ with the induction product $[M] \cdot [N] = [M \otimes_{\mathbb{C}} N \uparrow_{A_m \otimes A_n}^{A_{m+n}}]$, the map $\mathcal{K} \to \mathbf{PBT}$ sending the class of the module P_T on the polynomial $\mathbf{P}_T$ is a ring isomorphism.*

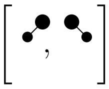
6 Conclusion

Since its discovery in the mid-seventies, the plactic monoid has, for a long time, been considered as a very singular object. It needed the discovery of quantum groups (independently due to Drinfeld and Jimbo about 1985) (see [14]), and

Kashiwara's theory of crystal bases (1991) (see [15]), to discover the plactic monoids associated with all semi-simple Lie algebras (see [19,22]). But even this point of view does not tell everything about plactic monoids. The hypoplactic monoid (see [18]), which is to quasi-symmetric functions what the ordinary plactic monoid is to ordinary symmetric functions, was obtained from a non standard version of the quantum linear group, and is not taken into account by the theory of crystal bases. This raises a first question, to find a quantum group interpretation of the sylvester monoid, and a second one, to characterize and classify all similar monoids.

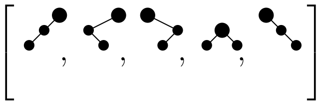
7 Tables

In this Section, we give the transition matrices between various bases in degree $n \leq 4$. Rows and columns of those matrices correspond to binary trees on n nodes arranged as follows:



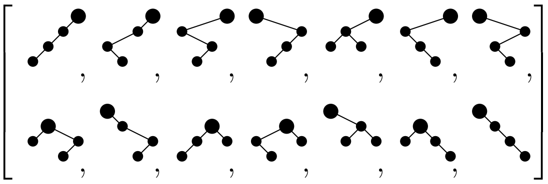
(72)

Fig. 11. Order on trees of size 2.



(73)

Fig. 12. Order on trees of size 3.



(74)

Fig. 13. Order on trees of size 4.

These orders correspond to the lexicographic order on canonical words:

$$[12, 21]; \qquad [123, 213, 231, 312, 321]; \tag{75}$$

$$\begin{bmatrix} 1234, 2134, 2314, 2341, 3124, 3214, 3241, \\ 3412, 3421, 4123, 4213, 4231, 4312, 4321\,. \end{bmatrix} \tag{76}$$

Now, let us give the matrices $M_{\mathbf{H},\mathbf{P}}$, $M_{\mathbf{E},\mathbf{P}}$, $M_{\mathbf{E},\mathbf{H}}$, $M_{\mathbf{Q}',\mathbf{Q}}$, $M_{\mathbf{M},\mathbf{Q}}$, and finally $M_{\mathbf{N},\mathbf{Q}}$ for $n = 2$, 3 and 4.

Notice that the matrix $M_{\mathbf{M},\mathbf{Q}}$ is the transpose of the inverse of $M_{\mathbf{H},\mathbf{P}}$. It is the same with $M_{\mathbf{N},\mathbf{Q}}$, that is the transpose of the inverse of $M_{\mathbf{E},\mathbf{P}}$.

$$\begin{pmatrix} 1 & 1 \\ . & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & . & 1 \\ . & . & 1 & . & 1 \\ . & . & . & 1 & 1 \\ . & . & . & . & 1 \end{pmatrix}$$

Fig. 14. The matrices $M_{\mathbf{H}_T,\mathbf{P}_T}$ for $n = 2, 3$.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
. & 1 & 1 & 1 & . & 1 & 1 & . & 1 & . & 1 & 1 & . & 1 \\
. & . & 1 & 1 & . & 1 & 1 & . & 1 & . & . & 1 & . & 1 \\
. & . & . & 1 & . & . & 1 & . & 1 & . & . & 1 & . & 1 \\
. & . & . & . & 1 & 1 & 1 & 1 & 1 & . & . & . & 1 & 1 \\
. & . & . & . & . & 1 & 1 & . & 1 & . & . & . & . & 1 \\
. & . & . & . & . & . & 1 & . & 1 & . & . & . & . & 1 \\
. & . & . & . & . & . & . & 1 & 1 & . & . & . & 1 & 1 \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & 1 \\
. & . & . & . & . & . & . & . & . & 1 & 1 & 1 & 1 & 1 \\
. & . & . & . & . & . & . & . & . & . & 1 & 1 & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}$$

Fig. 15. The matrix $M_{\mathbf{H}_T,\mathbf{P}_T}$ for $n = 4$.

$$\begin{pmatrix} 1 & . \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 1 & 1 & . & . \\ 1 & . & . & 1 & . \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Fig. 16. The matrices $M_{\mathbf{E}_T,\mathbf{P}_T}$ for $n = 2, 3$.

$$\begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . \\
1 & 1 & 1 & 1 & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\
1 & 1 & 1 & . & 1 & 1 & . & . & . & . & . & . & . & . \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & . & . & . & . & . & . & . \\
1 & . & . & . & 1 & . & . & 1 & . & . & . & . & . & . \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & . & . & . & . & . \\
1 & . & . & . & . & . & . & . & . & 1 & . & . & . & . \\
1 & 1 & . & . & . & . & . & . & . & 1 & 1 & . & . & . \\
1 & 1 & 1 & 1 & . & . & . & . & . & 1 & 1 & 1 & . & . \\
1 & . & . & . & 1 & . & . & 1 & . & 1 & . & . & 1 & . \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}$$

Fig. 17. The matrix $M_{\mathbf{E}_T,\mathbf{P}_T}$ for $n = 4$.

$$
\begin{pmatrix}
. & -1 \\
1 & 1
\end{pmatrix}
\qquad
\begin{pmatrix}
. & . & 1 & . & 1 \\
. & . & -1 & . & . \\
. & . & . & -1 & -1 \\
. & -1 & -1 & . & -1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

Fig. 18. The matrices $M_{\mathbf{E}_T,\mathbf{H}_T}$ for $n = 2, 3$.

$$
\begin{pmatrix}
. & . & . & -1 & . & . & -1 & . & -1 & . & . & -1 & . & -1 \\
. & . & . & 1 & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & 1 & . & . & . & 1 & 1 \\
. & . & . & 1 & . & . & 1 & . & 1 & . & . & . & . & . \\
. & . & . & -1 & . & . & -1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & -1 & -1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & 1 & . & 1 \\
. & . & . & . & . & . & . & . & . & -1 & -1 & -1 & -1 & -1 \\
. & . & 1 & 1 & . & 1 & 1 & . & 1 & . & . & 1 & . & 1 \\
. & . & -1 & -1 & . & . & . & . & . & . & . & -1 & . & . \\
. & . & . & . & -1 & -1 & -1 & -1 & -1 & . & . & . & -1 & -1 \\
. & -1 & -1 & -1 & . & -1 & -1 & . & -1 & . & -1 & -1 & . & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

Fig. 19. The matrix $M_{\mathbf{E}_T,\mathbf{H}_T}$ for $n = 4$.

$$
\begin{pmatrix} 1 & . \\ . & 1 \end{pmatrix}
\qquad\qquad
\begin{pmatrix}
1 & . & . & . & . \\
. & 1 & . & . & . \\
. & . & . & 1 & . \\
. & . & 1 & 1 & . \\
. & . & . & . & 1
\end{pmatrix}
$$

Fig. 20. The matrices $M_{\mathbf{Q}'_T,\mathbf{Q}_T}$ for $n = 2, 3$.

$$
\begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & 1 & . & 1 & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & 1 & . & . & . & 1 & . & 1 & . & . & . & . \\
. & . & . & . & . & . & 1 & 1 & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . \\
. & . & . & . & . & . & . & . & 1 & . & . & 1 & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}
$$

Fig. 21. The matrix $M_{\mathbf{Q}'_T,\mathbf{Q}_T}$ for $n = 4$.

$$\begin{pmatrix} 1 & . \\ -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & . & . & . & . \\ -1 & 1 & . & . & . \\ . & -1 & 1 & . & . \\ -1 & . & . & 1 & . \\ 1 & . & -1 & -1 & 1 \end{pmatrix}$$

Fig. 22. The matrices $M_{\mathbf{M}_T,\mathbf{Q}_T}$ for $n = 2, 3$.

$$\begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
-1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & -1 & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & -1 & 1 & . & . & . & . & . & . & . & . & . & . \\
-1 & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\
1 & . & -1 & . & -1 & 1 & . & . & . & . & . & . & . & . \\
. & . & 1 & -1 & . & -1 & 1 & . & . & . & . & . & . & . \\
. & . & . & . & -1 & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & 1 & . & -1 & -1 & 1 & . & . & . & . & . \\
-1 & . & . & . & . & . & . & . & . & 1 & . & . & . & . \\
1 & -1 & . & . & . & . & . & . & . & -1 & 1 & . & . & . \\
. & 1 & . & -1 & . & . & . & . & . & . & -1 & 1 & . & . \\
1 & . & . & . & . & . & . & -1 & . & -1 & . & . & 1 & . \\
-1 & . & . & 1 & . & . & . & 1 & -1 & 1 & . & -1 & -1 & 1
\end{pmatrix}$$

Fig. 23. The matrix $M_{\mathbf{M}_T,\mathbf{Q}_T}$ for $n = 4$.

$$\begin{pmatrix} 1 & -1 \\ . & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & . & -1 & 1 \\ . & 1 & -1 & . & . \\ . & . & 1 & . & -1 \\ . & . & . & 1 & -1 \\ . & . & . & . & 1 \end{pmatrix}$$

Fig. 24. The matrices $M_{\mathbf{N}_T,\mathbf{Q}_T}$ for $n = 2, 3$.

$$\begin{pmatrix}
1 & -1 & . & . & -1 & 1 & . & . & . & -1 & 1 & . & 1 & -1 \\
. & 1 & -1 & . & . & . & . & . & . & . & -1 & 1 & . & . \\
. & . & 1 & -1 & . & -1 & 1 & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & -1 & . & . & . & . & -1 & . & 1 \\
. & . & . & . & 1 & -1 & . & -1 & 1 & . & . & . & . & . \\
. & . & . & . & . & 1 & -1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & -1 & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & -1 & . & . & . & -1 & 1 \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & -1 \\
. & . & . & . & . & . & . & . & . & 1 & -1 & . & -1 & 1 \\
. & . & . & . & . & . & . & . & . & . & 1 & -1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & -1 \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & -1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}$$

Fig. 25. The matrix $M_{\mathbf{N}_T,\mathbf{Q}_T}$ for $n = 4$.

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