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The peak algebra and the Hecke–Clifford algebras at $q = 0$

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Abstract

Using the formalism of noncommutative symmetric functions, we derive the basic theory of the peak algebra of symmetric groups and of its graded Hopf dual. Our main result is to provide a representation theoretical interpretation of the peak algebra and its graded dual as Grothendieck rings of the tower of Hecke–Clifford algebras at $q = 0$.

Keywords: Peak algebra; Noncommutative Symmetric Functions; Hecke–Clifford algebra; Representations; Grothendieck rings

1. Introduction

Studies on the combinatorics of descents in permutations led to the discovery of a pair, $(QSym, \mathbf{Sym})$, of mutually dual graded Hopf algebras [8,9,17]. Here, $QSym$ is the graded Hopf algebra of quasi-symmetric functions, and its graded dual, $\mathbf{Sym}$, is the graded Hopf algebra of noncommutative symmetric functions. Recent investigations on the combinatorics of peaks in permutations resulted in the

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discovery of an interesting new pair, (Peak, Peak*), of graded Hopf algebras. The first one, Peak, originally due to Stembridge [22], is a subalgebra of *QSym*. As described in [3], its graded dual, Peak*, can therefore be identified as a homomorphic image of **Sym**. We shall see in the following that the existence of Peak* as well as many of its basic properties were already implicit in [13].

It is known that Peak can also be obtained as a quotient of *QSym*, in which case Peak* is realized as a subalgebra of **Sym**. On the other hand, each homogeneous component $\mathbf{Sym}_n$ of **Sym** is endowed with another multiplication, the internal product $*$, such that the resulting algebra is anti-isomorphic to Solomon's descent algebra of the symmetric group $\mathfrak{S}_n$. At this stage, a natural question arises. Is Peak_n^* stable under this operation? As shown in [17], the answer is yes (it is even a left ideal of $\mathbf{Sym}_n$), and the corresponding right ideal of the descent algebra is spanned by the sums of permutations having a given peak set. Recent developments [1,2,5,20] unveil many interesting properties and generalizations of Peak and Peak*. Most notably, we find in [2] that Peak is the terminal object in the category of combinatorial Hopf algebras satisfying generalized Dehn–Somerville relations. This reveals some of the significance of Peak and Peak*. Our main result demonstrates yet another facet of the importance of these graded Hopf algebras.

We shall start our presentation by showing that many of the basic results in the literature related to Peak and Peak* can be recovered in a very elegant and straightforward way by relying upon the techniques developed in [13]. This will be covered in Sections 2–4.

It is known that the dual pair of Hopf algebras (*QSym*, **Sym**) describes the representation theory of the 0-Hecke algebras of type A [14]. More precisely, *QSym* and **Sym** are, respectively, isomorphic to the direct sums of the Grothendieck groups $G_0(H_n(0))$ and $K_0(H_n(0))$. We provide here a similar interpretation for the pair (Peak, Peak*). This is done by replacing the Hecke algebras with the so-called Hecke–Clifford algebras, discovered by Olshanski [18]. This new result is presented in Section 5.

Our presentation is as self-contained as possible, but we encourage the diligent reader to be familiar with the content of [8,13].

2. The $(1-q)$-transform at $q=-1$

The main motivation for Stembridge's theory of enriched P-partitions, which led him to the quasi-symmetric peak algebra [22], was the study of the quasi-symmetric expansions of Schur's Q-functions [15,21]. As is well known, these symmetric functions correspond to the Hall–Littlewood functions with parameter $q=-1$. The peak algebra is therefore directly related to what we will call the "$(1-q)$-transform" at $q=-1$.

For our presentation, let *Sym* denote the graded Hopf algebra of (commutative) symmetric functions. There are several well-known bases for *Sym* [15]. It is algebraically generated by primitive elements, the power sums $\{p_n\}_{n\geqslant 1}$ where

$deg(p_n) = n$. In other words, the elements of *Sym* are polynomials in the power sums. Two other important sets of algebraic generators for *Sym* are the complete symmetric functions $\{h_n\}_{n\geqslant 1}$ and the elementary symmetric functions $\{e_n\}_{n\geqslant 1}$. It is often convenient to express these functions as series in a commutative alphabet X. That is, for a totally ordered countable set $X = \{x_1, x_2, \dots\}$ of commutative variables, we define $h_n(X)$ and $e_n(X)$ as the coefficients of t^n in

$$H_t(X) = \prod_{i\geqslant 1} \frac{1}{1-tx_i} \quad \text{and} \quad E_t(X) = \frac{1}{H_{-t}(X)},$$

respectively. The power sum $p_n(X)$ are then obtained as the coefficient of $\frac{1}{n}t^n$ in $P_t(X) = \log(H_t(X))$. Explicitly, that gives $p_n(X) = \sum_{i\geqslant 1} x_i^n$,

$$h_n(X) = \sum_{1\leqslant i_1\leqslant i_2\leqslant \cdots \leqslant i_n} x_{i_1}x_{i_2}\cdots x_{i_n} \quad \text{and} \quad e_n(X) = \sum_{1\leqslant i_1 < i_2 < \cdots < i_n} x_{i_1}x_{i_2}\cdots x_{i_n}.$$

In this classical setting, the $(1-q)$-transform θ_q is the algebra endomorphism of *Sym* defined on the power sums by $\theta_q(p_n) = (1-q^n)p_n$. In λ-ring notation, which is particularly convenient for dealing with such transformations, it reads $f(X) \mapsto f((1-q)X)$. One has to pay attention to the abuse of notation in using the same minus sign for the λ-ring and for scalars, though these operations are quite different. That is, θ_{-1} maps p_n to $2p_n$ if n is odd, and to 0 otherwise. Thus, $\theta_{-1}(f(X)) = f((1-q)X)_{q=-1}$ is not the same as $f((1+1)X) = f(2X)$.

The main results of [13] are concerned with the extension of the $(1-q)$-transform to the graded Hopf algebra **Sym** of noncommutative symmetric functions. As a noncommutative algebra, **Sym**, is freely generated by the noncommutative complete symmetric functions $\{S_n\}_{n\geqslant 1}$ where $deg(S_n) = n$. The comultiplication structure is given by $S_n \mapsto \sum_{i=0}^n S_i \otimes S_{n-i}$ with the convention that $S_0 = 1$. This algebra can be represented using series in a noncommutative alphabet A. More precisely, for a totally ordered countable set $A = \{a_1, a_2, \dots\}$ of noncommutative variables, we define $S_n(A)$ as the coefficient of t^n in the expression

$$\sigma_t(A) = \overrightarrow{\prod_{i\geqslant 1}} \frac{1}{1-ta_i} = \frac{1}{1-ta_1}\frac{1}{1-ta_2}\cdots,$$

where the parameter t commutes with all variables and the (noncommutative) product is taken in the natural order of the variables. The abelianization map χ: **Sym** $\to$ *Sym* which sends the noncommutative alphabet A to the commutative alphabet X is a Hopf homomorphism. In [13], we are interested in defining a $(1-q)$-transform on **Sym** which commutes with χ. A consistent definition of $\theta_q(F) = F((1-q)A)$ is proposed, and its fundamental properties are obtained. We briefly recall here the necessary steps. One first defines the complete symmetric functions $S_n((1-q)A)$ via their generating series [13, Definition 5.1]

$$\sigma_t((1-q)A) \coloneqq \sum_{n\geqslant 0} t^n S_n((1-q)A) = \sigma_{-qt}(A)^{-1}\sigma_t(A), \tag{1}$$

and then θ_q is defined as the ring homomorphism such that $\theta_q(S_n) = S_n((1-q)A)$.

To have a better understanding of the morphism θ_q we need to recall more facts about **Sym**. Given a sequence $(F_n)_{n\geqslant 1}$ of noncommutative symmetric functions and a composition $I=(i_1,i_2,\dots,i_r)$, we set $F^I=F_{i_1}F_{i_2}\cdots F_{i_r}$. By definition, the set $\{S^I\}$, where I runs over all compositions, is a homogeneous linear basis of **Sym**. For $I=(i_1,i_1,\dots,i_r)$, let $\ell(I)=r$ and given two compositions I and J we say that $J\leqslant I$ if I is a refinement of J. Also, for a composition $I=(i_1,i_2,\dots,i_r)$ of $n=i_1+i_2+\cdots+i_r$, let $Des(I)=\{i_1,i_1+i_1,\dots,i_1+\cdots+i_{r-1}\}\subseteq\{1,2,\dots,n-1\}$ denote the *descent* set of I. We define the *ribbon* noncommutative functions $R_I=\sum_{J\leqslant I}(-1)^{\ell(I)-\ell(J)}S^J$. Clearly, the set of all ribbon functions $\{R_I\}$ forms a linear basis of **Sym**. Consider now the algebra $\mathfrak{S}=\oplus_{n\geqslant 0}\mathbb{C}\mathfrak{S}_n$ where $\mathbb{C}\mathfrak{S}_n$ is the group algebra of the symmetric group $\mathfrak{S}_n$ on n elements. As seen in [13, Section 2.2], there is a linear isomorphism $\alpha^{-1}=\beta$: $\mathbf{Sym}\to\mathfrak{S}$ such that $\beta(R_I)=D_I=\{w\in\mathfrak{S}_n\mid w(i)>w(i+1)\Leftrightarrow i\in Des(I)\}$. The image of β is know as the Solomon descent algebra and is closed under composition of permutation in $\mathbb{C}\mathfrak{S}_n$. We define the *internal product* $*$ of **Sym** as the anti-pullback of the composition of permutations in $\mathbb{C}\mathfrak{S}_n$. That is $F*G=\alpha(\beta(G)\circ\beta(F))$. Specializing [13, Theorem 4.17] to our definition of the morphism θ_q, we obtain

$$F((1-q)A)=F(A)*\sigma_1((1-q)A). \tag{2}$$

The most important property of θ_q is its diagonalization [13, Theorem 5.14]: there is a unique family of Lie idempotents $\pi_n(q)$ (i.e., elements in the primitive Lie algebra such that $\chi(\pi_n(q))=\frac{1}{n}p_n$) with the property

$$\theta_q(\pi_n(q))=(1-q^n)\pi_n(q). \tag{3}$$

Moreover, θ_q is semi-simple, and its eigenvalues in the nth homogeneous components $\mathbf{Sym}_n$ of **Sym** are $p_\lambda(1-q)=\prod_i(1-q^{\lambda_i})$, where λ runs over the partitions of n. The projectors on the corresponding eigenspaces are the maps $F\mapsto F*\pi^I(q)$ [13, Section 3.4].

Another result [13, Section 5.6.4], which is just a translation of an important formula due to Blessenohl and Laue [6], gives $\theta_q(R_I)$ in closed form for any ribbon R_I. To be more in line with the current literature, we digress slightly from the notation of [8]. Let $[i,j]=\{i,i+1,i+2,\dots,j\}$. We let $A\Delta B=(A-B)\cup(B-A)$ be the symmetric difference of two sets. Given $A=\{a_1,a_2,\dots,a_{r-1}\}\subseteq[1,n-1]$ we let $A+1=\{a_1+1,a_2+1,\dots,a_{r-1}+1\}\subseteq[2,n]$. For a composition J of n, one defines $HP(J)=\{a\in Des(J)\mid a\neq 1, a-1\notin Des(J)\}\subseteq[2,n-1]$, and $hl(J)=|HP(J)|+1$. One usually refers to $HP(J)$ as the *peak* set of J. We are now in a position to give the formula for $\theta_q(R_I)$ [13, Lemma 5.38 and Proposition 5.41]:

$$R_I((1-q)A)=\sum_{HP(J)\subseteq Des(I)\Delta(Des(I)+1)}(1-q)^{hl(J)}(-q)^{b(I,J)}R_J(A), \tag{4}$$

where $b(I,J)$ is some explicit integer, which is not of any use when $q=-1$.

Setting $q=-1$ in the formulas above leads us immediately to the peak classes in **Sym**. We say that a set $P\subseteq[2,n-1]$ is a *peak set* when $a\in P\Rightarrow a-1\notin P$. For a peak set P let

$$\Pi_P = \sum_{HP(I)=P} R_I. \tag{5}$$

At $q=-1$, Eq. (4) now reads as

$$\theta_{-1}(R_I) = \sum_{P \subseteq Des(I) \Delta (Des(I)+1)} 2^{|P|+1} \Pi_P, \tag{6}$$

which is [20, Proposition 5.5] or [1, Proposition 5.8].

Let us denote for short θ_{-1} by a tilde, $\tilde{F} := \theta_{-1}(F)$, and let $\widetilde{\mathbf{Sym}}$ be its image. Since by definition

$$\widetilde{\mathbf{Sym}} = \{F((1-q)A)_{q=-1}\}, \tag{7}$$

it is immediate that $\widetilde{\mathbf{Sym}}$ is a graded Hopf subalgebra of **Sym**. Indeed, $F \mapsto F((1-q)A)$ is an algebra morphism, and also a coalgebra morphism, since [13, Section 5.1]

$$\Delta S_n((1-q)A) = \sum_{i+j=n} S_i((1-q)A) \otimes S_j((1-q)A) \tag{8}$$

for all values of q. Also, it is a left ideal for the internal product, since by Eq. (2)

$$\widetilde{\mathbf{Sym}} = \mathbf{Sym}(A) * \sigma_1((1-q)A)_{q=-1}. \tag{9}$$

We already know that $\widetilde{\mathbf{Sym}}$ is contained in the subspace $\mathscr{P}$ of **Sym** spanned by $\{\Pi_P\}$. The dimension of the homogeneous component $\mathscr{P}_n$ of $\mathscr{P}$ is easily seen to be equal to the Fibonacci number f_n (with the convention $f_0 = f_1 = f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$ for $n>0$). Indeed, the set $\{P \subseteq [2, n-1] \mid a \in P \Rightarrow a-1 \notin P\}$ has cardinality f_n. Remark that the number of compositions of n into odd parts is also f_n.

But, thanks to Eq. (3), we know that the elements

$$\pi^I(-1) = \pi_{i_1}(-1)\pi_{i_2}(-1)\cdots\pi_{i_r}(-1), \tag{10}$$

where $i = (i_1, \ldots, i_r)$ runs over compositions of n into odd parts, form a basis of $\widetilde{\mathbf{Sym}}_n$. Hence,

$$\widetilde{\mathbf{Sym}} = \mathscr{P}_n. \tag{11}$$

Also, since the commutative image of $\pi_n(q)$ is $\frac{1}{n}p_n$ for all q, this makes it clear that the commutative image of $\widetilde{\mathbf{Sym}}$ is the subalgebra of *Sym* generated by odd power-sums p_{2k+1}.

To summarize, we have shown that the peak classes Π_P in **Sym** form a linear basis of a graded Hopf subalgebra $\mathscr{P}$ of **Sym**, which is also a left ideal for the internal product, and we have described a basis of it, which is mapped onto products of odd power sums by the commutative image homomorphism. Since the $\pi_n(-1)$ are Lie idempotents, this also determines the primitive Lie algebra of $\mathscr{P}$ as the free Lie algebra generated by the $\pi_{2k+1}(-1)$. It is interesting to remark that all this has been obtained without much effort by setting $q=-1$ in a few formulas of [13].

3. The quasi-symmetric side

To recover Stembridge's algebra, we have to look at the dual of $\mathscr{P}$. Since $\mathscr{P}$ can be regarded either as a homomorphic image of **Sym** (under θ_{-1}) or as a subalgebra of **Sym** (spanned by the peak classes Π_P), the dual $\mathscr{P}^*$ can be realized either as a subalgebra, or as a quotient of $QSym$.

Recall that $QSym$ is the graded Hopf algebra of quasi-symmetric functions. A linear basis of this algebra is given by the complete quasi-symmetric functions $\{F_I\}$ where I runs over all compositions $n\geqslant 0$. The multiplication in $QSym$ is commutative, and F_I can be expressed in term of a commutative alphabet $X = \{x_1, x_2, \dots\}$ as

$$F_I(X) = \sum_{1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_n r\in Des(I)\Rightarrow i_r<i_{r+1}} x_{i_1}x_{i_2}\cdots x_{i_n}.$$

Recall that we have a nondegenerate duality between $QSym$ and **Sym** defined by [8,16]

$$\langle F_I, R_J\rangle = \delta_{IJ}. \tag{12}$$

This induces a duality of graded Hopf algebra between $QSym$ and **Sym**. The dual to the abelianization map χ: **Sym** $\rightarrow Sym$ is the inclusion $Sym\subseteq QSym$.

Let us first consider the noncommutative peak algebra $\mathscr{P} = \mathsf{Peak}^*$ as the image of the Hopf epimorphism $\varphi = \theta_{-1}$. Then, the adjoint map

$$\varphi^* : \mathscr{P}^* \rightarrow QSym \tag{13}$$

is an embedding of Hopf algebras. The duality between $\mathscr{P}$ and $\mathscr{P}^*$ is given by

$$\langle \varphi(F), G\rangle = \langle F, \varphi^*(G)\rangle. \tag{14}$$

Hence, if we denote by Π_P^* the dual basis of Π_P, we have for any ribbon R_I with descent set $D = Des(I)$

$$\langle \varphi^*(\Pi_P^*), R_I\rangle = \langle \Pi_P^*, \varphi(R_I)\rangle = \begin{cases} 2^{|P|+1} & \text{if } P\subseteq D\Delta(D+1), \\ 0 & \text{otherwise.}\end{cases} \tag{15}$$

Thus, in its realization as a subalgebra of $QSym$, $\mathscr{P}^*$ is spanned by Stembridge's quasi-symmetric functions

$$\Theta_P = \varphi^*(\Pi_P^*) = 2^{|P|+1} \sum_{P\subseteq Des(I)\Delta(Des(I)+1)} F_I. \tag{16}$$

Note also that thanks to the identity $(1+q)(1-q) = 1-q^2$, the kernel of φ is seen to be the ideal of **Sym** generated by the $S_n((1+q)A)_{q=-1}$ for $n\geqslant 1$. These are the χ_n of [5].

Finally, we can also consider $\mathscr{P}$ as an abstract algebra with basis (Π_P), and define a monomorphism $\psi : \mathscr{P}\rightarrow$ **Sym** by

$$\psi(\Pi_P) = \sum_{HP(I)=P} R_I. \tag{17}$$

Then, its adjoint $\psi^* : \ QSym \rightarrow \mathscr{P}^*$ is an epimorphism. The product map $\vartheta = \varphi^* \circ \psi^* : \mathscr{P}^* \rightarrow \mathscr{P}^*$ has been considered by Stembridge [22], and its diagonalization is given in [5]. We can easily recover its properties from the results of the previous section, since clearly $\vartheta = (\psi \circ \varphi)^*$ coincides with θ_{-1}. Its eigenvalues are then the integers $2^{\ell(\lambda)}$, where λ runs over partitions into odd parts. The spectral projectors are again constructed from the idempotents $\pi_\lambda(-1)$. Precisely, the projector onto the eigenspace associated with the eigenvalue 2^k of ϑ in $QSym_n$ is the adjoint of the endomorphism of $\mathbf{Sym}_n$ given by $F \mapsto F * U_k$ where $U_k = \sum \pi_\lambda(-1)$, the sum being over all odd partitions of n with exactly k parts. The dimensions of these eigenspaces can also be easily computed.

4. Miscellaneous related results

Here are some more results related to the recent literature. We choose to include them here for completeness.

4.1. Noncommutative tangent numbers

By definition, $\widetilde{\mathbf{Sym}}$ is generated by the $\tilde{S}_n$, $n \geqslant 1$. If we set $q = -1$ in [13, Proposition 5.2] we establish that $\tilde{S}_n = 2H_n$ for $n \geqslant 1$, where $H_0 = 1$ and

$$H_n = \sum_{k=0}^{n-1} R_{1^k, n-k}. \tag{18}$$

Then [8, Proposition 5.24] gives us that

$$H = \sum_{n \geqslant 0} H_n = (1 - \mathbf{t})^{-1}, \tag{19}$$

where $\mathbf{t}$ is the (left) noncommutative hyperbolic tangent

$$\mathbf{t} = \sum_{k \geqslant 0} (-1)^k T_{2k+1}, \quad T_{2k+1} = R_{1,2^k}. \tag{20}$$

Hence, $\widetilde{\mathbf{Sym}}$ is contained in the subalgebra generated by the T_{2k+1}, and since we already know that the dimension of $\widetilde{\mathbf{Sym}}_n$ is the number of odd compositions of n, we have in fact equality. Thus, the $T^I = T_{i_1} \cdots T_{i_r}$ (I odd) form a multiplicative basis of $\widetilde{\mathbf{Sym}}$ (this is the same as the basis Γ^P of [20]).

4.2. Peak Lie idempotents

An homogeneous element $L_n \in \mathbf{Sym}$ of degree n is called a Lie idempotent (see [19]) if it belong to the primitive Lie algebra of $\mathbf{Sym}$ and $\chi(L_n) = \frac{1}{n} p_n$. They are idempotent with respect of the internal product $*$. In [20], the images $\tilde{L}_n = \theta_{-1}(L_n)$ of some classical Lie idempotents L_n are calculated.

In [13] these families of Lie idempotents are readily identified and the computation of there image under θ_{-1} is for most of them straightforward. Let us start with $\pi_n(q)$ as discussed in Section 2. We have seen in Eq. (10), that the $\pi^I(-1)$, I odd, form a basis of $\widetilde{\mathbf{Sym}}$, so that $\widetilde{\mathbf{Sym}}_n$ contains Lie idempotents iff n is odd.

Let us now consider the family of Dynkin elements $\frac{1}{n}\Psi_n(A)$ where $\Psi_n(A)$ is the coefficient of t^{n-1} in $\sigma_t(A)^{-1}\frac{d}{dt}\sigma_t(A)$. The images $\tilde{\Psi}_n = \Psi_n((1-q)A)$ of the Dynkin elements Ψ_n are given in closed form for any q in [13, Proposition 5.34]. It suffices to set $q=-1$ in this formula to obtain [20, Proposition 7.3].

Next, consider the family of elements $\frac{1}{n}\Phi_n(A)$ defined by the coefficient of t^n in $\log\sigma_t(A)$. The expression for $\tilde{\Phi}_n$ in [20, Proposition 7.2] is an interesting new formula. The first part of the analysis can be simplified by applying Eq. (19) to the calculation of the generating series $\log\tilde{\sigma}_1$. Indeed,

$$\begin{aligned}\log\tilde{\sigma}_1 &= \log(1+\mathbf{t}) - \log(1-\mathbf{t})\\ &= 2\sum_{k\geqslant 0}\frac{\mathbf{t}^{2k+1}}{2k+1}\\ &= 2\sum_{I \text{ and } \ell(I) \text{ odd}}\frac{(-1)^{(|I|-\ell(I))/2}}{\ell(I)}T^I.\end{aligned}$$

Finally, to obtain the image of Klyachko's idempotent $\tilde{K}_n(q)$ (see [13, Proposition 6.3] for a definition of $K_n(q)$) one has to set $t=-1$ in [13, Proposition 8.2].

4.3. Structure of the peak algebras $(\mathscr{P}_n, *)$

Using the construction of [13, Section 3.4] restricted to odd partitions λ of n, it follows from Eq. (3) that the idempotents $E_\lambda(\pi(-1))$, associated to the sequence $\pi_n(-1)$, form a complete set of orthogonal idempotents of $\mathscr{P}_n$. Regarding $\mathscr{P}_n$ as a quotient of the descent algebra makes it clear that the left ideals $\mathscr{P}_n * E_\lambda(\pi(-1))$ are the indecomposable projectives modules of $\mathscr{P}_n$. We obtain explicitly the multiplicative structure of $(\mathscr{P}_n, *)$ by adapting [13, Lemma 3.10] to the sequence $\pi_n(-1)$ (instead of Ψ_n), and then imitating the rest of the argument presented there for the descent algebra.

4.4. Hall–Littlewood basis

The peak algebra $\mathscr{P} = \mathsf{Peak}^*$ can be regarded as a noncommutative version of the subalgebra of *Sym* spanned by the Hall–Littlewood functions $Q_\lambda(X;-1)$, where λ runs over strict partitions. Actually, it is easy to show that the noncommutative Hall–Littlewood functions of [4,10] at $q=-1$ yield two different analogous bases of $\mathscr{P}$. We do it here for [10] but a similar argument can be applied to [4].

Recall that the polynomials $H_I(A;q)$ of [10] are defined as noncommutative analogues of the $Q_\mu{}' = Q_\mu(X/(1-q);q)$. To obtain the correct analogues of Schur's q-functions, one has to apply the $(1-q)$ transform before setting $q=-1$.

More precisely, for two compositions I, J, let $\mathrm{Des}(I) = \{a_1 < a_2 < \cdots < a_{\ell(I)-1}\}$ and $\mathrm{Des}(J) = \{b_1 < b_2 < \cdots < b_{\ell(J)-1}\}$, and set $\mathrm{Bre}(I,J) = [1, \ell(I)-1] - \{\#\{a_j : a_j \leqslant b_i\} : 1 \leqslant i \leqslant \ell(J)-1\} \subseteq [1, \ell(I)-1]$. We have

$$H_I(A;q) = \sum_{J \leqslant I} q^{\left(\sum_{i \in \mathrm{Bre}(I,J)} i\right)} R_J(A).$$

Proposition 4.1. *The specialized noncommutative Hall–Littlewood functions*

$$\mathrm{Q}_I = H_J((1-q)A;q)_{q=-1}, \tag{21}$$

where I runs over all peak compositions, form a basis of $\mathcal{P}$.

Indeed, the factorization of H-functions at roots of unity imply that

$$\mathrm{Q}_I = \mathrm{Q}_{i_1 i_2} \mathrm{Q}_{i_3 i_4} \cdots Q_{i_{2k-1} i_{2k}} \mathrm{Q}_{i_{2k+1}} \tag{22}$$

(where $i_{2k+1} = 0$ if I is of even length), and simple calculations yield

- $\mathrm{Q}_n = 2\Pi_{\emptyset}$,
- $\mathrm{Q}_{n-1,1} = 2(\Pi_{\{n-1\}} + \Pi_{\emptyset})$,
- and for $2 \leqslant k \leqslant n-2$, $\mathrm{Q}_{k,n-k} = 4(\Pi_{\{k\}} + \Pi_{\{k+1\}} + \Pi_{\emptyset})$,

where Π_P is defined in Eq. (5). From this, it is straightforward to prove that the family Q_I is triangular with respect to the family Π_P, and hence the proposition follows.

5. Representation theory of the 0-Hecke–Clifford algebras

The character theory of symmetric groups (in characteristic 0), as worked out by Frobenius, can be summarized as follows. Let R_n denote the free abelian group spanned by isomorphism classes of irreducible representations of $\mathbb{C}\mathfrak{S}_n$. Endow the direct sum

$$R = \oplus_{n \geqslant 0} R_n \tag{23}$$

with the addition corresponding to direct sum, and multiplication $R_m \otimes R_n \to R_{m+n}$ corresponding to induction from $\mathfrak{S}_m \times \mathfrak{S}_n$ to $\mathfrak{S}_{m+n}$ via the natural embedding. The linear map sending the class of an irreducible representation $[\lambda]$ to the Schur function s_λ is then a ring isomorphism between R and *Sym* (see, e.g., [15]). Moreover, we can define a structure of graded Hopf algebra on R with comultiplication corresponding to restrictions from $\mathfrak{S}_n$ down to $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$ and summing over k. The linear map above gives rise to an isomorphism of graded Hopf algebras.

It is known that the pair of graded Hopf algebras (**Sym**, *QSym*) admits a similar interpretation, in terms of the tower of the 0-Hecke algebras $H_n(0)$ of type A_{n-1} (see [14]). Recall that the (Iwahori-) Hecke algebra $H_n(q)$ is the $\mathbb{C}$-algebra generated by elements T_i for $i<n$ with the relations:

$$\begin{aligned} &T_i^2 = (q-1)T_i + q && \text{for } 1 \leqslant i \leqslant n-1, \\ &T_i T_j = T_j T_i && \text{for } |i-j|>1, \\ &T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} && \text{for } 1 \leqslant i \leqslant n-2 \end{aligned} \tag{24}$$

(here, we assume that $q \in \mathbb{C}$). The 0-Hecke algebra is obtained by setting $q=0$ in these relations. Then, the first relation becomes $T_i^2 = -T_i$. If we denote by $G_n = G_0(H_n(0))$ the Grothendieck group of the category of finite dimensional $H_n(0)$-modules, and by $K_n = K_0(H_n(0))$ the Grothendieck group of the category of projective $H_n(0)$-modules, the direct sums $\mathscr{G} = \bigoplus_{n \geqslant 0} G_n$ and $\mathscr{K} = \bigoplus_{n \geqslant 0} K_n$, endowed with the same operations as above, are respectively isomorphic with *QSym* and **Sym**.

The aim of this final section is our main result: to provide a similar interpretation for the pair $(\mathscr{P}, \mathscr{P}^*)$. The relevant tower of algebras is the 0-Hecke–Clifford algebras, which are degenerate versions of Olshanski's Hecke–Clifford algebras.

5.1. *Hecke–Clifford algebra*

The complex Clifford algebra Cl_n is generated by n elements c_i for $i \leqslant n$ with the relations

$$c_i c_j = -c_j c_i \quad \text{for } i \neq j \quad \text{and} \quad c_i^2 = -1. \tag{25}$$

For each subset $D = \{i_1 < i_2 < \cdots < i_k\} \subset \{1, \dots, n\}$, we denote by c_D the product

$$c_D := \overrightarrow{\prod_{i \in D}} c_i = c_{i_1} c_{i_2} \cdots c_{i_k}. \tag{26}$$

It is easy to see that $(c_D)_{D \subset \{1,\dots,n\}}$ is a basis of the Clifford algebra.

The Hecke–Clifford superalgebra [18] is the unital $\mathbb{C}$-algebra generated by the c_i, and $n-1$ elements t_i satisfying the Hecke relations in the form

$$\begin{aligned} &t_i^2 = (q-q^{-1})t_i + 1 && \text{for } 1 \leqslant i \leqslant n-1, \\ &t_i t_j = t_j t_i && \text{for } |i-j|>1, \\ &t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} && \text{for } 1 \leqslant i \leqslant n-2 \end{aligned} \tag{27}$$

and the cross-relations

$$\begin{aligned} &t_i c_j = c_j t_i && \text{for } i \neq j, j+1, \\ &t_i c_i = c_{i+1} t_i && \text{for } 1 \leqslant i \leqslant n-1, \\ &(t_i + q - q^{-1}) c_{i+1} = c_i (t_i + q - q^{-1}) && \text{for } 1 \leqslant i \leqslant n-1. \end{aligned} \tag{28}$$

The Hecke–Clifford algebra has a natural $\mathbb{Z}_2$-grading, for which the t_i are even and the c_j are odd. Henceforth, it will be considered as a superalgebra.

Setting $t_i = q^{-1} T_i$ and taking the limit $q \to 0$ after clearing the denominators, we obtain the 0-Hecke–Clifford algebra $HCl_n(0)$, which is generated by the 0-Hecke algebra and the Clifford algebra, with the cross-relations

$$\begin{aligned} T_i c_j &= c_j T_i && \text{for } i \neq j, j+1, \\ T_i c_i &= c_{i+1} T_i && \text{for } 1 \leqslant i \leqslant n-1, \\ (T_i+1)c_{i+1} &= c_i(T_i+1) && \text{for } 1 \leqslant i \leqslant n-1. \end{aligned} \tag{29}$$

Let $\sigma = \sigma_{i_1} \cdots \sigma_{i_p}$ be a reduced word for a permutation $\sigma \in \mathfrak{S}_n$. The defining relations of $H_n(q)$ ensure that the element $T_\sigma := T_{i_1} \cdots T_{i_p}$ is independent of the chosen reduced word for σ. The family $(T_\sigma)_{\sigma \in \mathfrak{S}_n}$ is a basis of the Hecke algebra. Thus a basis for $HCl_n(q)$ is given by $(c_D T_\sigma)_{D \subset \{1,\dots,n\}, \sigma \in \mathfrak{S}_n}$, and consequently, the dimension of $HCl_n(q)$ is $2^n n!$ for all q.

5.2. *Quasi-symmetric characters of induced modules*

Since $H_n(0)$ is the sub-algebra of $HCl_n(0)$ generated by the T_i, our main tool in the sequel will be the induction process with respect to this inclusion. Let us recall some known facts about the representation theory of $H_n(0)$. There are 2^{n-1} simple $H_n(0)$-modules. These are all one dimensional and can be conveniently labelled by compositions I of n. The structure of the simple module $S_I := \mathbb{C}\varepsilon_I$ is given by

$$T_j \varepsilon_I = \begin{cases} -\varepsilon_I & \text{if } j \in Des(I), \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

As described in [14], there is an isomorphism $ch : \mathcal{G} \to QSym$ which we call the *Frobenius characteristic*. This maps the simple module S_I to the quasi-symmetric function F_I.

Let us define the $HCl_n(0)$-module M_I as the module induced by S_I through the natural inclusion map, that is

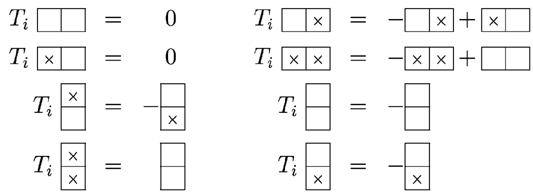
$$M_I := Ind_{H_n(0)}^{HCl_n(0)}(S_I) = HCl_n(0) \bigotimes_{H_n(0)} S_I. \tag{31}$$

A basis for M_I is given by $(c_D \varepsilon_I)_{D \subset \{1,\dots,n\}}$. A basis element can be depicted conveniently as follows. The boxes of the ribbon diagram associated with I are numbered from left to right and from top to bottom. We put a "$\times$" in the i-th box if $i \in D$. For example $c_{\{1,3,4,6\}}\, \varepsilon_{(2,1,3)} = c_1 c_3 c_4 c_6\, \varepsilon_{(2,1,3)}$ is depicted by

$$(2,1,3) = \begin{array}{cccc} \boxed{1} & \boxed{2} & & \\ & \boxed{3} & & \\ & \boxed{4} & \boxed{5} & \boxed{6} \end{array} \qquad c_{\{1,3,4,6\}}\, \epsilon_{(2,1,3)} = \begin{array}{cccc} \boxed{\times} & \boxed{} & & \\ & \boxed{\times} & & \\ & \boxed{\times} & \boxed{} & \boxed{\times} \end{array} \tag{32}$$

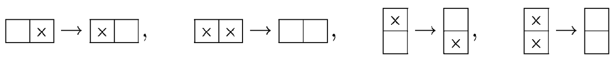
We can graphically view the set $Des(I)$ as the set of boxes with a box below, and the set $HP(I)$ as the set of boxes with boxes below and to the left. In the example above, $Des(2,1,3) = \{2,3\}$ are the boxes labeled 2 and 3, and $HP(2,1,3) = \{2\}$ is only the box 2.

We now remark that T_i acts only on the ith and $(i+1)$th boxes. On the graphical representation, drawing only the boxes i and $i+1$, rules (29) read



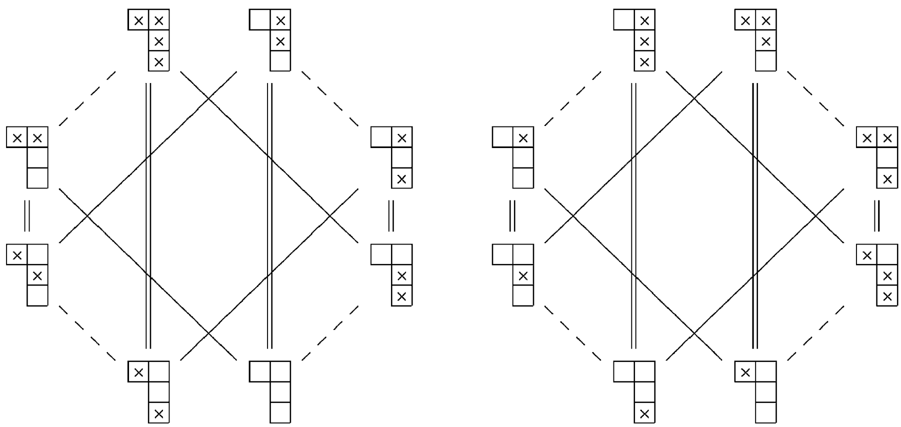
(33)

At this point, we can make a couple of useful remarks. Looking at the support of relation (33), we define



(34)

These relations can be interpreted as the cover relation of a (partial) order $\leq_I$ on the subset D of $\{1, \ldots, n\}$. Here is a picture of the Hasse diagram of this order for the composition $I = (2, 1, 1)$. The poset clearly has two components corresponding to the two $\mathbb{Z}_2$-graded homogeneous components of $M_{(211)}$.



(*)

The importance of this order comes from the following lemma, a direct consequence of Eq. (33).

Lemma 5.1. *The action of each T_i is triangular with respect to the order $\leqslant_I$, that is for all D,*

$$T_i \; c_D \varepsilon_I = \alpha(i, I, D) \; c_D \varepsilon_I + \text{smaller terms} \tag{35}$$

with $\alpha(i, I, D) \in \{0, -1\}$.

The $\alpha(i, I, D)$ are the eigenvalues of the T_i. They are equal to 0 in the leftmost columns of Eq. (33) and to -1 in the rightmost ones.

A second consequence of Eq. (33) is that in a vertical (two boxes) diagram, the eigenvalue depends only on the content of the upper box whereas in a horizontal diagram it depends only on the content of the rightmost box. Thus the content of the boxes without a box below or on the left does not matter for computing the eigenvalues. This can be translated into the following lemma.

Lemma 5.2. *Suppose that $k \in \{1, \dots, n\}$ is such that k is not a descent of I and has no box to its left. Then for all D, the eigenvalues $\alpha(i, I, D)$ satisfy*

$$\alpha(i, I, D) = \alpha(i, I, \{k\} \cup D). \tag{36}$$

Such a k is called a valley *of the composition I.*

Note that 1 and n can be valleys. There is obviously one more valley than the number of peaks.

Thanks to the order $\leqslant_I$, one can easily describe the structure of the restriction of M_I to $H_n(0)$. Our first goal is to get a composition series of $Res_{H_n(0)} M_I$ in order to compute its Frobenius characteristic. This can be done as follows. Let us choose a linear extension $D_1, D_2, \dots, D_{2^n}$ of $\leqslant_I$. For $k \geqslant 1$, define

$$M_I^k = \bigoplus_{l \leqslant k} \mathbb{C}\, c_{D_l}\, \varepsilon_I, \tag{37}$$

and $M_I^0 := \{0\}$. Then, thanks to Lemma 5.1, M_I^k is clearly a sub-module of $Res_{H_n(0)} M_I$, and

$$\{0\} = M_I^0 \subset M_I^1 \subset M_I^2 \subset \cdots \subset M_I^{2^n} = Res_{H_n(0)} M_I \tag{38}$$

is a composition series of the module $Res_{H_n(0)} M_I$. Let us compute the simple composition factors of the module $S_{D_i,I} = \mathbb{C}\varepsilon_{K(D_i,I)} := M_I^i / M_I^{i-1}$. For $1 \leqslant k < n$, the generator T_k acts as $T_k \varepsilon_{K(D_i,I)} = \alpha(k, I, D_i) \varepsilon_{K(D_i,I)}$. The eigenvalue $\alpha(k, I, D_i)$ equals -1 if

$$(k + 1 \in D_i \quad \text{and } k \notin Des(I)) \quad \text{or} \quad (k \notin D_i \text{ and } k \in Des(I)), \tag{39}$$

and 0 otherwise. Hence, according to Eq. (30), $\mathrm{ch}(S_{D_i,I}) = F_K$ where $Des(K) = Des(K(D_i, I)) = \{1 \leqslant k < n \mid \alpha(k, I, D_i) = -1\}$. When p is a peak of I, that is $p \neq 1$, $p - 1 \notin Des(I)$ and $p \in Des(I)$, then $|\{p - 1, p\} \cap Des(K)| = 1$. Indeed, if $p \in D_i$ then $p - 1 \in Des(K)$ and $p \notin Des(K)$ and if $p \notin D_i$ then $p - 1 \notin Des(K)$ and $p \in Des(K)$. Thus, $P = HP(I) \subseteq Des(K) \Delta (Des(K) + 1)$. Moreover, for $k \notin P \cup (P - 1)$, we can always find a D_i such that $k \in Des(K(D_i, I))$. All K such that $P \subseteq Des(K) \Delta (Des(K) + 1)$ can be obtained, and thanks to Lemma 5.2 there are $2^{|P|+1}$ sets D_i giving the same F_K. Thus, we have proved the following proposition.

Proposition 5.3. *The Frobenius characteristic of $Res_{H_n(0)} M_I$ depends only on the peak set P of the composition I and is given by Stembridge's Θ function*

$$ch(Res_{H_n(0)} M_I) = \Theta_P = 2^{|P|+1} \sum_{P \subseteq Des(K) \Delta (Des(K)+1)} F_K. \tag{40}$$

5.3. Homomorphisms between induced modules

The previous proposition suggests that $Res_{H_n(0)} M_I$ is isomorphic to $Res_{H_n(0)} M_J$ iff I and J have the same peak sets. This is actually true, and in fact, M_I and M_J are even isomorphic as $HCl_n(0)$-supermodules, as we will establish now.

Theorem 5.4. *Let I be a composition with valley set V, and let Cl_V be the subalgebra of Cl_n generated by $(c_v)_{v\in V}$. For $c\in Cl_V$ define a map f_c from M_I to itself by*

$$f_c(x\varepsilon_I) = xc\varepsilon_I \quad \text{for all } x\in Cl_n. \tag{41}$$

Then $c\mapsto f_c$ defines a right action of Cl_V on M_I which commutes with the left $HCl_n(0)$-action. Moreover the map $c\mapsto f_c$ is a graded isomorphism from Cl_V to $End_{HCl_n(0)}(M_I)$.

Proof. Since M_I is freely generated as a Cl_n-module by ε_I, a morphism $f\in End_{HCl_n(0)}(M_I)$ is determined by $f(\varepsilon_I) = x\varepsilon_I$ for $x\in Cl_n$. On the other hand, for $x\in Cl_n$, a map $f_x(\varepsilon_I) = x\varepsilon_I$ is in $End_{HCl_n(0)}(M_I)$ if and only if $T_j f_x(\varepsilon_I) = T_j x\varepsilon_I = xT_j\varepsilon_I$ for all $1\leqslant j<n$. Thus, to prove the theorem, it is sufficient to see that $f_x\in End_{HCl_n(0)}(M_I)$ if and only if $x\in Cl_V$. Equivalently,

$$\text{for all } 1\leqslant j<n, \quad T_j x\varepsilon_I = \begin{cases} -x\varepsilon_I & \text{if } j\in Des(I), \\ 0 & \text{otherwise,} \end{cases} \tag{42}$$

if and only if $x\in Cl_V$.

Let us first assume that $x = c_D$ for $D\subseteq V$. This means that in the graphical representation of $x\varepsilon_I$ there is no box below nor to the left of a box with a "×". If $j\in Des(I)$, the lower two equations of the right column of Eq. (33) then show that $T_j x\varepsilon_I = -x\varepsilon_I$. The top two equations on the left show that if $j\notin Des(I)$ then $T_j x\varepsilon_I = 0$. By linearity, we get that if $x\in Cl_V$ then Eq. (42) holds. Conversely, let $x\in Cl_n$ satisfy Eq. (42). Let $c_D\varepsilon_I$ be in the support of x, minimal with respect to $\preceq_I$. If $D\nsubseteq V$ then there is a box j with a "×" and a box below or to the left. Using Lemma 5.1 and Eq. (33) this would be a contradiction to Eq. (42). Hence $c_D\in Cl_V$. We can subtract it from x and repeat the argument above recursively to conclude that $x\in Cl_V$. □

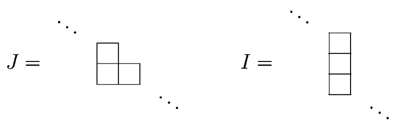
Theorem 5.5. *The induced supermodules M_I and M_J are isomorphic if and only if the peak sets of I and J coincide.*

Proof. One direction of this theorem is implied by the previous section. If M_I is isomorphic to M_J, then we must have that $Res_{H_n(0)} M_I$ is isomorphic to $Res_{H_n(0)} M_J$. In particular, they must have the same Frobenius characteristic. Thanks to Proposition 5.3, $ch(Res_{H_n(0)} M_I)$ depends only on the peak set of I. Thus, if M_I and M_J are isomorphic then I and J have the same peak sets.

The converse will follow once we construct explicit isomorphisms between any modules M_I and M_J in the same peaks class ($HP(I) = HP(J)$), such that I and J

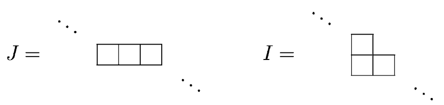
differ exactly by one descent. Isomorphisms between any modules M_I and M_J in the same peaks class in general will be obtained by composition of the constructed ones.

Let $I = J \cup \{k\}$ be such that $HP(I) = HP(J)$. Graphically, there are two possible cases to consider:



(43)

or



(44)

In case (43), we construct a map f which sends $\varepsilon_I \mapsto \eta = (c_{\{k,k+1\}} - 1)\varepsilon_J$. We remark that both η and ε_I are even. Furthermore,

$$\text{for all } 1 \leqslant i < n, \quad T_i\eta = \begin{cases} -\eta & \text{if } i \in Des(I), \\ 0 & \text{otherwise.} \end{cases} \tag{45}$$

Indeed, for $i \notin \{k-1, k\}$, the T_i commute with $c_{\{k,k+1\}}$. Moreover $i \in Des(I)$ if and only if $i \in Des(J)$, hence Eq. (45) follows in these cases. If $i = k-1 \in Des(I)$, then $T_{k-1}\eta = (c_{\{k-1,k+1\}}T_{k-1} + c_{\{k-1,k+1\}} - c_{\{k,k+1\}} - T_{k-1})\varepsilon_J = -\eta$, and if $i = k \in Des(I)$, then $T_k\eta = (-c_{\{k,k+1\}}T_k - c_{\{k,k+1\}} + 1 - T_k)\varepsilon_J = -\eta$. This allows us to define a nontrivial $HCl_n(0)$ supermorphism $f: M_I \to M_J$ where $f(c_D\varepsilon_I) = c_D\eta$. Thanks to Eq. (45), the submodule spanned by η in M_J is isomorphic to M_I. But since both spaces have the same dimension we have that f is an isomorphism.

For case (44), we proceed in the same way, sending $\varepsilon_J \mapsto (c_{\{k,k+1\}} + 1)\varepsilon_I$. This constructs a graded isomorphism from M_J to M_I. $\square$

5.4. Simples supermodules of $HCl_n(0)$

We are now in a position to construct the simple supermodules of $HCl_n(0)$. Our approach is similar to Jones and Nazarov [11]. Let I be a composition with peak set P and valley set $V = \{v_1, v_2, \ldots, v_k\}$. Choose a minimal even idempotent of the Clifford superalgebra Cl_V. For example

$$e_I := \frac{1}{2^l}\left(1 + \sqrt{-1}c_{v_1}c_{v_2}\right)\left(1 + \sqrt{-1}c_{v_3}c_{v_4}\right)\cdots\left(1 + \sqrt{-1}c_{v_{2l}}c_{v_{2l+1}}\right), \tag{46}$$

where $l := \lfloor \frac{k}{2} \rfloor = \lfloor \frac{|P|+1}{2} \rfloor$. Define $HClS_I := Cl_n e_I \varepsilon_I$ as the $HCl_n(0)$-module generated by $e_I\varepsilon_I$. One has to show that $HClS_I$ does not depend on the chosen minimal idempotent e_I, but this is an easy consequence of the representation theory of Cl_V which is know to be supersimple (see e.g. [12]). Suppose that e_I and e'_I are two minimal even idempotents of Cl_V. Since Cl_V, is supersimple, there exist

$x, y, x', y' \in Cl_V$ such that $e_I = x'e'_I y'$ and $e'_I = xe_I y$. Then by the representation theory of Cl_V, we know that f_y and $f_{y'}$ are two mutually reciprocal isomorphisms between $Cl_V e_I$ and $Cl_V e'_I$ and hence between $Cl_n e_I \varepsilon_I$ and $Cl_n e'_I \varepsilon_I$.

When n is even $Cl_V e_I$ has dimension $2^{\frac{|P|+1}{2}}$ and when n is odd the dimension is $2^{\frac{|P|+1}{2}-1}$. In short we can write $2^{\left\lfloor \frac{|P|+1}{2} \right\rfloor}$ for the dimension in both cases. Thus $HClS_I$ has dimension $2^{n-\left\lfloor \frac{|P|+1}{2} \right\rfloor}$.

A direct corollary to Theorem 5.4 is the following.

Corollary 5.6. *The induced module M_I is the direct sum of $2^{\left\lfloor \frac{|P|+1}{2} \right\rfloor}$ isomorphic copies of $HClS_K$, where K is the peak composition associated to I.*

We are now ready to show our main theorem and define the Frobenius characteristic map between $HCl_n(0)$ modules and $\mathscr{P}^*$, which we again denote by *ch*.

Theorem 5.7. *The set $\{HClS_I := Cl_n e_i \varepsilon_I\}$, where I runs over all compositions with distinct peak sets, is a complete set of pairwise nonisomorphic simple supermodules of $HCl_n(0)$. Moreover, there is a graded Hopf isomorphism defined by*

$$ch: \tilde{\mathscr{G}} \to \mathscr{P}^*$$

$$HClS_I \to 2^{-\left\lfloor \frac{|P|+1}{2} \right\rfloor} \Theta_{HP(I)}, \tag{47}$$

where $HP(I)$ is the peak set of I, and $\tilde{\mathscr{G}} = \oplus_{n \geqslant 0} G_0(HCl_n(0))$.

Thus the $(1-q)$-transform at $q=-1$ can be interpreted as the induction map from $G_0(H_n(0))$ to $G_0(HCl_n(0))$. This maps $\mathscr{G}$ to $\tilde{\mathscr{G}}$.

Proof. Suppose that S is a simple supermodule of $HCl_n(0)$. Decompose the $H_n(0)$-socle of S into simple modules and choose a non-zero vector v in one of these simple factors. Then v is a common eigenvector of all the T_i, so that there is a I such that $\varepsilon_I \mapsto v$ defines a $H_n(0)$-morphism $\phi : S_I \to S$. Then, since S is supersimple, v generates S under the action of $HCl_n(0)$. Thus by induction, there is a surjective morphism $M_I \to S$. Hence, each simple module of $HCl_n(0)$ must be a quotient of some M_I and consequently of some $HClS_I$.

Now, we know that given two $HClS_I$, either they are isomorphic (when they have the same peak sets) or else there is no morphism between them. Thus $HClS_I$ has to be simple. The multiplication and comultiplication structures are induced from $\mathscr{G}$ and $QSym$ and the Frobenius characteristic between them. □

By duality, we obtain $ch^*:\mathscr{P}\to\tilde{\mathscr{G}}^*$ where $\tilde{\mathscr{G}}^* = \oplus_{n\geqslant 0}K_0(HCl_n(0))$. We also remark that the dimension of the superradical of $HCl(0)$ is thus

$$2^n n! - \sum_P 2^{2n-(|P|+1)}, \tag{48}$$

where the sum is over all peak sets of n. It is still an open problem to find nice generators for this radical.

5.5. Decomposition matrices

The generic Hecke–Clifford algebra $HCl_n(q)$ has its simple modules U_λ labelled by partitions into distinct parts, and under restriction to $H_n(q)$, U_λ has as Frobenius characteristic

$$ch(U_\lambda) = 2^{-\lfloor \ell(\lambda)/2 \rfloor} Q_\lambda, \tag{49}$$

where Q_λ is Schur's Q-function (see [11,18]).

By Stembridge's formula [22], we have

$$ch(U_\lambda) = 2^{-\lfloor \ell(\lambda)/2 \rfloor} \sum_{T\in\mathscr{S}T^\lambda} \Theta_{\Lambda(T)}, \tag{50}$$

where $\mathscr{S}T^\lambda$ is the set of standard shifted tableaux of shape λ, and $\Lambda(T)$ the peak set of T. The quasisymmetric characteristics of the simple $HCl_n(0)$ modules are proportional to the Θ-functions, and the coefficients $d_{\lambda I}$ in the expression

$$ch(U_\lambda) = \sum_I d_{\lambda I}\, ch(HClS_I) \tag{51}$$

form the decomposition matrices of the Hecke–Clifford algebras at $q=0$.

For λ a strict partition of n, and I a peak composition of n with peak set P, one has explicitly

$$d_{\lambda I} = 2^{\lfloor \frac{1}{2}\ell(I)\rfloor - \lfloor \frac{1}{2}\ell(\lambda)\rfloor} |\{T\in\mathscr{S}T^\lambda \mid \Lambda(T) = P\}|. \tag{52}$$

This is the analog for $HCl_n(0)$ of Carter's combinatorial formula for the decomposition numbers of $H_n(0)$ [7].

Here are the decomposition matrices $[d_{\lambda I}]$ for $n\leqslant 9$. Note that for $n=2,3$, $HCl_n(0)$ is semi-simple.

	3	21
3	1	0
21	0	1

	4	31	22
4	1	0	0
31	0	1	1

	3	41	32	23	221
5	1	0	0	0	0
41	0	1	1	1	0
32	0	0	1	0	1

	6	51	42	33	321	24	231	222
6	1	0	0	0	0	0	0	0
51	0	1	1	1	0	1	0	0
42	0	0	1	1	1	0	1	1
321	0	0	0	0	1	0	0	1

	7	61	52	43	421	34	331	322	25	241	232	223	2221
7	1	0	0	0	0	0	0	0	0	0	0	0	0
61	0	1	1	1	0	1	0	0	1	0	0	0	0
52	0	0	1	1	1	1	1	1	0	1	1	1	0
43	0	0	0	1	0	0	1	1	0	0	1	0	2
421	0	0	0	0	1	0	1	2	0	0	1	1	2

	8	71	62	53	521	44	431	422	35	341	332	323	3221	26	251	242	233	2321	224	2231	2222
8	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
71	0	1	1	1	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0
62	0	0	1	1	1	1	1	1	1	1	1	1	0	0	1	1	1	0	1	0	0
53	0	0	0	1	0	1	1	1	0	1	2	1	2	0	0	1	1	2	0	2	2
521	0	0	0	0	1	0	1	2	0	1	2	2	2	0	0	1	1	2	1	2	2
431	0	0	0	0	0	0	1	1	0	0	2	1	4	0	0	0	1	2	0	2	4

	9	81	72	63	621	54	531	522	45	441	432	423	4221	36	351	342	333	3321	324	3231	3222	27	261	252	243	2421	234	2331	2322	225	2241	2232	2223	22221
9	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
81	0	1	1	1	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
72	0	0	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	0	1	0	0	0	1	1	1	0	1	0	0	1	0	0	0	0
63	0	0	0	1	0	1	1	1	1	1	2	1	2	0	1	2	2	2	1	2	2	0	0	1	1	2	1	2	2	0	2	2	2	0
621	0	0	0	0	1	0	1	2	0	1	2	2	2	0	1	2	2	2	2	2	2	0	0	1	1	2	1	2	2	1	2	2	2	0
54	0	0	0	0	0	1	0	0	0	1	1	1	0	0	0	1	1	2	0	2	2	0	0	0	1	0	0	2	2	0	0	2	0	2
531	0	0	0	0	0	0	1	1	0	1	3	2	4	0	0	2	3	6	1	6	8	0	0	0	1	2	1	4	6	0	2	6	4	4
432	0	0	0	0	0	0	0	0	0	0	1	0	2	0	0	0	1	2	0	2	4	0	0	0	0	0	0	2	2	0	0	2	2	2

(∗)

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