Hecke algebras, difference operators and quasi-symmetric functions

Florent Hivert

Institut Gaspard Monge – Université de Marne-la-Vallée
5, bd Descartes – Champs-sur-Marne
77454 Marne-la-Vallée cedex 2, FRANCE,
e-mail: Florent.Hivert@univ-mlv.fr

November 20, 1998

Abstract

We define a new action of the symmetric group and its Hecke algebra on polynomial rings whose invariants are exactly the quasi-symmetric polynomials. We interpret this construction in terms of a Demazure character formula for the irreducible polynomial modules of a degenerate quantum group. We use the action of the generic Hecke algebras to define quasi-symmetric and non-commutative analogues of Hall-Littlewood functions. We show that these generalized functions shares many combinatorial properties with the classical ones.

Résumé

Nous introduisons de nouvelles actions du groupe symétrique et de son algèbre de Hecke sur les polynômes, pour lesquelles les invariants sont les polynômes quasi-symétriques. Nous interprétons cette construction en termes de caractères de Demazure d'un groupe quantique dégénéré. Nous utilisons l'action de l'algèbre de Hecke générique pour définir des analogues quasi-symétriques et non commutatifs des fonctions de Hall-Littlewood. Nous montrons que ces fonctions généralisées ont un certain nombre de propriétés communes avec les fonctions classiques.

Contents

1 Introduction

The notion of symmetric function admits two interesting generalizations: quasi-symmetric functions [12], which are certain partially symmetric polynomials, and noncommutative symmetric functions [11], which are elements of a noncommutative algebra whose abelianization is the usual algebra of symmetric functions.

The main purpose of this article is to develop a theory of quasi-symmetric and noncommutative Hall-Littlewood functions. To motivate our construction, let us start with a brief history of the subject.

The Hall-Littlewood symmetric functions were introduced in 1961 by D.E. Littlewood, as a concrete realization of an algebra defined by P. Hall in some unpublished work in the theory of Abelian groups (see the reprint in [14]). Hall's algebra is spanned by isomorphism classes of finite Abelian p-groups, for some fixed prime p. An isomorphism class can be encoded by a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ also called the type of the group, recording the exponents in the decomposition

$$G_\lambda \approx \bigoplus_{i=1}^{r} (\mathbb{Z}/p^{\lambda_i}\mathbb{Z}). \tag{1}$$

Let u_λ be the isomorphism class of the above group. The multiplicative structure is defined by

$$u_\alpha u_\beta = \sum_\lambda g_{\alpha\beta}^\lambda u_\lambda, \tag{2}$$

where $g_{\alpha\beta}^\lambda$ is the number of subgroups H of G_λ which are of type α and such that G_λ/H is of type β. Hall showed that these numbers are expressed by polynomials in p, now called Hall polynomials. (Actually, the Hall algebra had been previously discovered by Steinitz [46], *cf.* [17]).

A finite Abelian p-group is the same as a finite module over the discrete valuation ring $\mathbb{Z}_p$ (p-adic integers) and the same construction works as well for the rings $\mathbb{K}[[t]]$ where $\mathbb{K} = \mathbb{F}_q$ is a finite field with q elements. Again the $g_{\alpha\beta}^\lambda$ are given by the same polynomials in q. This version of the Hall algebra was used in 1955 by J.A. Green to determine the character table of the finite linear group $GL(n, \mathbb{F}_q)$ [13].

It was known that the Hall algebra was isomorphic to the algebra of symmetric functions, but a basis of symmetric functions having the $g_{\alpha\beta}^\lambda$ as structure constants was not explicitly known, and this was needed to develop a practical algorithm for the calculation of character tables. This is the problem solved by Littlewood: he introduced the symmetric functions

$$P_\mu(x_1, \ldots, x_n; q) = \prod_{i>0} \frac{1}{[m_i]_q!} \sum_{\sigma \in \mathfrak{S}_n} \sigma \left(x_1^{\mu_1} \ldots x_n^{\mu_n} \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j} \right), \tag{3}$$

where m_i is the multiplicity of i in the partition μ. He proved that

$$u_\lambda = q^{-n(\lambda)} P_\lambda(x; q^{-1}) \tag{4}$$

realize the Hall algebra (here $n(\lambda) = \sum_{i>0}(i-1)\lambda_i$). Littlewood further observed that under the specialization $q=-1$, the P_μ reduced to a class of symmetric functions introduced by Schur in 1911 as generating functions of the spin characters of symmetric groups [44].

The Hall-Littlewood functions can be introduced, as in [38], by orthogonalizing the basis of monomial functions with respect to a deformed scalar product. Other deformations with two or more parameters lead to Macdonald's and Kerov's symmetric functions. One denotes by $\{Q_\mu\}$ the dual basis of P_μ for the deformed scalar product. Since P_μ is orthogonal, Q_μ is proportional to P_μ. It is more interesting to look at the dual basis $\{Q'_\mu\}$ of $\{P_\mu\}$ for the ordinary scalar product, for which Schur functions are orthonormal. Q'_μ is a q-analogue of the product h_μ of homogeneous symmetric functions, and the coefficient $K_{\lambda\mu}(q)$ of the expansion

$$Q'_\mu = \sum_\lambda K_{\lambda\mu}(q) s_\lambda \tag{5}$$

are q-analogues of the Kotska number $K_{\lambda\mu}$.

H.O. Foulkes conjectured in 1974 [9] that the $K_{\lambda\mu}(q)$ were polynomials with non-negative integer coefficients. This conjecture was proved in 1978 by A. Lascoux and M.P. Schützenberger [29], who introduced a statistic called *charge* on the set $\mathrm{Tab}(\lambda,\mu)$ of Young tableaux of shape λ and weight μ, and proved that

$$K_{\lambda\mu}(q) = \sum_{t\in\mathrm{Tab}(\lambda,\mu)} q^{\mathrm{charge}(t)}. \tag{6}$$

This statistic has now important applications in statistical mechanics, where it appears as the energy of certain quasi-particles in the Bethe Ansatz approach to quantum spin chains [18] or in the corner transfer matrix approach to solvable lattice models [42].

The reciprocal polynomials $\tilde{K}_{\lambda\mu}(q) = q^{n(\mu)}K_{\lambda\mu}(q^{-1})$ are the values of the unipotent characters of $GL(n,\mathbb{F}_q)$ on unipotent classes [45, 36], and their generating function $\tilde{Q}'_\mu = \sum \tilde{K}_{\lambda\mu}(q)s_\lambda$ are the graded characteristics of Springer's representations of symmetric groups in the cohomology of unipotent varieties [16]. They appear also in the harmonic polynomial in the flag manifold [23], and there exists a two parameters analogue [10].

Also, the Q'_μ have interesting specializations at roots of unity [24, 25, 38] which appear to be related to the representation theory of quantum affine algebras [28, 33] and of the q-Virasoro algebra [2].

It has been observed in [7] that Littlewood's definition (3) could be interpreted in terms of an action of the Hecke algebra $H_n(q)$ on $\mathbb{C}(q)[x_1,\ldots,x_n]$

(see also section 6). This action is obtained by lifting to $H_n(q)$ an action of the degenerate algebra $H_n(0)$, defined by means of the elementary symmetrizing operators involved in Demazure's character formula [5].

This point of view allows us to develop an analogous theory in the context of two recent generalizations of symmetric functions: Quasi-symmetric functions [12] and noncommutative symmetric functions [11]. These objects build up two Hopf algebra which are dual to each other [41], and are related to the representation theory of the degenerate Hecke algebra $H_n(0)$ in the same way as ordinary symmetric functions are related to the symmetric groups [20]. In particular Gessel's fundamental quasi-symmetric functions F_I have been shown in [22] to be the characters of the irreducible polynomial representations of a degenerate quantum group $\mathcal{U}_0(gl_N)$.

By means of a non-standard action of the symmetric group on $\mathbb{C}[X]$ (Section 3.1), we obtain a Weyl and a Demazure character formula for these representations (Section 4, Theorem 4.2 and 4.3). From this, we obtain an action of the degenerate Hecke algebra (Section 3.2), and left it to an action of $H_n(q)$. This allows us to define a q-analogue of the Weyl symmetrizer (Section 5), from which we obtain a notion of quasi-symmetric Hall-Littlewood functions G_I analogous to (3), (Section 6.2). By duality, we obtain noncommutative Hall-Littlewood functions H_I analogous to Q'_μ (Section 6.3). We investigate the properties of these generalized Hall-Littlewood functions, in particular, we describe explicitly their expressions on the generalized Schur bases and their structure constants (Theorems 6.1, 6.2 and 6.3). This allows us to prove that the H_I behave like the Q'_μ under specialization of q at roots of unity (Corollary 6.5).

Such a property was expected from the results of [24, 25] in the commutative case, and from the analogs of Klyachko's idempotent in [11]. Indeed the non-commutative function

$$K_n(q) = \sum_{|I|=n} q^{\mathrm{Maj}(I)} R_I \tag{7}$$

has for commutative image the Hall-Littlewood function $\tilde{Q}'_{(1^n)}$, whose specialization at $q = e^{2\pi i/n}$ is the power sum p_n, but as an element of the descent algebra becomes

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{Maj}(\sigma)}\,\sigma \tag{8}$$

whose specialization $q = e^{2\pi i/n}$ is, up to a scalar factor, Klyachko's idempotent [19].

Our non-standard action of the Hecke algebra can be extended to the affine Hecke algebra $\tilde{H}(q)$. In this way, it is possible to characterize some two parameter bases of quasi-symmetric and non-commutative symmetric functions which diagonalize the center of $\tilde{H}(q)$. But this does not seem to provide the proper generalization of Macdonald's functions [15].

Acknowledgements

This work is a part of my PhD-thesis. I would like to express all my gratitude to my advisor, Jean-Yves Thibon. Nothing would have been possible without his help. Many computations were done using Maple and especially with the NCSF package by B. C. V. Ung [48] wich can be found in ACE [49] `http://phalanstere.univ-mlv.fr/~ace`.

2 Background

2.1 Compositions

A *composition* $K = (k_1, \dots, k_p)$ is a p-tuple of positive integers. The integers k_i are called *the parts* of the composition. The number p is called the *length* of K and is denoted by $\ell(K)$. Let $n = |K| = k_1 + \cdots + k_p$ be the *sum* of the composition K. We say that K is a composition of n and we write it by $K \vDash n$. A weakly decreasing composition is called a *partition*. A p-tuple of non-negative integers of sum n is called a *pseudo composition*. Let $I = (i_1, \dots, i_q)$ and $J = (j_1, \dots, j_p)$ be two compositions. By $I \cdot J$ we mean the concatenation of the two compositions $I \cdot J = (i_1, \dots, i_q, j_1, \dots, j_p)$. We denote by $I \triangleright J$ the composition defined by $(i_1, \dots, i_q + j_1, \dots, j_p)$. Sometimes we need the composition of length r with all parts equal to i; it will be denoted by (i^r) or briefly by i^r.

Subsets of $\{1, \dots, n-1\}$ are in one-to-one correspondence with compositions of n:

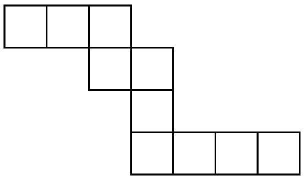
$$S = \{i_1 < i_2 < \cdots < i_p\} \longmapsto \mathrm{C}(S) = (i_1, i_2 - i_1, i_3 - i_2, \dots, n - i_p). \qquad (9)$$

The inverse bijection (*descent set of a composition*) is given by:

$$K = (k_1, \dots, k_p) \longmapsto \mathrm{Des}(K) = \{k_1 + \cdots + k_j,\ j = 1, \dots, p-1\}. \qquad (10)$$

For instance, the composition $(3, 1, 2, 1, 2, 2)$ of 11 corresponds to the subset $\{3, 4, 6, 7, 9\}$ of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

A composition can be represented by a skew Young diagram called a *ribbon diagram* of shape I (see [40]). For example, the ribbon diagram of the composition $I = (3, 2, 1, 4)$ is



The *conjugate* composition $I^{\sim}$ of I is obtained by reading from left to right the heights of the columns of the ribbon diagram of I. On the descent set, the conjugate is the complement in $\{1 \ldots n\}$. For example, the conjugate of $(3, 2, 1, 4)$ is $(1, 1, 2, 3, 1, 1, 1)$. These two compositions correspond to descent sets $\{3, 5, 6\}$ and $\{1, 2, 4, 7, 8, 9\}$.

Let I and J be two compositions of the same number n. We say that I is *finer* than J iff $\mathrm{Des}(I) \supseteq \mathrm{Des}(J)$. We will denote this by $I \succeq J$. This can be read on compositions in the following way: Let $J = (j_1, \ldots, j_p)$. The composition I is finer than J iff there exist compositions I_1 of j_1 , I_2 of j_2, $\ldots$, I_p of j_p such that $I = I_1 \cdot I_2 \cdots I_p$ is the composition obtained by gluing $I_1, \ldots, I_p$ one after another. In this case we call the composition $\#(I, J) = (\ell(I_1), \ldots, \ell(I_p))$ the *refining composition*.

Keeping the previous notations, let us write $I = (i_1, \ldots, i_q) = I_1 \cdot I_2 \cdots I_p$. We regard I as a q-tuple in which some parts are separated by "," and some other by "$\cdot$". Then if we replace all subsequences between the "," by the sum of their parts minus the number of "$\cdot$", we get a composition of $n - \ell(J) + 1$ which we denote by I/J. The composition I/J is the composition associated with the complement of the set $\mathrm{Des}(J)$ in $\mathrm{Des}(I)$, considered as a subset of the complement of $\mathrm{Des}(J)$ in $\{1, \ldots, n-1\}$. Let J be a fixed composition of n. The above algorithm defines an order preserving one-to-one correspondence between compositions finer than J and compositions of $n - \ell(J) + 1$.

Example 2.1. Let $I = (2, 2, 1, 2, 1, 1, 1, 2, 1, 1)$ and $J = (2, 3, 5, 2, 1, 1)$, we write first $I = (2 \cdot 2, 1 \cdot 2, 1, 1, 1 \cdot 2 \cdot 1 \cdot 1)$ and then we get $I/J = (3, 2, 1, 1, 2)$ and $\#(I, J) = (1, 2, 4, 1, 1, 1)$. On descent sets, $\mathrm{Des}(J) = \{2, 5, 10, 12, 13\} \subset \{1 < \cdots < 13\}$ and $\mathrm{Des}(I) = \{2, 4, 5, 7, 8, 9, 10, 12, 13\}$. Then $\mathrm{Des}(I) \setminus \mathrm{Des}(J) = \{4, 7, 8, 9\}$ considered as a subset of $\{1 < 3 < 4 < 6 < 7 < 8 < 9 < 11\}$ which is, after renumbering, $\{3, 5, 6, 7\}$ considered as a subset of $\{1 < \cdots < 8\}$ and so is associated to the composition $(3, 2, 1, 1, 2)$.

The following properties are immediate consequences of the definitions.

Proposition 2.1. *Let I and J be two compositions of the same number n. Suppose that $I \succeq J$. Then $J^{\sim} \succeq I^{\sim}$ and $(\#(I, J))^{\sim} = J^{\sim}/I^{\sim}$.*

Example 2.2. With the notations of the former example, we have

$$I^{\sim} = (1,2,3,5,3) \text{ and } ({}^{\sim}J) = (1,2,1,2,1,1,1,2,3).$$

We verify that

$$\begin{aligned} J^{\sim}/I^{\sim} &= (2,2,1,1,4) = (1,2,4,1,1,1)^{\sim} = (\#(I,J))^{\sim} \\ \#(J^{\sim},I^{\sim}) &= (1,1,2,4,1) = (3,2,1,1,2)^{\sim} = (I/J)^{\sim}. \end{aligned}$$

Finally let $K = (k_1, \ldots, k_p)$ be a composition of n. The *major index* of Mac-Mahon [40] is defined by

$$\mathrm{Maj}(K) = \sum_{i \in \mathrm{Des}(K)} i = (p-1)k_1 + (p-2)k_2 + \cdots + 2k_{p-2} + k_{p-1}. \tag{11}$$

It verifies the equation

$$\mathrm{Maj}(K) + \mathrm{Maj}(K^{\sim}) = \frac{n(n-1)}{2}. \tag{12}$$

For example, if $J = (2,3,5,2,1,1)$, then $\mathrm{Maj}(J) = 2+5+10+12+13 = 42$. and $\mathrm{Maj}(J^{\sim}) = 1+3+4+6+7+8+9+11 = 49$ and $42+49 = 91 = \frac{13\cdot 14}{2}$.

2.2 Quasi-symmetric functions

Let $X = \{x_1 < x_2 < \cdots < x_n\}$ denote a totally ordered set of commutative indeterminates. X is called the alphabet. By $\mathcal{P}(X)$ (resp. $\mathcal{P}_k(X)$), we mean the set of the subsets (resp. k-elements subsets) of the alphabet X.

Let m be the monomial $x_1^{m_1} \ldots x_n^{m_n}$ where the m_i are possibly zero. For readability, we identify m with the pseudo-composition $[m_1, m_2, \ldots, m_n]$. We define the *support* of m as the set $A \in \mathcal{P}(X)$ of the x_i whose exponent is non-zero and the composition K obtained by removing the zeros in the sequence $(m_1, m_2, \ldots, m_n)$. In the sequel we write A^K in place of the monomial m. For example if $X = \{x_1 < x_2 < x_3 < x_4\}$, we write $x_1^2 x_3 = [2,0,1,0] = \{x_1, x_3\}^{(2,1)}$ and $x_1^3 x_2^5 x_4 = [3,5,0,1] = \{x_1, x_2, x_4\}^{(3,5,1)}$.

A polynomial $f \in \mathbb{C}[X]$ is said to be *quasi-symmetric* iff for each composition $I = (i_1, \ldots, i_r)$ the coefficient of the monomial A^I is independent of the set of variables $A \in \mathcal{P}_r(X)$. The quasi-symmetric polynomials form a subalgebra of $\mathbb{C}[X]$ denoted by $\mathbf{Qsym}_n$. It is often convenient to let $n \to \infty$ and to take the inverse limit in the category of graded ring. Hence we get an algebra called the algebra of *quasi-symmetric functions* [12]. Such functions can be seen as formal sums of monomials on an infinite alphabet $X = \{x_1 < x_2 < \cdots < x_n < \cdots\}$.

It is clear that the family of so-called *quasi-monomial functions* defined by

$$M_I = \sum_{A \in \mathcal{P}_r(X)} A^I = \sum_{j_1 < \cdots < j_r} x_{j_1}^{i_1} \cdots x_{j_r}^{i_r} = \sum_{k \to I} x^k \tag{13}$$

labeled by compositions $I = (i_1 \ldots, i_r)$ form a basis of **Qsym**. The last sum is extended to all pseudo compositons $k \to I$ of length n obtained by inserting zeros in the composition I. For example

$$M_{(2,1)} = \{x_1, x_2\}^{(2,1)} + \{x_1, x_3\}^{(2,1)} + \{x_1, x_4\}^{(2,1)} + \{x_2, x_3\}^{(2,1)} + \{x_2, x_4\}^{(2,1)} + \{x_3, x_4\}^{(2,1)}$$

and for readability, we prefer to write

$$M_{(2,1)} = [2,1,0,0] + [2,0,1,0] + [2,0,0,1] + [0,2,1,0] + [0,2,0,1] + [0,0,2,1]$$

instead of $M_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_1^2 x_4 + x_2^2 x_3 + x_2^2 x_4 + x_3^2 x_4$

Another important basis of **Qsym** is given by the *quasi-ribbon functions.* In [12], Gessel defined them as characteristic functions of permutations with given descent set. They appear to play the same role as the Schur functions in the theory of symmetric function [21, 22]. One can define them by

$$F_I = \sum_{I \preceq J} M_J, \tag{14}$$

e.g., $F_{122} = M_{122} + M_{1112} + M_{1211} + M_{11111}$. It is important to note that F_I is the commutative image of the sum of all quasi-ribbon words of shape I ([12], see also Section 4). Their characteristic properties are now explained by the fact that the algebra of quasi-symmetric functions is in natural duality with the Hopf algebra of noncommutative symmetric functions, which is nothing but the direct sum of the descent algebras of all symmetric groups [41].

2.3 Noncommutative symmetric functions

The algebra of *noncommutative symmetric functions* [11] is the free associative algebra $\mathbf{Sym} = \mathbb{C}\langle S_1, S_2, \ldots \rangle$ generated by an infinite sequence of noncommutative indeterminates S_k, called *complete* symmetric functions. For a composition $I = (i_1, i_2, \ldots, i_r)$, one sets $S^I = S_{i_1} S_{i_2} \ldots S_{i_r}$. The family (S^I) is a linear basis of **Sym**. A useful realization can be obtained by taking an infinite alphabet $A = \{a_1, a_2, \ldots\}$ and defining its complete homogeneous symmetric functions by the generating function

$$\sum_{n \geq 0} t^n S_n(A) = (1 - t\, a_1)^{-1} (1 - t\, a_2)^{-1} (1 - t\, a_3)^{-1} \ldots \tag{15}$$

Then $S_n(A)$ appears as the sum of all nondecreasing words of length n. Note that these functions are not symmetric in the usual sense. These are invariant for a more subtle action due to Lascoux and Schützenberger [29], (see also [11]). The role of Schur functions is played by the noncommutative *ribbon Schur functions* R_I defined by

$$R_I = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)} S^J \ . \tag{16}$$

The family (R_I) form a basis of **Sym**. In the realization of **Sym** given by equation (15), R_I reduces to the sum of all words of shape I [11].

The pairing $\langle \cdot, \cdot \rangle$ between **Qsym** and **Sym** is defined by $\langle M_I, S^J \rangle = \delta_{IJ}$ or equivalently $\langle F_I, R_J \rangle = \delta_{IJ}$ (*cf.* [41, 11]). This duality can be interpreted as the canonical duality between the Grothendieck groups respectively associated with finite dimensional and projective modules over 0-Hecke algebras ([8, 20], see also Section 4).

3 Quasi-symmetrizing actions

Let $\mathfrak{S}_n$ denote the n-th symmetric group. For $i = 1, 2, \dots, n-1$, let σ_i denotes the elementary transposition $(i, i+1)$. It is a well known fact that $\mathfrak{S}_n$ is generated by the $(\sigma_i)_{i=1,\dots,n-1}$ with the presentation:

$$\begin{aligned} \sigma_i^2 &= 1 && \text{for } 1 \le i \le n-1, \\ \sigma_i\, \sigma_j &= \sigma_j\, \sigma_i && \text{for } |i-j| > 1, \\ \sigma_i\, \sigma_{i+1}\, \sigma_i &= \sigma_{i+1}\, \sigma_i\, \sigma_{i+1} && \text{for } 1 \le i \le n-2. \end{aligned} \tag{17}$$

These relations are called Moore-Coxeter relations. The last two relations are called braid relations. A *reduced word* $\sigma = \sigma_{i_1} \dots \sigma_{i_p}$ for the permutation σ is a minimal length decomposition. The length $\ell(\sigma) = p$ of such a decomposition is called the *length of the permutation* σ. It is equal to the number of inversions of σ. The number $\epsilon(\sigma) = (-1)^{\ell(\sigma)}$ is called the sign of σ.

The permutation of minimal length is the identity Id. The permutation of maximal length is the permutation $n, n-1, \dots, 1$ (such that i is sent to $n+1-i$). We will denote this permutation by ω. It is of length $n(n-1)/2$ and it has many reduced decompositions, among them:

$$\omega = \sigma_1 (\sigma_2\, \sigma_1)(\sigma_3\, \sigma_2\, \sigma_1) \dots (\sigma_{n-1} \dots \sigma_2\, \sigma_1) \tag{18}$$
$$= (\sigma_1\, \sigma_2 \dots \sigma_{n-1}) \dots (\sigma_1\, \sigma_2\, \sigma_3)(\sigma_1\, \sigma_2)\, \sigma_1 \tag{19}$$

Definition 3.1. *We say that the permutation σ admits a* descent *at position i if $\sigma(i) > \sigma(i+1)$. Otherwise, we say that σ admits a* rise *at position i.*

This is equivalent to the fact that there exists a reduced word for σ ending with σ_i. Therefore this notion makes sense for arbitrary Coxeter groups, in this context one would rather say that σ_i is a descent for σ. Another way to express this definition is the following: an elementary transposition σ_i is a descent for σ if and only if there exists a permutation σ' such that $\sigma = \sigma' \sigma_i$ and $\ell(\sigma) = \ell(\sigma') + 1$.

Definition 3.2. *Symmetrically, one says that the permutation σ admits a* recoil *at i if $\sigma^{-1}(i) > \sigma^{-1}(i+1)$.*

For all $i = 1, \ldots, n-1$, there is a descent and a recoil at position i in the maximal permutation ω. This means that we can start or end a reduced word for the maximal permutation by any elementary transposition.

3.1 The action of the symmetric group

This section is devoted to the study of an unusual action of the symmetric group on polynomials whose invariants are the quasi-symmetric polynomials. After defining this action, we give some character formulas. In particular, we show that there are only very few irreducible representations occurring in it.

As usual permutations act on polynomials by the formula:

$$(\sigma f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \tag{20}$$

or with the notations of the previous sections,

$$\sigma[m_1, \ldots, m_n] = [m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)}]. \tag{21}$$

Let us define the quasi symmetrizing action.

Definition 3.3. *Let $m = x_1^{k_1} \ldots x_n^{k_n} = [k_1, \ldots, k_n]$ be a monomial. The operator $\boldsymbol{\sigma}_i$ acts on m by*

$$\boldsymbol{\sigma}_i[k_1, \ldots, k_i, k_{i+1}, \ldots, k_n] = \begin{cases} [k_1, \ldots, k_{i+1}, k_i, \ldots, k_n] & \text{if } k_i = 0 \text{ or } k_{i+1} = 0 \\ [k_1, \ldots, k_i, k_{i+1}, \ldots, k_n] & \text{if } k_i \neq 0 \text{ and } k_{i+1} \neq 0. \end{cases} \tag{22}$$

The map $\sigma_i \to \boldsymbol{\sigma}_i$ defines a faithful action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{C}[X]$ called the quasi-symmetrizing action.

To see this we recast (22) in the following form:

Proposition 3.1. *Let $m = [k_1, \dots, k_n] = A^I$ be a monomial. The quasi-symmetrizing action of a permutation $\sigma \in \mathfrak{S}_n$ is given by the following operator $\boldsymbol{\sigma}$:*

$$\boldsymbol{\sigma}(A^I) = (\sigma\, A)^I = \{x_{\sigma(i)} \mid x_i \in A\}^I. \tag{23}$$

Proof — It is easy to see that both expressions give the same operator when $\sigma = \sigma_i$. For example, on $[k_1, k_2, 0] = \{x_1, x_2\}^{(k_1,k_2)}$, the transposition σ_1 gives $[k_1, k_2, 0]$, which is $(\sigma_1\{x_1, x_2\})^{(k_1,k_2)}$ and σ_2 gives $[k_1, 0, k_2]$ which is $(\sigma_2\{x_1, x_2\})^{(k_1,k_2)} = (\{x_1, x_3\})^{(k_1,k_2)}$. Since it is obvious that the expression (23) defines an action of the symmetric group, the operators defined by (22) verify the defining relations of the symmetric group. So, both expressions define the same action of $\mathfrak{S}_n$. Moreover, this action is clearly faithful, since the action on monomials of degree 1, that is the variables, allows to reconstruct the permutation. $\square$

Example 3.1. From equation (23), we get that

$$\begin{aligned}
\boldsymbol{\sigma}_1(x_1^6x_2^2) = \boldsymbol{\sigma}_1[6,2,0] = (\sigma_1\{x_1,x_2\})^{(6,2)} = \{x_1,x_2\}^{(6,2)} = x_1^6x_2^2 \\
\boldsymbol{\sigma}_1(x_1^6x_3^2) = \boldsymbol{\sigma}_1[6,0,2] = (\sigma_1\{x_1,x_3\})^{(6,2)} = \{x_2,x_3\}^{(6,2)} = x_2^6x_3^2
\end{aligned}$$

which agrees with (22). Let σ be the permutation which exchanges 1 and 4, and fixes 2 and 3. Let $\boldsymbol{\sigma}$ its associated quasi-symmetrizing operator, then

$$\boldsymbol{\sigma}[1,0,2,0] = (\sigma\{x_1,x_3\})^{(1,2)} = \{x_3,x_4\}^{(1,2)} = [0,0,1,2].$$

so that $\boldsymbol{\sigma}(x_1x_3^2) = x_3x_4^2$.

Note 3.1. In the sequel, we will see that the quasi-symmetrizing action is not a faithful action of the *algebra* of the symmetric group. Thus we have to distinguish between abstract permutations and quasi-symmetrizing operators acting on polynomials. We use the following convention: The abstract elements of the symmetric group or Hecke algebra will be denoted by normal-type letter for example σ or T. The associated quasi-symmetrizing operators will appear in bold-type such as $\boldsymbol{\sigma}$ or $\boldsymbol{T}$. Moreover since the classical actions are faithful, we don't need to distinguish if from abstract permutations, so we keep the normal-type for it.

Note 3.2. This action of $\mathfrak{S}_n$ is different from the classical one, for example: $\sigma_1(x_1^2x_2) = x_1x_2^2$ whereas $\boldsymbol{\sigma}_1(x_1^2x_2) = x_1^2x_2$. It is important to see that the quasi-symmetrizing action is an action on the vector space of polynomials, with no relation with the algebra structure: $(\boldsymbol{\sigma}_1\, x_1^2)(\boldsymbol{\sigma}_1\, x_2) = x_1x_2^2$ whereas $\boldsymbol{\sigma}_1(x_1^2x_2) = x_1^2x_2$. In particular, though symmetric functions are scalars for the classical σ_i, in our case, the only polynomials f such that $\boldsymbol{\sigma}_i(fg) = f\,\boldsymbol{\sigma}_i(g)$ for all g are the constant ones.

Let us have a closer look at this representation of the symmetric group. It is possible to give the character of this representation. Our notations for symmetric functions are those of [38].

Recall that the direct sum $\bigoplus_{n\geq 0} R(\mathfrak{S}_n)$ of Grothendieck rings of all symmetric groups is in natural isomorphism with the ring of symmetric functions, by the so-called Frobenius characteristic map *ch*. It sends the irreducible character χ^λ to the Schur function s_λ. The product of symmetric functions corresponds to induction from $\mathfrak{S}_n \times \mathfrak{S}_p$ to $\mathfrak{S}_{n+p}$. One can get the value of the character χ on the conjugacy class (cycle type) indexed by the partition $\mu = (\mu_1, \dots, \mu_p)$ by the formula:

$$\chi(\mu) = \langle \mathrm{ch}(\chi), p_\mu \rangle . \tag{24}$$

where p_μ is product of power sum symmetric functions. The quasi-symmetrizing action is compatible with the usual grading of polynomial rings. Hence, one can define the graded characteristic ch_t of the representation on $\mathbb{C}[X]$ as the generating series of the characteristics of the representations on the homogeneous components $\mathbb{C}_i[X]$:

$$\mathrm{ch}_t(\mathbb{C}[X]) = \sum_{i=0}^{\infty} \mathrm{ch}(\mathbb{C}_i[X])t^i. \tag{25}$$

Proposition 3.2. *The graded characteristic of the quasi symmetrizing action of $\mathfrak{S}_n$ over the space of polynomials is given by*

$$\mathrm{ch}_t(\mathbb{C}[x_1, \dots, x_n]) = \sum_{m=0}^{n} \frac{t^m}{(1-t)^m} h_{(m,n-m)}. \tag{26}$$

This proposition relies on the following well known lemma

Lemma 3.1. *Let $m \leq \frac{n}{2}$. Consider $\mathfrak{S}_n$ acting on $\mathcal{P}_m(X)$. This representation is isomorphic to the induction of the trivial representation of the Young subgroup $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$ to $\mathfrak{S}_n$.*

Proof of the lemma — By definition, the induced representation of the trivial representation of $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$ to $\mathfrak{S}_n$ is given by the right action of $\mathfrak{S}_n$ on the cosets of $\mathfrak{S}_n/(\mathfrak{S}_m \times \mathfrak{S}_{n-m})$. Let us show that there is an isomorphism between this representation and $\mathcal{P}_m(X)$.

Let c be a coset in $\mathfrak{S}_n/(\mathfrak{S}_m \times \mathfrak{S}_{n-m})$. The image of $\{1, \dots, m\}$ under any permutation $\sigma \in c$ is independent of the choice of σ. Moreover, it characterizes c. Hence it makes sense to define the mapping ϕ which sends each coset to the image of $\{1, \dots, m\}$ under any permutation $\sigma \in c$. Its easy to see that this is an isomorphism of representations. □

Proof of proposition 3.2 — The character formula for representations induced from Young subgroups is well known (see *e.g.* [38] section I.7 page 112–114). For $m \leq \frac{n}{2}$ it gives:

$$\mathrm{ch}(\mathcal{P}_m(X)) = \mathrm{ch}(\mathfrak{S}_n/(\mathfrak{S}_m \times \mathfrak{S}_{n-m})) = h_{(n-m,m)} = \sum_{k \leq m} s_{(n-k,k)}.$$

Note that the first two equalities are still true even if $m > \frac{n}{2}$, since $h_{(n-m,m)} = h_{(m,n-m)}$. The representation $\mathcal{P}_m(X)$ occur as many times as the number of compositions of length m. The generating series of such compositions counted by their sum is

$$\sum_{i=1}^{\infty} \left(\sum_{K \vDash i,\ \ell(K)=m} t^i \right) = \frac{t^m}{(1-t)^m}. \tag{27}$$

It remains to see that the representation on the constant polynomials is the trivial one. Its character is $s_n = h_n$. This corresponds to the case $m = 0$. □

Corollary 3.1. *The only irreducible representations occurring in $\mathbb{C}[X]$ under the quasi-symmetrizing action are the trivial one $\chi^{(n)}$ and the two parts representations $\chi^{(n-m,m)}$ for $m \leq \frac{n}{2}$.*

But, the alternating sum of the permutations of a Young subgroup corresponding to a partition λ kills all irreducible representation indexed by partition μ such that the conjugate of λ is not finer than μ. In our case, for a subgroup of type $(1, \ldots, 1, 3, 1, \ldots, 1)$, one gets the following equality :

$$\boldsymbol{\sigma}_i\, \boldsymbol{\sigma}_{i+1}\, \boldsymbol{\sigma}_i - \boldsymbol{\sigma}_{i+1}\, \boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i\, \boldsymbol{\sigma}_{i+1} + \boldsymbol{\sigma}_i + \boldsymbol{\sigma}_{i+1} - 1 = 0 \tag{28}$$

It allows to express the action of every permutation as a linear combination of the action of permutations such that no reduced word has a sequence $\sigma_i\, \sigma_{i+1}\, \sigma_i$. These are exactly the permutations that avoid the pattern 321. Recall that a permutation avoids the pattern 321 if there does not exist $i < j < k$ such that $\sigma(i) > \sigma(j) > \sigma(k)$. These are the permutations whose insertion tableau under the Robinson-Schensted algorithm has at most 2 rows.

As a consequence one has :

Theorem 3.1. *The image of $\mathbb{C}[\mathfrak{S}_n]$ in $\mathrm{End}(\mathbb{C}[X])$ is the quotient of $\mathbb{C}[\mathfrak{S}_n]$ by the ideal generated by*

$$(\sigma_i\, \sigma_{i+1}\, \sigma_i - \sigma_{i+1}\, \sigma_i - \sigma_i\, \sigma_{i+1} + \sigma_i + \sigma_{i+1} - 1)_{i=1\ldots n-2} \tag{29}$$

The family $(\boldsymbol{\sigma})_\sigma$ where σ goes along the set of all permutations avoiding the pattern 321 is a basis of the image of $\mathbb{C}\langle\mathfrak{S}_n\rangle$ in $\mathrm{End}(\mathbb{C}[X])$.

Note that this element admits the following factorization due to Young:

$$(\sigma_i - 1)(\sigma_{i+1}\,\sigma_i - \sigma_{i+1} + 1) \tag{30}$$

We will give another proof of this theorem in the case of the generic Hecke algebra (see Theorem 5.5).

Note 3.3. Let A be a p-subset of $\{1, \dots, n\}$. There are exactly $p!(n-p)!$ permutations fixing m. Thus, for each B of cardinal p, there are exactly $p!(n-p)!$ permutations σ such that $\sigma A = B$. Among them there is a shortest one. Is it easy to see that the set of such permutations is the set of permutations avoiding the pattern 321.

Since the cosets $\mathfrak{S}_n/(\mathfrak{S}_m \times \mathfrak{S}_{n-m})$ play an important role in the sequel, let us record some of their properties.

Proposition 3.3. *Let $A = \{1, \dots, p\}$.*

- *The image of A under σ depends only on the class of σ in $\mathfrak{S}_n/(\mathfrak{S}_p \times \mathfrak{S}_{n-p})$. This defines a bijection between p-subsets of $\{1, \dots, n\}$ and cosets in $\mathfrak{S}_n/(\mathfrak{S}_p \times \mathfrak{S}_{n-p})$.*
- *Let c be a coset in $\mathfrak{S}_n/(\mathfrak{S}_p \times \mathfrak{S}_{n-p})$. It contains a unique shortest permutation σ_c.*
- *The set of permutations σ_c is exactly the set of the permutations occurring in the shuffle*
$$12\dots p \sqcup\!\sqcup\, p+1\dots n.$$
We denote this set by $\mathfrak{S}_{n/p}$.

There are also the two useful encodings for a coset : one can replace the p-subset A by a word w of weight $1^p 0^{n-p}$, putting a $w_i = 1$ if $i \in A$. Another way is given by partition λ fitting in $(n-p)^p$. Reading the upper right border of the Ferrers diagram (in french notations) as a word on the letter "south" or "east".

Proof — The first part has been already proved during the proof of Proposition 3.2. The shortest permutation in the coset c is the unique permutation of c such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \cdots < \sigma(n). \tag{31}$$

By definition, there are exactly the permutations occurring in the shuffle of $12\dots p$ with $p+1\dots n$. $\square$

The main property of the quasi-symmetrizing action is the following:

Proposition 3.4. *A polynomial $f \in \mathbb{C}[x_1 < \cdots < x_n]$ is quasi-symmetric if and only if $\boldsymbol{\sigma}\, f = f$ for all permutations $\sigma \in \mathfrak{S}_n$.*

Proof — By definition, a polynomial is quasi-symmetric iff the coefficient of A^K is independent of $A \in \mathcal{P}_p(X)$. But the symmetric group acts transitively on $\mathcal{P}_p(X)$. Hence, if A and B are two p-subsets of X, there is a permutation σ such that $\sigma\, A = B$ and therefore $\boldsymbol{\sigma}\, A^I = B^I$. □

Is is easy to see that the quasi-monomial polynomial M_K is the sum of the orbit of X^K under the quasi-symmetrizing action. Moreover we can get it in the following way:

Proposition 3.5. *Let $K = (k_1, \ldots, k_p)$ be a composition. Let $m = X^K = [k_1, \ldots, k_p, 0, \ldots, 0]$. Then*

$$M_K = \frac{1}{p!(n-p)!} \sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{\sigma}\, m = \sum_{\sigma \in \mathfrak{S}_n / (\mathfrak{S}_p \times \mathfrak{S}_{n-p})} \boldsymbol{\sigma}\, m \tag{32}$$

Proof — It is sufficient to see that $p!(n-p)$ is the number of permutations which fix the monomial $[k_1, \ldots, k_p, 0, \ldots, 0]$. This is the size of a coset of $\mathfrak{S}_n / (\mathfrak{S}_p \times \mathfrak{S}_{n-p})$. □

3.2 Hecke algebra, quasi-symmetrizing divided differences

Let q be a formal or complex parameter. The Hecke algebra $H_n(q)$ of the symmetric group $\mathfrak{S}_n$ (type A_{n-1}) is the $\mathbb{Q}[q, q^{-1}]$ algebra generated by elements $(T_i)_{i=1,\ldots,n-1}$ with the relations:

$$\begin{aligned} T_i^2 &= (q-1)\, T_i + q && \text{for } 1 \le i \le n-1, \\ T_i\, T_j &= T_j\, T_i && \text{for } |i-j| > 1, \\ T_i\, T_{i+1}\, T_i &= T_{i+1}\, T_i\, T_{i+1} && \text{for } 1 \le i \le n-2. \end{aligned} \tag{33}$$

For generic q (different from 0 or a non-trivial root of unity) $H_n(q)$ is isomorphic to $\mathbb{Q}[\mathfrak{S}_n]$.

Let $\sigma = \sigma_{i_1} \ldots \sigma_{i_p}$ be a reduced word. The defining relations of $H_n(q)$ ensure that the element $T_{i_1} \ldots T_{i_p}$ is independent of the chosen reduced word for σ. We denote this element by T_σ. By convention, $T_{\mathrm{Id}} = 1$, where Id is the identity of the symmetric group. The family $(T_\sigma)_{\sigma \in \mathfrak{S}_n}$ is a basis of the Hecke algebra.

With this choice for the quadratic relation, it is possible to put $q = 0$ in the definition of the Hecke Algebra. We obtain the degenerate Hecke algebra $H_n(0)$. The relations become

$$\begin{aligned} T_i^2 &= -\,T_i && \text{for } 1 \le i \le n-1, \\ T_i\,T_j &= T_j\,T_i && \text{for } |i-j| > 1, \\ T_i\,T_{i+1}\,T_i &= T_{i+1}\,T_i\,T_{i+1} && \text{for } 1 \le i \le n-2. \end{aligned} \tag{34}$$

The goal of the remainder of this section is to show that there exists an action of the degenerate Hecke algebra on polynomials which fixes the quasi-symmetric polynomials. On the way, we get a new expression for the quasi-ribbon polynomials. The method is similar to the one of Lascoux and Schützenberger. In [32], these authors described several families of operators acting on $\mathbb{C}[X]$ which satisfy the braid and Hecke relations. In particular the so-called isobaric divided differences, given by

$$\pi_i\,f = (1+\sigma_i)\frac{(x_i f)}{x_i - x_{i+1}}$$

define an action of the degenerate Hecke algebra $H_n(0)$ (see, *e.g.* [39, 5, 1]).

Definition 3.4. *Let f be a polynomial and $i < n$. The quasi-symmetrizing isobaric divided differences are defined by*

$$\boldsymbol{\pi}_i\,f = \frac{x_i f - x_{i+1}\,\boldsymbol{\sigma}_i\,f}{x_i - x_{i+1}} \quad \textit{and} \quad \overline{\boldsymbol{\pi}}_i = \boldsymbol{\pi}_i - \mathrm{Id}\,. \tag{35}$$

Proposition 3.6. *The actions of $\boldsymbol{\pi}_i$ and $\overline{\boldsymbol{\pi}}_i$ are given by*

$$\boldsymbol{\pi}_i[k_1,\ldots,k_i,k_{i+1},\ldots,k_n] = \begin{cases} \displaystyle\sum_{u=0}^{k_i} [k_1,\ldots,k_i-u,u,\ldots,k_n] & \textit{if } k_i \neq 0 \textit{ and } k_{i+1} = 0 \\ [k_1,\ldots,k_n] & \textit{if } k_i \neq 0 \textit{ and } k_{i+1} \neq 0 \\ \displaystyle -\sum_{u=1}^{k_{i+1}-1} [k_1,\ldots,u,k_{i+1}-u,\ldots,k_n] & \textit{if } k_i = 0 \textit{ and } k_{i+1} > 1 \\ [k_1,\ldots,k_n] & \textit{if } k_i = 0 \textit{ and } k_{i+1} = 0 \\ 0 & \textit{if } k_i = 0 \textit{ and } k_{i+1} = 1 \end{cases} \tag{36}$$

$$\overline{\boldsymbol{\pi}}_i[k_1,\ldots,k_i,k_{i+1},\ldots,k_n] =$$
$$\begin{cases} \displaystyle\sum_{u=1}^{k_i} & [k_1,\ldots,k_i-u,u,\ldots,k_n] & \textit{if } k_i\neq 0 \textit{ and } k_{i+1}=0 \\ & 0 & \textit{if } k_i = 0 \textit{ and } k_{i+1}=0 \\ -\displaystyle\sum_{u=0}^{k_{i+1}-1} & [k_1,\ldots,u,k_{i+1}-u,\ldots,k_n] & \textit{if } k_i = 0 \textit{ and } k_{i+1}\neq 0 \\ & 0 & \textit{if } k_i\neq 0 \textit{ and } k_{i+1}\neq 0 \end{cases} \tag{37}$$

These formulas are obvious consequences of the identity:

$$x^n - y^n = (x-y)\sum_{u+v=n-1} x^u y^v. \tag{38}$$

For example : $\boldsymbol{\pi}_1[1,2,3] = [1,2,3]$ and $\boldsymbol{\pi}_2[1,0,3] = -[1,2,1]-[1,1,2]$. Using this we derive the following:

Lemma 3.2. *One has:*

$$\boldsymbol{\sigma}_i\,\boldsymbol{\pi}_i = \boldsymbol{\pi}_i \qquad \textit{and} \qquad \boldsymbol{\sigma}_i\,\overline{\boldsymbol{\pi}}_i = \boldsymbol{\pi}_i - \boldsymbol{\sigma}_i \tag{39}$$
$$\overline{\boldsymbol{\pi}}_i\,\boldsymbol{\sigma}_i = -\overline{\boldsymbol{\pi}}_i \qquad \textit{and} \qquad \boldsymbol{\pi}_i\,\boldsymbol{\sigma}_i = -\overline{\boldsymbol{\pi}}_i + \boldsymbol{\sigma}_i \tag{40}$$

where fg is the composition $f\circ g$ of the two operators f and g (the operators act on their right). We will still use this notation in the sequel.

Proof — The first formula is a consequence of the definition of the operators (equations (22) and (37)). It says that the image of a polynomial by $\boldsymbol{\pi}_i$ is invariant under the action of $\boldsymbol{\sigma}_i$. Using the fact that $\overline{\boldsymbol{\pi}}_i = \boldsymbol{\pi}_i - \mathrm{Id}$, we get the second one. The other two formulas are proved in the same way. □

The first formula permits to deduce the following useful lemma:

Lemma 3.3. *A polynomial f is invariant under $\boldsymbol{\sigma}_i$ if and only if f is invariant under $\boldsymbol{\pi}_i$.*

These lemmas will be used in the proof the following theorem.

Theorem 3.2. *The $\boldsymbol{\pi}_i$ operators satisfy:*

$$\begin{aligned} \boldsymbol{\pi}_i^2 &= \boldsymbol{\pi}_i && \textit{for } 1\leq i\leq n-1, \\ \boldsymbol{\pi}_i\,\boldsymbol{\pi}_j &= \boldsymbol{\pi}_j\,\boldsymbol{\pi}_i && \textit{for } |i-j|>1, \\ \boldsymbol{\pi}_i\,\boldsymbol{\pi}_{i+1}\,\boldsymbol{\pi}_i &= \boldsymbol{\pi}_{i+1}\,\boldsymbol{\pi}_i\,\boldsymbol{\pi}_{i+1} && \textit{for } 1\leq i\leq n-2. \end{aligned} \tag{41}$$

The $\overline{\boldsymbol{\pi}}_i$ operators satisfy:

$$\begin{aligned} \overline{\boldsymbol{\pi}}_i^2 &= -\,\overline{\boldsymbol{\pi}}_i && \textit{for } 1\leq i\leq n-1, \\ \overline{\boldsymbol{\pi}}_i\,\overline{\boldsymbol{\pi}}_j &= \overline{\boldsymbol{\pi}}_j\,\overline{\boldsymbol{\pi}}_i && \textit{for } |i-j|>1, \\ \overline{\boldsymbol{\pi}}_i\,\overline{\boldsymbol{\pi}}_{i+1}\,\overline{\boldsymbol{\pi}}_i &= \overline{\boldsymbol{\pi}}_{i+1}\,\overline{\boldsymbol{\pi}}_i\,\overline{\boldsymbol{\pi}}_{i+1} = 0 && \textit{for } 1\leq i\leq n-2. \end{aligned} \tag{42}$$

Before proving this theorem, we give a more concise formulation. Let $\sigma = \sigma_{i_1}, \ldots, \sigma_{i_p}$ be a reduced word. The braid relations ensure that the operators $\boldsymbol{\pi}_\sigma = \boldsymbol{\pi}_{i_1} \cdots \boldsymbol{\pi}_{i_p}$ and $\overline{\boldsymbol{\pi}}_\sigma = \overline{\boldsymbol{\pi}}_{i_1} \cdots \overline{\boldsymbol{\pi}}_{i_p}$ are independent of the reduced word for σ. As usual, we set that $\boldsymbol{\pi}_{\mathrm{Id}} = \mathrm{Id}_{\mathbb{C}[X]}$.

Corollary 3.2. *The mappings $T_\sigma \mapsto (-1)^{\ell(\sigma)} \boldsymbol{\pi}_\sigma$ and $T_\sigma \mapsto \overline{\boldsymbol{\pi}}_\sigma$ defines two actions of the degenerate Hecke algebra.*

The classical theory of the Hecke algebra says that if one of these two families of operators satisfies the Hecke relations then the other satisfies it again. Indeed $\boldsymbol{\pi}$ and $\overline{\boldsymbol{\pi}}$ are the image one of each other under the classical involution of the degenerate Hecke algebra. This involution corresponds to the exchange of the two roots $q_1 = 0$ and $q_2 = 1$ in the quadratic Hecke equation

$$(T_i - q_1)(T_i - q_2) = 0. \tag{43}$$

Thus we only need to prove the formulas for $\overline{\boldsymbol{\pi}}$.

Proof — The first remark is that the operator $\overline{\boldsymbol{\pi}}_i$ acts only on the two variables x_i and x_{i+1}. So it is sufficient to prove the few following formulas: $\overline{\boldsymbol{\pi}}_1 \overline{\boldsymbol{\pi}}_1[i,j] = -\overline{\boldsymbol{\pi}}_1[i,j]$ and $\overline{\boldsymbol{\pi}}_1 \overline{\boldsymbol{\pi}}_2 \overline{\boldsymbol{\pi}}_1[i,j,k] = \overline{\boldsymbol{\pi}}_2 \overline{\boldsymbol{\pi}}_1 \overline{\boldsymbol{\pi}}_2[i,j,k] = 0$ for all integers i, j, k.

Let us consider the first equality: $\overline{\boldsymbol{\pi}}_1 \overline{\boldsymbol{\pi}}_1[i,j] = -\overline{\boldsymbol{\pi}}_1[i,j]$. If $i \neq 0$ and $j \neq 0$ the both sides are zero by (37). And formula (39) allows us to deduce the case $i = 0$ and $j \neq 0$ from the cases $i \neq 0$ and $j = 0$. So, we only need to prove $\overline{\boldsymbol{\pi}}_1 \overline{\boldsymbol{\pi}}_1[i,0] = -\overline{\boldsymbol{\pi}}_1[i,0]$.

By equation (37),

$$\overline{\boldsymbol{\pi}}_1 \overline{\boldsymbol{\pi}}_1[i,0] = \overline{\boldsymbol{\pi}}_1 \sum_{j=1}^{i} [j, i-j] = \sum_{j=1}^{i-1} \overline{\boldsymbol{\pi}}_1[j, i-j] + \overline{\boldsymbol{\pi}}_1[0,i].$$

But if $1 \leq j \leq i-1$, equation (37) gives $\overline{\boldsymbol{\pi}}_1[j, i-j] = 0$ and by (39) $\overline{\boldsymbol{\pi}}_1[0,i] = -\overline{\boldsymbol{\pi}}_1[i,0]$. So the announced formula is true.

To prove the second formula, let us compute $\overline{\boldsymbol{\pi}}_1 \overline{\boldsymbol{\pi}}_2 \overline{\boldsymbol{\pi}}_1[0,i,0]$ on one hand and $\overline{\boldsymbol{\pi}}_2 \overline{\boldsymbol{\pi}}_1 \overline{\boldsymbol{\pi}}_2[0,i,0]$ on the other hand, for $i \neq 0$. From (37), we get :

$$\overline{\boldsymbol{\pi}}_1[0,i,0] = -\sum_{\substack{u+v=i \\ v \neq 0}} [u,v,0] \quad \text{and} \quad \overline{\boldsymbol{\pi}}_2[0,i,0] = \sum_{\substack{u+v=i \\ v \neq 0}} [0,u,v].$$

It follows that:

$$\overline{\boldsymbol{\pi}}_2 \overline{\boldsymbol{\pi}}_1[0,i,0] = -\sum_{\substack{u+v=i \\ v \neq 0}} \sum_{\substack{s+t=v \\ t \neq 0}} [u,s,t] = -\sum_{\substack{u+v+w=i \\ w \neq 0}} [u,v,w]$$

But this formula is symmetric in x_1 and x_2, and therefore its image under $\overline{\boldsymbol{\pi}}_1$ is zero. Likewise

$$\overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2[0,i,0] = -\sum_{\substack{u+v=i\\ u\neq 0\\ v\neq 0}}\ \sum_{\substack{s+t=v\\ t\neq 0}}[s,t,v] = -\sum_{\substack{u+v+w=i\\ v\neq 0\\ w\neq 0}}[u,v,w]$$

which is symmetric in x_2 and x_3. Finally

$$\overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2\,\overline{\boldsymbol{\pi}}_1[0,i,0] = \overline{\boldsymbol{\pi}}_2\,\overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2[0,i,0] = 0. \tag{44}$$

Then, by virtue of (39)

$$\overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2\,\overline{\boldsymbol{\pi}}_1[i,0,0] = \overline{\boldsymbol{\pi}}_2\,\overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2[0,0,i] = 0.$$

Since $\overline{\boldsymbol{\pi}}_1[0,0,i] = \overline{\boldsymbol{\pi}}_2[i,0,0] = 0$, the equality (44) still holds for the cases $[i,0,0]$ and $[0,0,i]$.

Let us do the cases $[i,j,0]$, $[i,0,j]$ and $[0,i,j]$, for $i\neq 0$ and $j\neq 0$.

$$\overline{\boldsymbol{\pi}}_1[i,0,j] = \sum_{\substack{u+v=i\\ v\neq 0}}[u,v,j] \quad \text{hence} \quad \overline{\boldsymbol{\pi}}_2\,\overline{\boldsymbol{\pi}}_1[i,0,j] = 0$$

because if v and j are non-zero then $\overline{\boldsymbol{\pi}}_2[u,v,j] = 0$. Likewise

$$\overline{\boldsymbol{\pi}}_2[i,0,j] = -\sum_{\substack{u+v=j\\ v\neq 0}}[i,u,v] \quad \text{hence} \quad \overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2[i,0,j] = -\sum_{\substack{u+v=i\\ v\neq 0}}[u,v,j]$$

And finally

$$\overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2\,\overline{\boldsymbol{\pi}}_1[i,0,j] = \overline{\boldsymbol{\pi}}_2\,\overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2[i,0,j].$$

We conclude with (39) which gives

$$\overline{\boldsymbol{\pi}}_1[0,i,j] = -\,\overline{\boldsymbol{\pi}}_1[i,0,j] \quad \text{and} \quad \overline{\boldsymbol{\pi}}_2[i,j,0] = -\,\overline{\boldsymbol{\pi}}_1[i,0,j]$$

We have shown the equality

$$\overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2\,\overline{\boldsymbol{\pi}}_1\,m = \overline{\boldsymbol{\pi}}_2\,\overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2\,m = 0$$

for all monomials m in three variables. $\square$

3.3 Maximal symmetrizer, quasi-ribbon polynomials

Let ω be the maximal permutation $n, n-1, \ldots, 1$. The operator

$$\boldsymbol{\pi}_\omega = \sum_{\sigma \in \mathfrak{S}_n} \overline{\boldsymbol{\pi}}_\sigma \tag{45}$$

is called the maximal quasi-symmetrizer or total symmetrizer. It has this nice property:

Proposition 3.7. *$\boldsymbol{\pi}_\omega$ is a projector whose image is the space of quasi-symmetric polynomial.*

Proof — First, if f is quasi symmetric, $\boldsymbol{\pi}_i\, f = f$, for all $i \in \{1, \ldots, n-1\}$. It follows that $\boldsymbol{\pi}_\sigma\, f = f$ for all σ and in particular for ω.

Conversely, we have to show that $\boldsymbol{\pi}_\omega\, f$ is quasi-symmetric for all f. But there is a reduced word for ω ending by any elementary transposition. Consequently, for any i, there exists a permutation σ' such that $\boldsymbol{\pi}_\omega = \boldsymbol{\pi}_i\, \boldsymbol{\pi}_{\sigma'}$ and hence by formula (39)

$$\boldsymbol{\sigma}_i\, \boldsymbol{\pi}_\omega = \boldsymbol{\sigma}_i\, \boldsymbol{\pi}_i\, \boldsymbol{\pi}_{\sigma'} = \boldsymbol{\pi}_i\, \boldsymbol{\pi}_{\sigma'} = \boldsymbol{\pi}_\omega$$

This proves that $\boldsymbol{\pi}_\omega\, f$ is invariant by any elementary transposition and consequently by any permutation. By proposition 3.4 it is quasi-symmetric. □

Corollary 3.3. *Let f be a polynomial. The following three properties are equivalent:*

(i) *f is quasi-symmetric,*

(ii) *$\boldsymbol{\pi}_\sigma\, f = f$ for all $\sigma \in \mathfrak{S}_n$.*

(iii) *$\boldsymbol{\pi}_\omega\, f = f$.*

Now, we want to express the image of a monomial by the maximal symmetrizer. If $K = (k_1, \ldots, k_p)$ is a composition, we say that the monomial $X^K = [x_1, \ldots, x_p, 0, \ldots, 0]$ is dominant. The following formula is of great interest:

Theorem 3.3. *Let K be a composition and $M = X^K = [x_1, \ldots, x_p, 0, \ldots, 0]$ its associated dominant monomial. Its image under the total symmetrizer is the quasi-ribbon function:*

$$\boldsymbol{\pi}_\omega\, X^K = F_K. \tag{46}$$

We will not prove this theorem now, since there is a stronger statement in Theorem 3.4.

In the case where the monomial m is not dominant, the straightening rule is given by the fact that $\boldsymbol{\pi}_\omega = \boldsymbol{\pi}_\omega \boldsymbol{\pi}_i$, for any i.

Proposition 3.8 (Straightening rule for quasi-ribbon functions). *Let $m = [k_1, \ldots, k_n]$ be a monomial such that $k_p = 0$. Then*

$$\boldsymbol{\pi}_\omega m = \begin{cases} -\sum\limits_{\substack{u+v=k_{p+1} \\ u\neq 0,\ v\neq 0}} \boldsymbol{\pi}_\omega[k_1, \ldots, u, v, \ldots, k_n] & \text{if } k_{p+1} > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{47}$$

Proof — Since $\omega^{-1} = \omega$, a reduced word for the maximal permutation can end with any permutation. Therefore, for any i, there exists a σ' such that $\boldsymbol{\pi}_\omega = \boldsymbol{\pi}_{\sigma'} \boldsymbol{\pi}_i$. Then $\boldsymbol{\pi}_\omega \boldsymbol{\pi}_i = \boldsymbol{\pi}_{\sigma'} \boldsymbol{\pi}_i^2 = \boldsymbol{\pi}_\omega$, because $\boldsymbol{\pi}_i^2 = \boldsymbol{\pi}_i$. The end of the proof comes from formula (37). □

By an easy induction, we get the final formula:

Proposition 3.9. *Suppose that $m = [k_1, \ldots, k_p, 0, \ldots, 0]$ is a monomial such that $k_p \neq 0$, the k_i for $i < p$ are possibly zero, Then*

$$\boldsymbol{\pi}_\omega m = (-1)^{C_m} \sum_J F_J \tag{48}$$

where C_m is the number of zero parts occurring before k_p. The sum is extended to all compositions obtained from m, by replacing maximal sequences of zeros followed by a non-zero part by a composition of this part. If such a composition does not exist, i.e. if the number of zeros is greater than the part, then the sum is null.

For example, let us compute $\boldsymbol{\pi}_\omega[0, 0, 5, 0, 2, 2, 0]$. There are three zeroes before the last non-zero part, so that $C_m = -1$. The 5 is preceded by two zeroes, so we replace the three parts $0, 0, 5$ by a composition of 5 into three parts, the possible choice are $(1, 1, 3)$, $(1, 2, 2)$, $(1, 3, 1)$, $(2, 1, 2)$, $(2, 2, 1)$, $(3, 1, 1)$. And the first 2 is preceded by one 0, so that the $0, 2$ is replaced by $1, 1$. Finally we get that

$$\begin{aligned}\boldsymbol{\pi}_\omega[0, 0, 5, 0, 2, 2, 0] = -\big(&F_{(1,1,3,1,1,2)} + F_{(1,2,2,1,1,2)} + F_{(1,3,1,1,1,2)} + \\ &F_{(2,1,2,1,1,2)} + F_{(2,2,1,1,1,2)} + F_{(3,1,1,1,1,2)}\big)\end{aligned}$$

Also, $\boldsymbol{\pi}_\omega[0, 0, 2] = 0$ since it is not possible to break 2 into three non-zero parts.

Note that this can we viewed as a definition for F_m where m is not a dominant monomial.

3.4 Partial symmetrizers

In this subsection, we present an explicit formula for the partial symmetrization of a dominant monomial. In particular this will prove Theorem 3.3. Before stating the formula, we need some combinatorial definitions.

Definition 3.5. *We say that the monomial $m = A^I$ fits in a composition K if the composition I is finer than K.*

This is equivalent to the fact that each partial sum of K appears as a partial sum of m:

$$\text{for all } i \leq p, \text{ there exists a } j \leq n \text{ such that } \sum_{l\leq i} k_l = \sum_{l \leq j} m_l. \tag{49}$$

For example $[1, 0, 2, 0, 2]$ fits in $(3, 2)$ but not in $(2, 1, 2)$. The quasi-ribbon function F_K is the sum of all monomials fitting in K.

Definition 3.6. *Let $m = [m_1, \dots, m_n]$ and $m' = [m'_1, \dots, m'_n]$ be two monomials. We say that m is bigger than m' and we write $m \gg m'$, if for all $i < n$ one has*

$$\sum_{j\leq i} m_j \geq \sum_{j\leq i} m'_j. \tag{50}$$

This defines a partial ordering on monomials.

Note that this order extends the natural ordering of partitions. Let us look at the restriction of this order on the monomials fitting in K. The greatest monomial is $X^K = [x_1, \dots, x_p, 0, \dots, 0]$, the smallest monomial is $[0, \dots, 0, x_1, \dots, x_p]$. The so called quasi-crystal graph [22] (see figure 1, 2) shows the graph of this ordering. It allows to give a simple expression for the partial symmetrizer:

Theorem 3.4. *Let K be a composition and $m = X^K = [k_1, \dots, k_p, 0, \dots, 0]$ its associated dominant monomial. Let $\sigma \in \mathfrak{S}_n$. The image of m under the partial symmetrizer $\boldsymbol{\pi}_\sigma$ is given by*

$$\boldsymbol{\pi}_\sigma\, m = \sum_{\substack{m' \gg \boldsymbol{\sigma}\, m, \\ m \text{ fits in } K}} m'. \tag{51}$$

Hence, for $K = (2, 2)$ with 4 variables, $m = [2, 2, 0, 0]$. Let $\sigma = \sigma_1\, \sigma_3\, \sigma_2$, so that $\boldsymbol{\sigma}\, m = [0, 2, 0, 2]$. Then figure 1 shows that :

$$\begin{aligned}\boldsymbol{\pi}_\sigma[2, 2, 0, 0] = {} & [2, 2, 0, 0] + [2, 1, 1, 0] + [2, 0, 2, 0] + [2, 0, 1, 1] + [1, 1, 2, 0] + \\ & [2, 0, 0, 2] + [1, 1, 1, 1] + [0, 2, 2, 0] + [1, 1, 0, 2] + [0, 2, 1, 1] + [0, 2, 0, 2].\end{aligned}$$

The following proposition will simplify the proof.

Proposition 3.10. *Let K a composition of length p and $m = X^K$ the associated dominant monomial. The image of m under $\boldsymbol{\pi}_\sigma$ depend only on the coset of σ in the quotient $\mathfrak{S}_n/(\mathfrak{S}_p \times \mathfrak{S}_{n-p})$.*

Proof — The monomial m is invariant by all the permutation from the Young subgroup $(\mathfrak{S}_p \times \mathfrak{S}_{n-p})$. But, by lemma 3.3, if m is invariant under σ_i, it is invariant under $\boldsymbol{\pi}_i$. The proof follows. $\square$

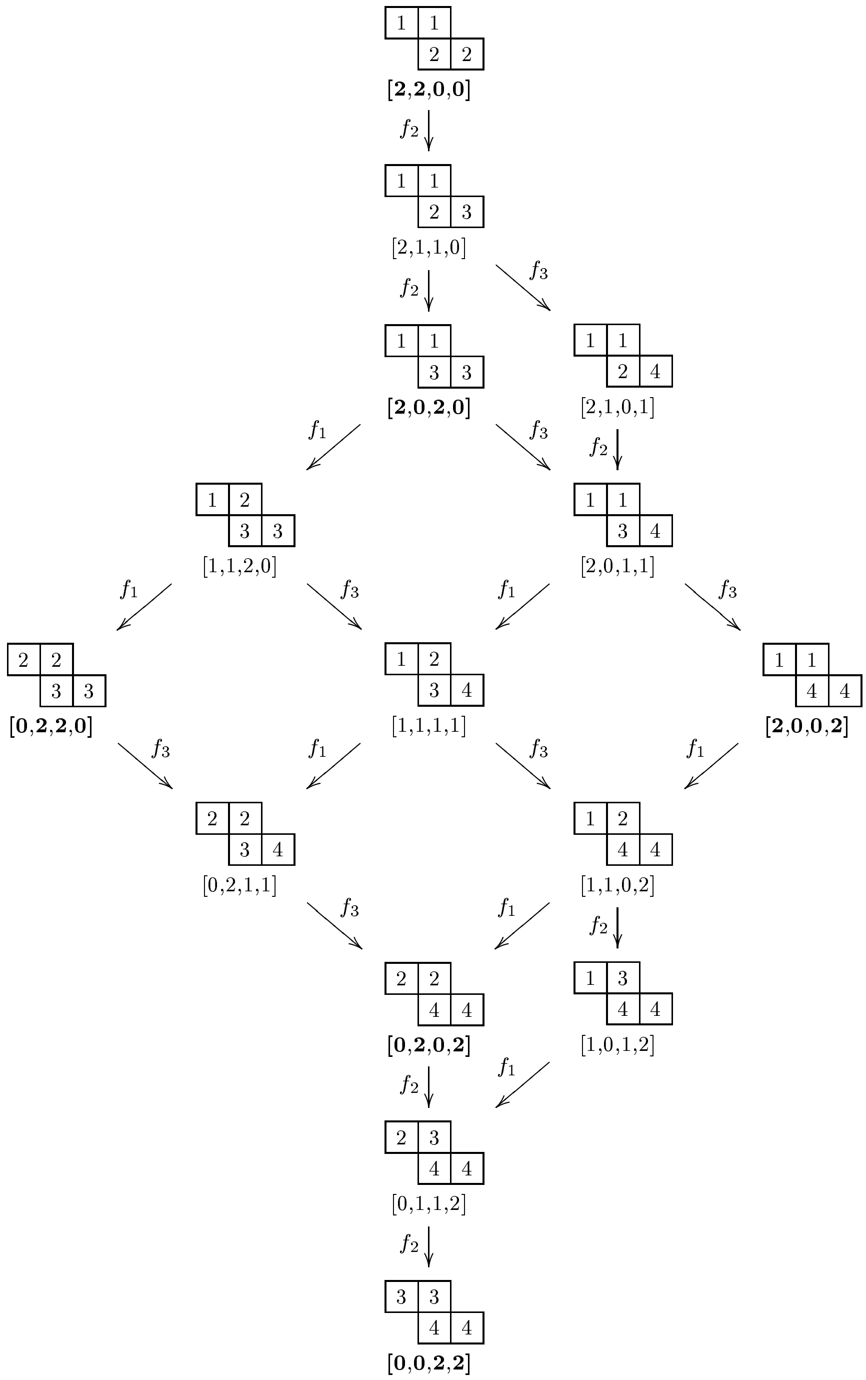


Figure 1: Quasi-crystal graph of $\mathbf{D}_{22}$ for $\mathcal{U}_0(gl_4)$

Let us prove the theorem 3.4.

Proof — By induction on the length of the permutation σ. First, if $\sigma = \mathrm{Id}$ the formula is true since $m = X^K$ is the greatest monomial fitting in K. Now, suppose that the formula is true for the permutation σ. We have to show the property for any $\sigma_i\,\sigma$ such that $\ell(\sigma_i\,\sigma) = 1 + \ell(\sigma)$ (i is a rise of σ^{-1}).

Lemma 3.4. *Let $A = \{1, 2, \ldots, p\}$ for some $p \leq n$ and $\sigma \in \mathfrak{S}_n$. Suppose that i is a rise of σ^{-1}. Then $i + 1$ is in $\sigma\,A$ implies that i is in $\sigma\,A$.*

Proof — Recall that

$$\sigma\,A = \{\sigma(j) \mid j \in A\} = \{i \mid \sigma^{-1}(i) \in A\}.$$

By hypothesis i is a rise of σ^{-1} and thus $\sigma^{-1}(i) < \sigma^{-1}(i+1)$. But if $i+1 \in A$, then $\sigma^{-1}(i + 1) \leq p$ and then $\sigma^{-1}(i) \leq p$. □

Let us go Back to the proof of the theorem 3.4. Write

$$\boldsymbol{\sigma}\,m = \boldsymbol{\sigma}\,X^k = l = [l_1, \ldots, l_n]. \tag{52}$$

By definition of the quasi-symmetrizing action

$$l = (\sigma\{x_1, \ldots, x^p\})^K. \tag{53}$$

It follows that if the exponent of x_{i+1} in $\boldsymbol{\sigma}\,X^K$ is non-zero, then the exponent of x_i is also non-zero.

Notice that in the two cases $l_i = l_{i+1} = 0$ on one hand and $l_i \neq 0$ and $l_{i+1} \neq 0$ on the other hand, the transposition $\boldsymbol{\sigma}_i$ leave l invariant. That is

$$\sigma\{x_1, \ldots, x_p\} = \sigma_i\,\sigma\{x_1, \ldots, x_p\}. \tag{54}$$

By the proposition 3.3 the two permutations σ et $\sigma_i\,\sigma$ belongs to the same coset modulo $\mathfrak{S}_p \times \mathfrak{S}_{n-p}$. As a consequence of the proposition 3.10, we get that $\boldsymbol{\pi}\,m = \boldsymbol{\pi}_i\,\boldsymbol{\pi}\,m$ which is the desired formula. It just remains the case $l_i \neq= 0$ and $l_{i+1} = 0$.

We suppose that $\boldsymbol{\sigma}\,m = l = [l_1, \ldots, l_p]$ with $l_i \neq= 0$ and $l_{i+1} = 0$ and that

$$\boldsymbol{\pi}_\sigma\,m = \sum_{\substack{m' \gg \boldsymbol{\sigma}\,m,\\ m \text{ fits in } K}} m'. \tag{55}$$

This sum breaks into two sorts of monomials.

- If m' is a monomial such that

$$m'_1 + \cdots + m'_{i-1} + m'_i + m'_{i+1} > l_1 + \cdots + l_i \tag{56}$$

then the fact that m' fits in K gives that $l_1 + \cdots + l_i$ is a partial sum of m'. Consequently, either $m_{i+1} \neq 0$ and then $\boldsymbol{\sigma}_i\, m'$ is bigger than l (independently of the fact that $m_i = 0$ or not), or else

$$m'_1 + \cdots + m'_{i-1} \geq l_1 + \cdots + l_i \tag{57}$$

and then

$$[m'_1, \ldots, 0, m'_i, \ldots, m'_n] \gg l. \tag{58}$$

It follows that $\boldsymbol{\sigma}_i$ stabilizes the monomials that verify (56) and thus the action of $\boldsymbol{\pi}_i$ is trivial on the sum of this sort of monomials. Note that this is precisely the sum of the monomials q such that

$$q \gg \boldsymbol{\sigma}_i\, l \text{ and } q_1 + \cdots + q_{i+1} > l_1 + \cdots + l_i \tag{59}$$

- It remains the sum of the monomials such that

$$m'_1 + \cdots + m'_{i-1} + m'_i = l_1 + \cdots + l_i, \text{ and } m'_{i+1} = 0. \tag{60}$$

These are the only cases where the action of $\boldsymbol{\pi}_i$ is non-trivial. The action of $\boldsymbol{\pi}_i$ on such monomials is to send

$$[m'_1, \ldots, m'_i, 0, \ldots, m'_n] \tag{61}$$

to the sum

$$\sum_{u+v=m'_i} [m'_1, \ldots, u, v, \ldots, m'_n]. \tag{62}$$

In this way, we get the sum of all monomials q such that:

$$\begin{aligned} & q_1 + \cdots + q_{i-1} + q_i + q_{i+1} = l_1 + \cdots + l_i \\ & \qquad \text{and} \quad q_1 + \cdots + q_k \geq l_1 + \cdots + l_k, \text{ for } k < i \text{ or } k > i+1. \end{aligned} \tag{63}$$

This is precisely the sum of the monomials q verifying

$$q \gg \boldsymbol{\sigma}_i\, l \text{ and } q_1 + \cdots + q_{i+1} = l_1 + \cdots + l_i \tag{64}$$

The sum of the two preceding kinds of monomials is the sum of all monomials such that $m \gg \boldsymbol{\sigma}_i\, l$. This proves the result for the permutation $\boldsymbol{\pi}_i\, \boldsymbol{\sigma}\, X^K$, so that the proof follows by induction. $\square$

3.5 Characteristic

As in the case of the symmetric group, this representation of the Hecke algebra is not faithful if $n > 2$, since the equality

$$T_1\, T_2\, T_1 = T_2\, T_1\, T_2 = 0 \tag{65}$$

does not holds in the Hecke algebra.

In [8], Krob and Thibon showed that there is an analogue of the Frobenius characteristic for the degenerate Hecke Algebra. The characteristic of a module is no longer a symmetric function but a quasi-symmetric one. The reader is referred to [21] for the details. The Frobenius characteristic of this representation is given by the formula:

$$\mathrm{ch}_t(\mathbb{C}[x_1, \ldots, x_n]) = \sum_{m=0}^{n} \frac{t^m}{(1-t)^m} h_{(m,n-m)}. \tag{66}$$

It does not give the decomposition of the representation into irreducibles, but rather the composition factor of the module, which is not semi-simple. Since this is a specialization $q = 0$ of an action of the generic Hecke algebra, the characteristic is the same as the one of the symmetric group (see [20]).

Moreover, it is possible to give an explicit composition sequence for this representation : let d be an integer. Let $V^d = \mathbb{C}^d[X]$ the homogeneous component of degree d of $\mathbb{C}[X]$. If $m = A^I \in V$ is a monomial, let us call the length of m the cardinal of the support of m. Denote it by $\ell(m)$. Let $V^d_{=l}$ (*resp.* $V^d_{\leq l}$) the subspace of V^d generated by the nomomials of length equal to l (*resp.* smaller than l). By definition of $\overline{\boldsymbol{\pi}}_i$ (equation 37) $V^d_{\leq l}$ is clearly stable under $H_n(0)$. Thus $V^d_{\leq l+1}$ is a submodule of $V^d_{\leq l}$.

Proposition 3.11. *Let d and l be two integers. Choose a total order $\geq$ on the monomial of $V^d_{=l}$ extending the partial order $\gg$:*

$$m^1 \geq m^2 \geq \cdots \geq m^q. \tag{67}$$

Let w^i be the submodule of $V^d_{\leq l}$ defined by

$$W^i = \left(V^d_{\leq l+1} \oplus \bigoplus_{j \geq i} \mathbb{C} m^j \right) / V^d_{\leq l+1}. \tag{68}$$

Then the sequence $(W^i)_i$ is a composition sequence for $V^d_{\leq l}/v^d_{\leq l+1}$.

Proof — Let us look at the quotient $V^d_{\leq l}/V^d_{\leq l+1}$ Let $m = [m_1, \dots, m_n] \in V^d_{=l}$. The only case where $\boldsymbol{\pi}_i\, m$ is different of m and not in $\in V^d_{\leq l+1}$ is $m_i \neq 0$ and $m_{i+1} = 0$. In this case

$$\boldsymbol{\pi}_i[\dots, m_i, 0, \dots] = [\dots, m_i, 0, \dots] + [\dots, 0, m_i, \dots] \mod V^d_{\leq l+1}. \tag{69}$$

Notice that $[\dots, m_i, 0, \dots] \gg [\dots, 0, m_i, \dots]$. This show that W^i is a submodule of $V^d_{\leq l}/V^d_{\leq l+1}$. Moreover W^i/W^{i+1} is one dimensional and thus simple. □

Theorem 3.5. *The image of $H_n(0)$ in $\mathrm{End}(\mathbb{C}[X])$ is the quotient of $H_n(0)$ by the two sided ideal $\mathbb{C}\langle T_\sigma\rangle$ where σ runs over the set of permutations with the pattern* 321.

The proof of this theorem and of the following corollary is postponed to the case of the generic Hecke algebra (see Theorem 5.5).

Corollary 3.4. *The set of permutations σ such that $\overline{\boldsymbol{\pi}}_\sigma \neq 0$ is the set of permutations avoiding the pattern* 321.

4 The degenerate quantum enveloping algebra

The aim of this section is to give a representation theoretical interpretation of the two main results of the preceding section. It is provided by the degenerate quantum group $\mathcal{U}_0(gl_N)$ studied by Krob and Thibon in [20, 21, 22]. This bialgebra is a specialization of a non-standard two parameter analogue of the universal enveloping algebra of gl_N, which was originally defined by Takeuchi in [47]. The main result of Krob and Thibon is that quasi-symmetric functions are in some sense characters for $\mathcal{U}_0(gl_N)$, in particular the irreducible characters are the quasi-ribbon functions F_I. The reader referred to these papers for formal definitions.

Let us recall some properties of this algebra. Like the enveloping algebra $U(gl_N)$, the algebra $\mathcal{U}_0(gl_N)$ is generated by three kinds of elements called Chevalley generators:

- raising generators : $(e_i)_{1\leq i\leq N-1}$,
- lowering generators : $(f_i)_{1\leq i\leq N-1}$,
- diagonal generators : $(k_i)_{1\leq i\leq N}$.

The subalgebra $\mathcal{U}_0(\mathfrak{h})$ generated by the k_i is the Cartan subalgebra. We denote by $\mathcal{U}_0(\mathfrak{b}_+)$ the so-called Borel subalgebra generated by e_i and k_i.

Let $(\xi_i)_{1\leq i\leq N}$ be the canonical basis of $V = \mathbb{C}^N$. As described in [22], there is a natural morphism from $\mathcal{U}_0(gl_N)$ to $\mathrm{End}_{\mathbb{C}}(V)$ which is called the *fundamental representation* (or *vector representation*) of $\mathcal{U}_0(gl_N)$, denoted by (ρ_V, V). The bialgebra structure on $\mathcal{U}_0(gl_N)$ makes it possible to define tensor products of representations. Let $(\rho_{n,N}, V^{\otimes n})$ be the n-th tensor product of the fundamental representation. Note that $\mathcal{U}_0(gl_N)$ is not a Hopf algebra (there is no antipode).

Definition 4.1. *A representation of* $\mathcal{U}_0(gl_N)$ *is said to be* polynomial of degree n *if it is isomorphic to some sub-representation of* $(\rho_{n,N}, V^{\otimes n})$.

4.1 Quasi-crystal graph of an irreducible module, Weyl character formula for $\mathcal{U}_0(gl_N)$

In [22], using the fact that the action of $\mathcal{U}_0(gl_N)$ commutes with Jimbo's action of $H_n(0)$ on the tensor product, and the classification of the irreducible representations of $H_n(0)$ by Carter [3], Krob and Thibon give a complete description of the irreducible polynomial representation of $\mathcal{U}_0(gl_N)$. Their construction makes use analogues of Young's idempotents in $H_n(0)$. The irreducible modules of $\mathcal{U}_0(gl_N)$ are the image in the tensor product of these analogues. On the way they get a canonical basis for such a module. Let us describe the structure of theses modules. First, we need some combinatorial definitions.

Let A be a finite set called the alphabet and let I denote a composition.

Definition 4.2. *A* quasi-ribbon tableau *of shape I is obtained by filling the the ribbon diagram associated to I by letters of A in such a way that each row is non-decreasing from left to right and each column is strictly increasing from* top to bottom*. A word is said to be a* quasi-ribbon word *of shape I if it can be obtained by reading from* bottom to top *and from left to right the columns of a quasi-tableau of shape I.*

The set of all quasi-ribbon words of shape I is denoted by $\mathrm{QR}(I)$.

For example the word $\mathbf{u} = abdcbdfef$ is a quasi-ribbon of shape $(3, 1, 3, 2)$ since it is the reading of the first following quasi-ribbon tableau. Conversely the word $aacbabdcd$ is not a quasi-ribbon word. since the diagram obtained by writing its decreasing factors is not a quasi-ribbon tableau.

quasi-ribbon : $\begin{array}{cccccc} a & b & b & & & \\ & & c & & & \\ & & d & d & e & \\ & & & & f & f \end{array}$ not quasi-ribbon : $\begin{array}{cccccc} a & a & a & & & \\ & & b & & & \\ & & c & \mathbf{b} & c & \\ & & & & d & d \end{array}$

We are now in a position to describe the irreducible polynomial modules of $\mathcal{U}_0(gl_N)$. The modules $\mathbf{D}_K$ of degree n are indexed by compositions $K \vDash n$.

The basis $\xi_{\mathbf{u}}$ of the module $\mathbf{D}_K$ is indexed by quasi-ribbon $\mathbf{u}$ words of shape K, over the alphabet $A = \{1, \ldots, n\}$.

Let us describe the action of Chevalley generators on the quasi-ribbon basis. Suppose $\mathbf{u} = u_1 \ldots u_n$ is a quasi-ribbon word. The diagonal generator k_i sends the vector $\xi_{\mathbf{u}}$ to 0 if $\mathbf{u}$ contain the letter i, and otherwise keeps $\xi_{\mathbf{u}}$ unchanged. Let $\mathbf{u}^+$ (*resp.* $\mathbf{u}^-$) be the word obtained from $\mathbf{u}$ by replacing the last i by a $i+1$ (*resp.* the last $i+1$ by a i). If there is no such letter $\mathbf{u}^+$ is not defined. The raising operator e_i sends $\xi_{\mathbf{u}}$ to $\xi_{\mathbf{u}^-}$ if $\mathbf{u}^-$ exists and is a quasi-ribbon word of shape K, otherwise it sends $\xi_{\mathbf{u}}$ to 0. The lowering operator f_i send $\xi_{\mathbf{u}}$ to $\xi_{\mathbf{u}^+}$ if $\mathbf{u}^+$ exists and is a quasi-ribbon word of shape K, otherwise it sends $\mathbf{u}$ to 0.

Example 4.1. In the module $\mathbf{D}_{(1,2)}$ for $\mathcal{U}_0(gl_4)$ the vector 212 is sent to 0 by f_1 because 222 is not a quasi-ribbon word of shape $(1, 2)$. On the other hand f_2 sends it to 213 (see figure 2).

This describes completely the irreducible $\mathcal{U}_0(gl_N)$-module $\mathbf{D}_K$. Figure 2 shows the structure of a $\mathcal{U}_0(gl_N)$-module. We call this graph the quasi-crystal graph $\Gamma_N(I)$. For simplification we only show the action of the f_i. The action of e_i reverses that of the f_i.

Remark that although crystal graphs describe in general only the combinatorial skeleton of a generic module, the quasi-crystal graph $\Gamma_N(I)$ encode the full structure of the $\mathcal{U}_0(gl_N)$-module $\mathbf{D}_K$.

Theorem 4.1 (Krob-Thibon [22]). *The $\mathcal{U}_0(gl_N)$-module $\mathbf{D}_K$ module is irreducible. Its character is the quasi-ribbon function $F_I(x_1, \ldots, x_n)$.*

The $(\mathbf{D}_K)$ form a complete family of irreducible polynomial $\mathcal{U}_0(gl_N)$-modules.

Recall that in the classical case the character of the irreducible gl_N module $\mathbf{D}_\lambda$ is the Schur function s_λ. Though it has been originally defined as the quotient of two alternants, the Schur function is given by the symmetrization formula [39, 5]: $s_\lambda = \pi_\omega X^\lambda$. In our case theorem 3.3 can be restated as follows:

Theorem 4.2 (Weyl character formula for $\mathcal{U}_0(gl_N)$). *Let K be a composition. The character of the irreducible module $\mathbf{D}_K$ is given by the formula*

$$\mathrm{ch}(\mathbf{D}_K) = \boldsymbol{\pi}_\omega X^K = F_K. \tag{70}$$

Note 4.1 (Hypoplactic characters). In fact in [22], Krob and Thibon define characters of $\mathcal{U}_0(gl_N)$ as elements of a quotient of the plactic algebra called the hypoplactic algebra, rather than as quasi-symmetric functions.

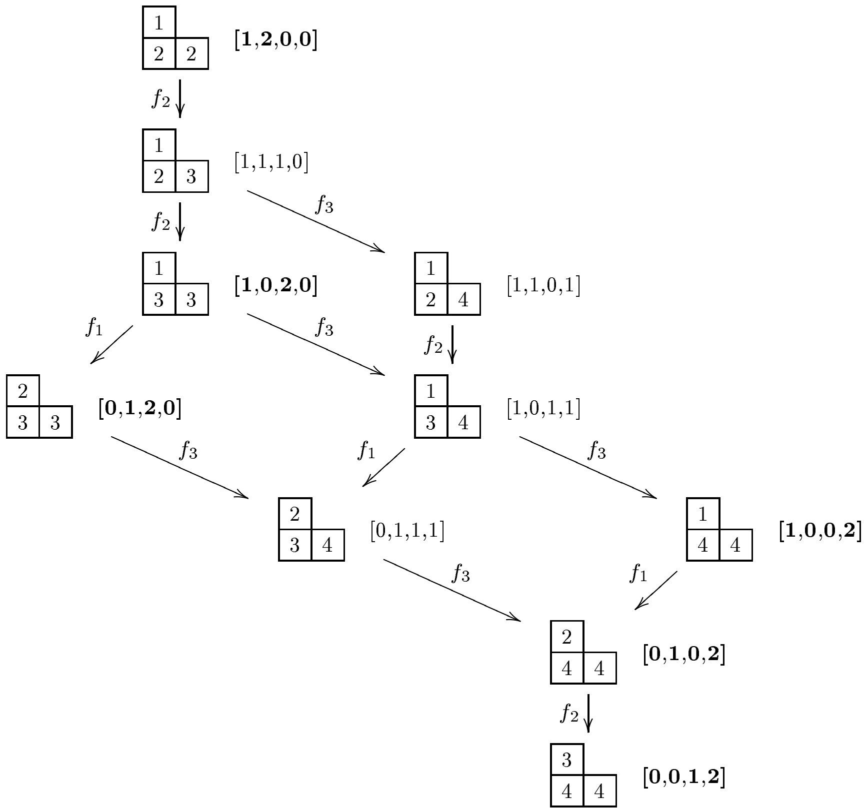


Figure 2: Quasi-crystal graph of $\mathbf{D}_{12}$ for $\mathcal{U}_0(gl_4)$

The hypoplactic algebra admit quasi-ribbon words for basis. They play the same role as tableaux in the plactic algebra. Krob and Thibon show that the subalgebra generated by the hypoplactic quasi-ribbon functions is commutative and isomorphic to the algebra of the quasi-symmetric functions. The reader may look at [43] for a combinatorial study of the hypoplactic algebra.

Proposition 4.1 of the next section allows to lift the action of the Hecke algebra on polynomials in the hypoplactic algebra. That is, we construct hypoplactic divided differences such that the hypoplactic quasi-ribbon function F_K is the image of the unique quasi-ribbon word of shape and evaluation K under the maximal symmetrizer $\boldsymbol{\pi}_\omega$.

In the classical case the symmetrizing formula has been stated by Demazure in [5]. The partial symmetrizer gives the characters of the so-called Demazure module. The following subsection shows that this has a complete analogue in our case.

4.2 Demazure character formula for $\mathcal{U}_0(gl_N)$

The Cartan subalgebra $\mathcal{U}_0(\mathfrak{h})$ acts diagonally on polynomial modules. In the classical case, the eigenspaces for this subalgebra are called weight spaces and the linear forms in $\mathcal{U}_0(\mathfrak{h})^*$ which give the eigenvalues are called weights. In our case, due to the degeneracy, the only possible eigenvalues for k_i are 0 or 1, so that the action of the Cartan subalgebra in not sufficient to define a good notion of weight. We have to use the action of the e_i and f_i and to count multiplicities is some sense. Here we only give a combinatorial definition of the weight of a vector taken out of the previous basis of $\mathbf{D}_K$.

Definition 4.3. *Let K be a composition and $\mathbf{u} = u_1 \dots u_n$ a quasi-ribbon word of shape K. The weight of the vector $\xi_{\mathbf{u}}$ is the evaluation* $\mathrm{Eval}(u)$ *of the word $\mathbf{u}$, that is, the commutative image of the word $\mathbf{u}$. Otherwise said, if $|\mathbf{u}|_i$ denotes the number of occurrences of the letter i in $\mathbf{u}$, the weight of $\xi_{\mathbf{u}}$ is the commutative monomial $x_1^{|\mathbf{u}|_1} \dots x_N^{|\mathbf{u}|_N}$. As usual it will be identified with the pseudo composition $\big[|\mathbf{u}|_1, \dots, |\mathbf{u}|_N\big]$.*

Recall that a monomial $m = A^I$ fits in K if the composition I is finer than K.

Proposition 4.1. *Let K be a composition. Then,* Eval *is a bijective correspondence between quasi-ribbon word of shape K and monomials fitting in K.*

In terms of representation theory, there is a vector of weight m in $\mathbf{D}_K$ if and only if m fits K, and in this case, it is unique, up to constants. Otherwise said, in the irreducible representations the weight spaces are one dimensional.

This allows one to transport the combinatorial construction on monomials to vectors. For example the action of the symmetric group is transported to vectors. This can be seen as the action of the Weyl group $W = \mathfrak{S}_N$.

The ordering of the monomials defined in (3.4) makes sense in representation theory. Indeed, the weight of the image of a vector $\xi_{\mathbf{u}}$ by a raising generator is bigger than the weight of $\xi_{\mathbf{u}}$ itself. It allows us to define highest weights and highest weight vectors: The highest weight of the module $\mathbf{D}_K$ is X^K and the associated vector is $\xi_{\mathbf{u}}$ where $\mathbf{u} = 1^{k_1}2^{k_2} \dots N^{k_N}$. By abuse of notation it is denoted by ξ_K. The above properties show that this ordering has a nice expression on words: the ordering of vectors of shape K is nothing but the product order on quasi-ribbon words, that is, if $\mathbf{u}$ and $\mathbf{u}'$ are two quasi-ribbons of shape K

$$u_1 \dots u_N \gg u'_1 \dots u'_N \qquad \text{iff} \qquad u_i > u'_i \text{ for all } i \geq N. \tag{71}$$

Consequently the quasi-crystal graph is the graph of the product order restricted to the set of quasi-ribbons of shape K.

Gessel shows in [12] that the quasi-ribbon function F_K is the sum of all evaluations of quasi-ribbon words of shape K. It means that the character of an irreducible module is the generating series of the dimensions of its weight spaces, as in the classical cases. In the classical case, this property has a refinement due to Demazure [5]:

Definition 4.4. *Let $\mathbf{D}_K$ be an irreducible module of $\mathcal{U}_0(gl_N)$. An* extremal weight *is the image of the highest weight under any element of the Weyl group $W = \mathfrak{S}_N$. To each extremal weight is associated a unique vector up to a constant. These vectors are called* extremal vectors*.*

In our case, these are the vectors of weight $m = A^K$ for all $A \in \mathcal{P}_k(X)$ where k is the length of K. They appear in bold-type in the quasi-crystal graphs. Recall that the Borel subalgebra $\mathcal{U}_0(\mathfrak{b}_+)$ is the algebra generated by e_i and k_i.

Definition 4.5. *Let ξ be an extremal vector. The $\mathcal{U}_0(\mathfrak{b}_+)$-module generated by ξ is called a Demazure module.*

The following theorem is a degenerate analogue of the classical Demazure formula. Since the weight spaces are one dimensional, it gives also a characterisation of bases of Demazure modules analogue to those of [34] (see also [31] for the type A_n).

Theorem 4.3 (Demazure character formula for $\mathcal{U}_0(gl_N)$).
Let $\mathbf{D}_K$ an irreducible $\mathcal{U}_0(gl_N)$ module. Suppose that $\sigma \in W$ is a permutation and that $\xi = \boldsymbol{\sigma}\,\xi_K$ is its associated extremal vector. The generating series of the dimension of the weight spaces, also called character of the Demazure module is given by

$$\mathrm{ch}\,(\mathcal{U}_0(\mathfrak{b}_+)\xi) = \boldsymbol{\pi}_\sigma\, X^K. \tag{72}$$

Example 4.2. For the algebra $\mathcal{U}_0(gl_4)$ the module $\mathbf{D}_{(1,2)}$ is of dimension 10. Its basis is indexed by the words 212, 213, 313, 214, 323, 314, 324, 414, 424, 434, of respective weights $[1,2,0,0]$, $[1,1,1,0]$, $[1,0,2,0]$, $[1,1,0,1]$, $[0,1,2,0]$, $[1,0,1,1]$, $[0,1,1,1]$, $[1,0,0,2]$, $[0,1,0,2]$, $[0,0,1,2]$.
The extremal weights are the following:
$[1,2,0,0]$, $[1,0,2,0]$, $[0,1,2,0]$, $[1,0,0,2]$, $[0,1,0,2]$, $[0,0,1,2]$.
Fix $\sigma = (1423)$. The vector ξ of weight $[1,0,0,2] = \boldsymbol{\sigma}[1,2,0,0]$ generate a Demazure module of dimension 6 whose character is given by:
$\mathrm{ch}\,(\mathcal{U}_0(\mathfrak{b}_+)\xi) = \boldsymbol{\pi}_\sigma[1,2,0,0] = [1,2,0,0] + [1,1,1,0] + [1,0,2,0] + [1,1,0,1] + [1,0,1,1] + [1,0,0,2]$.

Proof — The work has been almost done in the proof of Theorem 3.4. It remains to identify the weight vectors of the Demazure module. Suppose that $m \gg m'$ are two weights fitting in K and ξ, ξ' are their associated vectors. It is easy to see that ξ is in the image of ξ' under the action of $\mathcal{U}_0(\mathfrak{b}_+)$, so that the Demazure module is the space generated by all vectors of weight greater than $\boldsymbol{\sigma}\,\xi_K$. The generating series of such weights is well $\boldsymbol{\pi}_\sigma\, X^K$ according to Theorem 3.4. □

5 Action of the generic Hecke algebra

5.1 Main theorem

The goal of this section is to construct an action of the generic Hecke algebra generalizing the two preceding actions. Recall that, in the classical case, the generic Hecke algebra $H_n(q)$ acts on $\mathbb{C}[X]$ by the formula ([4, 37, 30]):

$$T_i \mapsto T_i \;\text{ with }\; T_i\, f = (q-1)\,\pi_i\, f + \sigma_i\, f. \tag{73}$$

The idea is simply to interpolate between the two preceding action of the symmetric group and the Hecke degenerate algebra:

Theorem 5.1. *The operators $\boldsymbol{T}_i$ defined by*

$$\boldsymbol{T}_i = (1-q)\,\overline{\boldsymbol{\pi}}_i + q\,\boldsymbol{\sigma}_i = \overline{\boldsymbol{\pi}}_i + q(\boldsymbol{\sigma}_i - \overline{\boldsymbol{\pi}}_i) \tag{74}$$

verify the Hecke relations.

Before proving the theorem let us give the computation rule. To simplify the notation, we only write the rule for $\boldsymbol{T}_1$, since $\boldsymbol{T}_i$ acts only on x_i and x_{i+1}.

Proposition 5.1. *Let i, j be two non-zero integers. then*

$$\boldsymbol{T}_1[0,0] = q[0,0] \quad \textit{and} \quad \boldsymbol{T}_1[i,j] = q[i,j] \tag{75}$$

$$\boldsymbol{T}_1[i,0] = (1-q)\sum_{u=1}^{i-1}[i-u,u] + [0,i] \tag{76}$$

$$\boldsymbol{T}_1[0,i] = (q-1)\sum_{u=1}^{i-1}[i-u,u] + q[i,0] + (q-1)[0,i] \tag{77}$$

Note 5.1. In the classical case, the divided difference operators commute with multiplication by symmetric polynomials. So it is sufficient to check these identities on a basis of $\mathbb{C}[X]$ as a free module over the ring of symmetric functions. The Grothendieck and Schubert polynomials are helpful in this case [39, 30]. In our case, the quasi-symmetrizing action does not commute with the product, and the ring $\mathbb{C}[X]$ considered as a **Qsym**-module is not free. So the proof is done by a direct calculation, checking the braid relation for all monomials over three variables.

Proof — With the help of the expression (39) and (42) of the operators $Qddb_i$ and $\boldsymbol{\sigma}_i$, we get:

$$\begin{aligned}\boldsymbol{T}_i^2 &= (1-q)^2\,\overline{\boldsymbol{\pi}}_i^2 + q(1-q)(\boldsymbol{\sigma}_i\,\overline{\boldsymbol{\pi}}_i + \overline{\boldsymbol{\pi}}_i\,\boldsymbol{\sigma}_i) + q^2\\ &= -(1-q)^2\,\overline{\boldsymbol{\pi}}_i + q(q-1)\,\boldsymbol{\sigma}_i + q\\ &= (q-1)\,\boldsymbol{T}_i + q,\end{aligned}$$

which proves the quadratic relation. The following lemma allows to show the braid relation for $\boldsymbol{T}_i$.

Lemma 5.1. *For $i < n-2$ one has:*

$$\left.\begin{aligned} &\boldsymbol{\sigma}_i\,\overline{\boldsymbol{\pi}}_{i+1}\,\overline{\boldsymbol{\pi}}_i\\ +&\overline{\boldsymbol{\pi}}_i\,\boldsymbol{\sigma}_{i+1}\,\overline{\boldsymbol{\pi}}_i\\ +&\overline{\boldsymbol{\pi}}_i\,\overline{\boldsymbol{\pi}}_{i+1}\,\boldsymbol{\sigma}_i\end{aligned}\right\} = \left\{\begin{aligned} &\boldsymbol{\sigma}_{i+1}\,\overline{\boldsymbol{\pi}}_i\,\overline{\boldsymbol{\pi}}_{i+1}\\ +&\overline{\boldsymbol{\pi}}_{i+1}\,\boldsymbol{\sigma}_i\,\overline{\boldsymbol{\pi}}_{i+1}\\ +&\overline{\boldsymbol{\pi}}_{i+1}\,\overline{\boldsymbol{\pi}}_i\,\boldsymbol{\sigma}_{i+1}\end{aligned}\right. \tag{78}$$

$$\left.\begin{aligned} &\boldsymbol{\sigma}_i\,\boldsymbol{\sigma}_{i+1}\,\overline{\boldsymbol{\pi}}_i\\ +&\boldsymbol{\sigma}_i\,\overline{\boldsymbol{\pi}}_{i+1}\,\boldsymbol{\sigma}_i\\ +&\overline{\boldsymbol{\pi}}_i\,\boldsymbol{\sigma}_{i+1}\,\boldsymbol{\sigma}_i\end{aligned}\right\} = \left\{\begin{aligned} &\boldsymbol{\sigma}_{i+1}\,\boldsymbol{\sigma}_i\,\overline{\boldsymbol{\pi}}_{i+1}\\ +&\boldsymbol{\sigma}_{i+1}\,\overline{\boldsymbol{\pi}}_i\,\boldsymbol{\sigma}_{i+1}\\ +&\overline{\boldsymbol{\pi}}_{i+1}\,\boldsymbol{\sigma}_i\,\boldsymbol{\sigma}_{i+1}\end{aligned}\right. \tag{79}$$

Indeed, assuming the lemma, the expression

$$\begin{aligned}\boldsymbol{T}_i\,\boldsymbol{T}_{i+1}\,\boldsymbol{T}_i =&(1-q)^3\,\overline{\boldsymbol{\pi}}_i\,\overline{\boldsymbol{\pi}}_{i+1}\,\overline{\boldsymbol{\pi}}_1 +\\ &q(1-q)^2(\boldsymbol{\sigma}_i\,\overline{\boldsymbol{\pi}}_{i+1}\,\overline{\boldsymbol{\pi}}_i + \overline{\boldsymbol{\pi}}_i\,\boldsymbol{\sigma}_{i+1}\,\overline{\boldsymbol{\pi}}_i + \overline{\boldsymbol{\pi}}_i\,\overline{\boldsymbol{\pi}}_{i+1}\,\boldsymbol{\sigma}_i) +\\ &q^2(1-q)(\boldsymbol{\sigma}_i\,\boldsymbol{\sigma}_{i+1}\,\overline{\boldsymbol{\pi}}_i + \boldsymbol{\sigma}_i\,\overline{\boldsymbol{\pi}}_{i+1}\,\boldsymbol{\sigma}_i + \overline{\boldsymbol{\pi}}_i\,\boldsymbol{\sigma}_{i+1}\,\boldsymbol{\sigma}_i) +\\ &q^3\,\boldsymbol{\sigma}_i\,\boldsymbol{\sigma}_{i+1}\,\boldsymbol{\sigma}_i\,.\end{aligned}$$

appears to be symmetric in i and $i+1$.

Let us prove the lemma. Recall that

$$\overline{\boldsymbol{\pi}}_i[m_0,\ldots,m_i,m_{i+1},\ldots,m_n] = 0$$

unless only one of the m_i and m_{i+1} in non-zero. It is easy to see that the two sides of the two equations of the lemma are null on the constant monomial and on $[i,j,k]$ provided i, j and k are three strictly positive integers.

Now we have to compute the image of the 6 other monomials under the 12 operators appearing in Lemma 5.1. These calculations are straightforward, applying the rule (37) and Definition 3.3. The following arrays display the results.

	$\boldsymbol{\sigma}_1\,\overline{\boldsymbol{\pi}}_2\,\overline{\boldsymbol{\pi}}_1$	$\overline{\boldsymbol{\pi}}_1\,\boldsymbol{\sigma}_2\,\overline{\boldsymbol{\pi}}_1$	$\overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2\,\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2\,\overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2$	$\overline{\boldsymbol{\pi}}_2\,\boldsymbol{\sigma}_1\,\overline{\boldsymbol{\pi}}_2$	$\overline{\boldsymbol{\pi}}_2\,\overline{\boldsymbol{\pi}}_1\,\boldsymbol{\sigma}_2$
$[i,j,0]$	0	0	$\Sigma_i+[0,i,j]$	$\Sigma_i+[0,i,j]$	0	0
$[i,0,j]$	0	$-(\Sigma_i+[0,i,j])$	0	$-(\Sigma_i+[0,i,j])$	0	0
$[0,i,j]$	0	$\Sigma_i+[0,i,j]$	$-(\Sigma_i+[0,i,j])$	0	0	0
$[i,0,0]$	Σ	Σ'	$-\Sigma'$	0	0	Σ
$[0,i,0]$	$-\Sigma$	$-\Sigma'$	0	$-\Sigma'$	$-\Sigma$	0
$[0,0,i]$	0	0	Σ'	Σ'	Σ	$-\Sigma$

	$\boldsymbol{\sigma}_1\,\boldsymbol{\sigma}_2\,\overline{\boldsymbol{\pi}}_1$	$\boldsymbol{\sigma}_1\,\overline{\boldsymbol{\pi}}_2\,\boldsymbol{\sigma}_1$	$\overline{\boldsymbol{\pi}}_1\,\boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_1\,\overline{\boldsymbol{\pi}}_2$	$\boldsymbol{\sigma}_2\,\overline{\boldsymbol{\pi}}_1\,\boldsymbol{\sigma}_2$	$\overline{\boldsymbol{\pi}}_2\,\boldsymbol{\sigma}_1\,\boldsymbol{\sigma}_2$
$[i,j,0]$	0	$\Sigma_j+[0,i,j]$	$\Sigma_i+[0,i,j]$	$\Sigma_j+[0,i,j]$	$\Sigma_i+[0,i,j]$	0
$[i,0,j]$	$\Sigma_i+[i,0,j]$	0	$-(\Sigma_i+[0,i,j])$	$-(\Sigma_j+[0,i,j])$	0	$\Sigma_j+[i,0,j]$
$[0,i,j]$	$-(\Sigma_i+[i,0,j])$	$-(\Sigma_j+[0,i,j])$	0	0	$-(\Sigma_i+[0,i,j])$	$-(\Sigma_j+[i,0,j])$
$[i,0,0]$	$[0,\ast,\ast]$	$[\ast,0,\ast]$	0	0	$[\ast,0,\ast]$	$[0,\ast,\ast]$
$[0,i,0]$	$-[0,\ast,\ast]$	0	$[\ast,\ast,0]$	$[\ast,\ast,0]$	0	$-[0,\ast,\ast]$
$[0,0,i]$	0	$-[\ast,0,\ast]$	$-[\ast,\ast,0]$	$-[\ast,\ast,0]$	$-[\ast,0,\ast]$	0

with the notations:

$$\Sigma = \sum_{\substack{u+v+w=i\\ w\neq 0}} [u,v,w], \qquad \Sigma' = \sum_{\substack{u+v+w=i\\ v\neq 0,\ w\neq 0}} [u,v,w],$$

$$\Sigma_i = \sum_{\substack{u+v=i\\ u\neq 0,\ v\neq 0}} [u,v,j], \qquad \Sigma_j = \sum_{\substack{u+v=j\\ u\neq 0,\ v\neq 0}} [i,u,v],$$

$$[*,*,0] = \sum_{\substack{u+v=i\\ v\neq 0}} [u,v,0], \quad [*,0,*] = \sum_{\substack{u+v=i\\ v\neq 0}} [u,0,v], \quad [0,*,*] = \sum_{\substack{u+v=i\\ v\neq 0}} [0,u,v].$$

For example let us compute $\boldsymbol{\sigma}_2\,\overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2[i,0,j]$. First

$$\overline{\boldsymbol{\pi}}_2[i,0,j] = -\sum_{u+v=j,\ v\neq 0} [i,u,v].$$

Now under the action of $\overline{\boldsymbol{\pi}}_1$ the only monomial which does not vanish is $-[i,0,j]$. Thus

$$\overline{\boldsymbol{\pi}}_1\,\overline{\boldsymbol{\pi}}_2[i,0,j] = -\sum_{u+v=i,\ v\neq 0} [u,v,j].$$

Finally, $\boldsymbol{\sigma}_2$ does not change anything. This sum is exactly $-\Sigma_i - [0,i,j]$, as recorded in the array.

It possible to check the results. Indeed, between two horizontal line of the arrays, the sum in each column has to be zero, because the sum of the monomials on the left is quasi-symmetric. For example

$$\boldsymbol{\sigma}_1\,\boldsymbol{\sigma}_2\,\overline{\boldsymbol{\pi}}_1([i,j,0]+[i,0,j]+[0,i,j]) = 0 + (\Sigma_i + [i,0,j]) - (\Sigma_i + [i,0,j]) = 0$$

Now, in a row, the sums on the left and on the right are equal, so that the lemma is true. This ends the proof. □

5.2 q-idempotents and the Yang-Baxter equation

We want now to define a q-analog of the symmetrizing operator $\boldsymbol{\pi}$. For that, we define the following two operators:

Definition 5.1. *One defines the q-symmetrizing and q-anti-symmetrizing elementary operators by:*

$$\square_i = \boldsymbol{T}_i + 1 \quad \textit{and} \quad \boldsymbol{\nabla}_i = \boldsymbol{T}_i - q. \tag{80}$$

These operators act on polynomials according to (74). In the sequel, we will show many relations between these operators, most of them being true in the Hecke algebra. When this is the case, we use normal-type operator:

$$\Box_i = T_i + 1 \quad \text{and} \quad \nabla_i = T_i - q. \tag{81}$$

When the relation are only true in the quasi-symmetric case we use the bold-type operator. Of course, the bold-type version of a normal-type equality is true.

As an immediate consequence of the definition we get the following commutation rules:

Proposition 5.2. *Let $i < n$ be an integer. We have*

$$\Box_i\, T_i = T_i\, \Box_i = q\, \Box_i \quad \textit{and} \quad \nabla_i\, T_i = T_i\, \nabla_i = -\, \nabla_i \tag{82}$$

In [7], Duchamp et al. make a detailed study of the classical version of these operators. The classical divided differences admit symmetric functions as scalars. But the module $\mathbb{C}[X]$ over the ring of symmetric functions Sym is known to be free. Using a suitable basis of this module, such as Schubert polynomials, they prove useful relations between operators. Since the classical action of the Hecke algebra is faithful, their relations are not only true between operators on polynomials, but also in the Hecke algebra. Thus the relations the write hold in our case. Let us recall their presentation.

We have the following relations, which will be useful to construct q-idempotents:

$$\Box_i^2 = (q+1)\, \Box_i \quad \text{and} \quad \nabla_i^2 = -(q+1)\, \nabla_i \tag{83}$$

However, these operator do not satisfy the braid relation but a deformation of it, called the Yang Baxter relation:

$$\Box_i(\Box_{i+1} - \frac{q}{1+q})\, \Box_i = \Box_{i+1}(\Box_i - \frac{q}{1+q})\, \Box_{i+1}, \tag{84}$$

$$\nabla_i(\nabla_{i+1} + \frac{q}{1+q})\, \nabla_i = \nabla_{i+1}(\nabla_i + \frac{q}{1+q})\, \nabla_{i+1}\,. \tag{85}$$

Using this relation, one can defines operators $\Box_\sigma$ and ∇_σ for any permutation σ. Let us present the construction. Let $[c]_q$ denote the q-integers $(1-q^c)/(1-q)$. It follows from the last two equations that the operators defined by

$$R_i(u,v) = \Box_i - q\frac{[v-u-1]_q}{[v-u]_q} \quad \text{or} \quad R_i(u,v) = \nabla_i + q\frac{[v-u-1]_q}{[v-u]_q} \tag{86}$$

are two solutions of the following equation, which is the standard form of the Yang-Baxter equation [50]:

$$R_i(u,v)R_{i+1}(u,w)R_i(v,w) = R_{i+1}(w,v)R_i(u,w)R_{i+1}(u,v). \tag{87}$$

It allows us to define $\square_\sigma$ and ∇_σ by induction. To begin with, let $\square_{\mathrm{Id}} = \nabla_{\mathrm{Id}} = 1$ where Id is the identity of the symmetric group. Now, suppose that i is a rise of ${\sigma'}^{-1}$, so that $\sigma = \sigma_i\,\sigma'$ is a reduced expression for σ. Let $u = (\sigma')^{-1}(i)$ and $v = (\sigma')^{-1}(i+1)$. Then, thanks to Yang-Baxter relations:

$$\square_\sigma = \left(\square_i - q\frac{[v-u-1]_q}{[v-u]_q}\right)\square_{\sigma'} \quad \text{and} \quad \nabla_\sigma = \left(\nabla_i + q\frac{[v-u-1]_q}{[v-u]_q}\right)\nabla_{\sigma'} \tag{88}$$

are independent of the choice of σ' and i.

For example, let $\sigma = \sigma_2\,\sigma_1\,\sigma_3\,\sigma_2$. Then

$$\begin{aligned} \sigma_2[1,2,3,4] &= [1,3,2,4] \quad u=2,\ v=3,\\ \sigma_1\,\sigma_2[1,2,3,4] &= [3,1,2,4] \quad u=1,\ v=3,\\ \sigma_3\,\sigma_1\,\sigma_2[1,2,3,4] &= [3,1,4,2] \quad u=2,\ v=4,\\ \sigma_2\,\sigma_3\,\sigma_1\,\sigma_2[1,2,3,4] &= [3,4,1,2] \quad u=1,\ v=4, \end{aligned}$$

and finally

$$\begin{aligned} \square_{3412} &= \square_2\left(\square_3 - \frac{q}{1+q}\right)\left(\square_1 - \frac{q}{1+q}\right)\left(\square_2 - \frac{q+q^2}{1+q+q^2}\right)\\ \nabla_{3412} &= \nabla_2\left(\nabla_3 + \frac{q}{1+q}\right)\left(\nabla_1 + \frac{q}{1+q}\right)\left(\nabla_2 + \frac{q+q^2}{1+q+q^2}\right) \end{aligned}$$

Let us go back to the maximal permutation ω. Recall that, reduced words for ω can start or end with any transposition σ_i. Hence we get the following factorization:

Proposition 5.3. *Let i be a integer. There exist four permutations σ, σ', δ and δ' such that :*

$$\square_\omega = \square_i\,\square_\sigma = \square_{\sigma'}\,\square_i \quad \textit{and} \quad \nabla_\omega = \nabla_i\,\nabla_\delta = \nabla_{\delta'}\,\nabla_i$$

The interest of the so-defined operators $\square_\omega$ and ∇_ω is that there are actually q-analogues of the full symmetrizer and anti-symmetrizer (*cf.* [7]):

Theorem 5.2. *Let ω be the maximal permutation of $\mathfrak{S}_n$. Then*

$$\square_\omega = \sum_{\sigma\in\mathfrak{S}_n} T_\sigma \quad \textit{and} \quad \nabla_\omega = \sum_{\sigma\in\mathfrak{S}_n} (-q)^{\ell(\omega\,\sigma)}\,T_\sigma \tag{89}$$

We will also makes use of the following two factorizations:

$$\Box_\omega = \Box_{\omega'} (1 + T_{n-1} + T_{n-1} T_{n-2} + \cdots + T_{n-1} \ldots T_1) , \tag{90}$$

$$\Box_\omega = (1 + T_{n-1} + T_{n-2} T_{n-1} + \cdots + T_1 T_2 \ldots T_{n-1}) \Box_{\omega'} . \tag{91}$$

where ω' is the maximal permutation of $\mathfrak{S}_{n-1}$.

Let us go back to quasi-symmetric case.

Theorem 5.3. *The full q-symmetrizing operator $\boldsymbol{\Box}_\omega$ considered as acting on polynomials with coefficients in $\mathbb{Z}[q]$ has for image the space of quasi-symmetric functions.*

Moreover, if we take coefficients in $\mathbb{C}(q)$, the operator $\frac{1}{[n]_q!} \boldsymbol{\Box}_\omega$ is an idempotent with range the space of quasi-symmetric functions.

As usual, the q-factorial if given by $[n]_q! = \prod_{i=1}^{n} [i]_q$.

Proof — Let i be an integer. By Proposition 5.3, we have $\Box_\omega = \Box_i \Box_{\sigma'}$ for some σ'.

Lemma 5.2. *Let $i \in \{1, \ldots, n-1\}$. Then $\boldsymbol{\sigma}_i \boldsymbol{\Box}_i = \boldsymbol{\Box}_i$.*

This is easy, since $\boldsymbol{\Box}_i = (1-q)\,\overline{\boldsymbol{\pi}}_i + q\,\boldsymbol{\sigma}_i + 1 = \boldsymbol{\pi}_i + q(1 + \boldsymbol{\sigma}_i - \boldsymbol{\pi}_i)$, and $\boldsymbol{\sigma}_i \boldsymbol{\pi}_i = \boldsymbol{\pi}_i$.

It follows from the lemma that $\boldsymbol{\sigma}_i \boldsymbol{\Box}_\omega f = \boldsymbol{\Box}_\omega f$, for any f and i, and consequently that $\boldsymbol{\Box}_\omega f$ is quasi-symmetric.

Conversely, if f is quasi-symmetric, then $\boldsymbol{T}_i f = qf$. And thus,

$$\boldsymbol{\Box}_\omega f = \sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{T}_\sigma f = \sum_{\sigma \in \mathfrak{S}_n} q^{\ell(\sigma)} f. \tag{92}$$

Since $[n]_q!$ is the generating function of permutations counted by length, the proof is done. □

As a consequence of the proof we get the following characterization of quasi-symmetric polynomials:

Proposition 5.4. *A polynomial f if quasi-symmetric if and only if $\boldsymbol{T}_i f = qf$ for all $i \in \{1, \ldots, n\}$.*

The case of $\boldsymbol{\nabla}_\omega$ is totally different.

Theorem 5.4. *Suppose that $n > 2$. Then $\boldsymbol{\nabla}_\omega = 0$.*

Proof — First, the operator $\boldsymbol{\nabla}_\omega$ factorizes as

$$\nabla_1(\nabla_2 + \frac{q}{1+q})\,\nabla_1(\nabla_3 + \frac{1+q}{1+q+q^2})(\nabla_2 + \frac{q}{1+q})\ldots. \tag{93}$$

thus it is sufficient to prove the theorem for $n = 3$.

Next, the quadratic relation for T_i can be rewritten as

$$(T_i + 1)(T_i - q) = (T_i - q)(T_i + 1) = 0, \tag{94}$$

or, with the q-symmetrizer and q-anti-symmetrizer :

$$\square_i\,\nabla_i = \nabla_i\,\square_i = 0. \tag{95}$$

Expanding $\square_i$ we get :

$$(\boldsymbol{\overline{\pi}}_i + q(1 + \boldsymbol{\sigma}_i + \boldsymbol{\overline{\pi}}_i))\,\boldsymbol{\nabla}_i = 0. \tag{96}$$

Hence we have proven the following lemma:

Lemma 5.3. *Let $i \in \{1, \ldots, n-1\}$. Then $\boldsymbol{\sigma}_i\,\boldsymbol{\nabla}_i = -\,\boldsymbol{\nabla}_i$.*

To continue, remark that a reduced expression for $\boldsymbol{\nabla}_\omega$ can start with either $\boldsymbol{\nabla}_1$ or $\boldsymbol{\nabla}_2$. Thus $\boldsymbol{\sigma}_1\,\boldsymbol{\nabla}_\omega = \boldsymbol{\sigma}_2\,\boldsymbol{\nabla}_\omega = -\,\boldsymbol{\nabla}_\omega$. But also $\boldsymbol{\sigma}_1\,\boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_1\,\boldsymbol{\nabla}_\omega = -\,\boldsymbol{\nabla}_\omega$. Now suppose that A is a subset of $\{1, 2, 3\}$. Then, either two of the integers $1, 2, 3$ are in A or else two of them are in the complementary of A. Thus the transposition which exchanges these two integers fixes A. A a consequence each monomial is fixed at least by the quasi-symmetrizing action of one of the three transpositions $\sigma_1 = (1, 2)$, $\sigma_2 = (2, 3)$ or $\sigma_1\,\sigma_2\,\sigma_1 = (1, 3)$. We conclude that for all $f \in \mathbb{C}[X]$, we have $\boldsymbol{\nabla}_\omega\,f{=}0$. □

Since this relation does not hold in the Hecke algebra, once more, this is not a faithful representation. It is possible to give a complete characterization of the kernel:

Theorem 5.5. *The image of $H_n(q)$ in $\mathrm{End}(\mathbb{C}[X])$ is the quotient of $H_n(q)$ by the ideal generated by $(\nabla_{(i,i+2)})$ where $i = 1 \ldots n-2$.*

The family $(\boldsymbol{\nabla}_\sigma)$ where σ runs over the set of all permutations avoiding the pattern 321 is a basis of the image of $H_n(q)$ in $\mathrm{End}(\mathbb{C}[X])$.

Proof — The equality $\boldsymbol{\nabla}_{(i,i+2)} = 0$ is already proved in Theorem 5.4. Hence the family $(\boldsymbol{\nabla}_\sigma)$ with σ avoiding the pattern 321 generate the image.

Let E be the space of polynomial in X such that no monomials contain any squared variables. With our notations there are monomials of the form A^K with $A \in \mathcal{P}(X)$ and $K = (1, 1, \ldots, 1)$. As a Hecke algebra module, it is isomorphic to the space generated by subsets of X or to the space $V^{\otimes n}$ where $V = \mathbb{C}\xi_0 + \mathbb{C}\xi_1$ is a two dimensional space. It is known that the dimension of the image of $H_n(q)$ in $\mathrm{End}(V^{\otimes n})$ is the number of permutations avoiding the pattern 321 [3]. Thus the announced family is a basis.

Finally, the first part of the theorem follows immediately, since the equality $\boldsymbol{\nabla}_{(i,i+2)} = 0$ allows to express all the actions on the basis. □

Note that the number of permutation avoiding 321 in $\mathfrak{S}_n$ is the Catalan number $C_n = \frac{1}{n-1}\binom{2n}{n}$. For example, in 4 variables, the image of $H_n(q)$ in $\mathrm{End}(\mathbb{C}[X])$ is of dimension 14. A basis for this image is $(\boldsymbol{\nabla}_\sigma)$ for $\sigma \in \{1234,$ $1243, 1324, 1342, 1423, 2134, 2143, 2314, 2341, 2413, 3124, 3142, 3412, 4123\}$. Remark that $\boldsymbol{\nabla}_{4231}$ is not in the kernel (There is no deduced word starting on ending with a sequences $\sigma_i\,\sigma_{i+1}\,\sigma_i$).

As an equivalent property we can give the character of this action of the Hecke algebra:

Proposition 5.5. *The Frobenius characteristic of the quasi-symmetrizing action of the Hecke algebra is*

$$\mathrm{ch}_t(\mathbb{C}[x_1,\ldots,x_n]) = \sum_{m=0}^{n} \frac{t^m}{(1-t)^m} h_{(m,n-m)}. \tag{97}$$

Proof — This is a well known fact that the characteristic of an action of the generic Hecke algebra is the characteristic of the action of the symmetric group obtained by letting $q = 1$. Let us recall the argument.

Let $V(\lambda)$ the q-Specht module associated with the partition λ. As in the case of the symmetric group the the character formula is given by (see [3])

$$\mathrm{ch}^\lambda_\mu(q) = \mathrm{tr}(T_{\omega_\mu}) = \langle s_\lambda, C_\mu(q)\rangle \tag{98}$$

where s_λ is the Schur function, ω_μ is the maximal element of the Young subgroup $\mathfrak{S}_\lambda$ and $C_\mu(q) = (q-1)^{\ell(\mu)} h^\mu((q-1)X)$ is a q analogue of the cycle indicator of the symmetric group. Note that the characteristic s_λ does not depend of q. But if E is a module, its decomposition into irreducible is obtained by decomposing its character as

$$\chi_E(q) = \sum_\lambda c_\lambda \chi^\lambda(q). \tag{99}$$

The integers c_λ do not depend on q. So they can be computed by specifying $q = 1$. Thus the characteristic of the representation of generic Hecke algebra is equal to the characteristic of the action of the symmetric group which is given by Proposition 3.2. □

Note that, as shown in [8], this formula is still valid for $q = 0$. It does not give the decomposition of the representation in irreducible, but rather the factor composition of the module.

6 Hall-Littlewood functions

6.1 Outline of the classical theory

We first recall some facts of the classical theory. Our notations will be essentially those of [38], to which the reader is referred for more details.

Let $\Delta_n(q)$ denote the q-alternant $\prod\limits_{i<j\leq n}(qx_i - x_j)$. Then, on the one hand, the Hall-Littlewood polynomial $Q_\lambda(x_1, \ldots, x_n; q)$ indexed by a partition λ of length $\leq n$ is defined by [35]

$$Q_\lambda = \frac{(1-q)^{\ell(\lambda)}}{[m_0]_q!} \sum_{\sigma \in \mathfrak{S}_n} \sigma_\sigma \left(x^\lambda \frac{\Delta_n(q)}{\Delta_n(1)} \right) \tag{100}$$

where $m_0 = n - \ell(\lambda)$ and σ_σ is the usual action of the permutation σ on polynomials.

On the other hand, the classical action of $H_n(q)$ on $\mathbb{C}[X]$ is given by the formula $T_i = (q-1)\,\pi_i + \sigma_i$ and it is shown in [7] that if $\square_\omega$ denote the classical q-symmetrizing operator $\square_\omega = \sum_{\sigma\in\mathfrak{S}_n} T_\sigma$, then

$$Q_\lambda(x_1, \ldots, x_n; q^{-1}) = q^{-\binom{N}{2}} \frac{(1-q^{-1})^{\ell(\lambda)}}{[m_0]_{q^{-1}}!} \square_\omega(x^\lambda). \tag{101}$$

The normalization factor $1/[m_0]_q!$ is here to ensure stability with respect to the adjunction of variables, and if we denote by X the infinite set $X = \{x_1, x_2, \ldots, \}$ then $Q_\lambda(X; q) = \lim_{n\to\infty} Q_\lambda(x_1, \ldots, x_n; q)$.

The P-functions are defined by

$$P_\lambda(X; q) = \frac{1}{(1-q)^{\ell(\lambda)}[m_1]_q! \cdots [m_n]_q!} Q_\lambda(X; q) \tag{102}$$

where m_i is the multiplicity of the part i in λ.

We have the specializations: $P_\lambda(X; 0)$ is equal to the Schur function s_λ and $P_\lambda(X; 1)$ is equal to the monomial function m_λ,

We consider these functions as elements of the algebra $\mathrm{Sym} = \mathrm{Sym}(X)$ of symmetric functions with coefficients in $\mathbb{C}(q)$. There is a scalar product $\langle\, , \rangle$ on Sym, for which the Schur functions s_λ form an orthonormal basis. We denote by $(Q'_\mu(X; q))$ the adjoint basis of $P_\lambda(X; q)$ for this scalar product. It is easy to see that $Q'_\mu(X;q)$ is the image of $Q_\mu(X;q)$ by the ring homomorphism $p_k \mapsto (1-q^k)^{-1}p_k$ (in λ-ring notation, $Q'_\mu(X;q) = Q(X/(1-q);q)$). In the Schur basis,

$$Q'_\mu(X;q) = \sum_\lambda K_{\lambda\mu}(q) s_\lambda(X) \tag{103}$$

where the $K_{\lambda\mu}(q)$ are the so-called Kotska-Foulkes polynomials. Lascoux and Schützenberger showed that the polynomial $K_{\lambda\mu}(q)$ is the generating function of a statistic c called *charge* on the set $\mathrm{Tab}(\lambda,\mu)$ of Young tableaux of shape λ and weight μ [38].

6.2 Quasi-symmetric Hall-Littlewood functions

By analogy, we introduce the following quasi-symmetric analogues of the P-functions:

Definition 6.1. *Let $K = (k_1, \ldots, k_p)$ be a composition and $X^K = x_1^{k_1} \cdots x_p^{k_p}$. The* quasi-symmetric Hall-Littlewood function G_K *is defined by*

$$G_K(x_1, \ldots, x_n; q) = \frac{1}{[p]_q! \ [n-p]_q!} \square_\omega(X^K). \tag{104}$$

We have the following specializations: $G_K(X;0)$ is the quasi-ribbon function F_K and $G_K(X;1)$ is the quasi-monomial function M_K.

For instance $G_{(2,1)}(x_1,x_2,x_3;q) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + (1-q)x_1x_2x_3$. On this example, we verify that at $q=0$, one has $G_{(2,1)}(x_1,x_2,x_3;0) = F_{(2,1)}$ and at $q=1$, one has $G_{(2,1)}(x_1,x_2,x_3;1) = M_{(2,1)}$.

Proposition 6.1. *Let K be a composition of length p. Then*

$$G_K(x_1, \ldots, x_n; q) = \sum_{\sigma \in \mathfrak{S}_{n/p}} \boldsymbol{T}_\sigma X^K. \tag{105}$$

Proof — Recall that $\mathfrak{S}_{n/p}$ is exactly the set of permutations σ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \cdots < \sigma(n). \tag{106}$$

But a permutation σ factorizes in a unique way as $\sigma = \mu\mu'$ with $\mu \in \mathfrak{S}_{n/p}$, $\mu' \in \mathfrak{S}_p \times \mathfrak{S}_{n-p}$. Consequently we have the following factorization of the full q-symmetrizer:

$$\square_\omega = \left(\sum_{\sigma \in \mathfrak{S}_{n/p}} \boldsymbol{T}_\sigma \right) \left(\sum_{\sigma \in \mathfrak{S}_p \times \mathfrak{S}_{n-p}} \boldsymbol{T}_\sigma \right). \tag{107}$$

Now, since K is of length p, for each $\sigma \in \mathfrak{S}_p \times \mathfrak{S}_{n-p}$ we have $\boldsymbol{T}_\sigma X^K = q^{\ell(\sigma)} X^K$. This ends the proof since $[n]_q!$ is the generating series of the permutations of $\mathfrak{S}_n$ enumerated by their length. $\square$

Corollary 6.1. *The polynomial $G_K(x_1, \dots, x_n; q)$ is quasi-symmetric with coefficients in $\mathbb{Z}[q]$. More precisely, the expansion of the quasi-symmetric Hall-Littlewood polynomials on the monomial basis is of the form $G_K = M_K + \sum_{J \succeq K} a_J M_J$ where a_J is a polynomial in q with integer coefficients.*

Corollary 6.2. *The family $(G_K)_{\ell(K) \le n}$ is a basis of the space of quasi-symmetric polynomials with coefficients in $\mathbb{C}(q)$.*

We will prove in the next section that in fact (G_K) remains a basis even if we take coefficients in $\mathbb{Z}[q]$. The transition matrix is upper unitriangular (*i.e.* triangular with 1 on the diagonal). We will give an explicit formula for the polynomial a_J in the proof of Theorem 6.1.

The coefficient $\frac{1}{[p]_q! \; [n-p]_q!}$ is here to ensure that G_K has the stability property:

Proposition 6.2. *Let K be of length $p \le n$. Then,*

$$G_K(x_1, \dots, x_n, 0; q) = G_K(x_1, \dots, x_n; q). \tag{108}$$

Proof — Let us denote $\square_\omega^{(i)}$ the q-symmetrizing operator associated with the maximal permutation of $\mathfrak{S}_i$. The proposition is equivalent to the expression

$$(\square_\omega^{(n+1)} X^K)_{/x_{n+1}=0} = [n-p+1]_q (\square_\omega^{(n)} X^K). \tag{109}$$

Since we need this in the sequel of the proof, let us start by showing the property in the particular case $n = p$.

Let $m = X^K = [k_1, \dots, k_p, 0]$. Then $\square_\omega^{(p)} m = [p]_q! m$. Using the factorization (90) of $\square_\omega^{(p+1)}$, we have

$$\square_\omega^{(p+1)} m = [p]_q!(1 + \boldsymbol{T}_p + \boldsymbol{T}_{p-1} \boldsymbol{T}_p + \dots + \boldsymbol{T}_1 \boldsymbol{T}_2 \dots \boldsymbol{T}_p) m.$$

But, $$\boldsymbol{T}_p m = (1-q) \sum_{v=1}^{k_n} [k_1, \dots, k_n - v, v] \; + \; q[k_1, \dots, 0, k_p].$$

Thus for $r \le p$, we get $(\boldsymbol{T}_r \dots \boldsymbol{T}_{p-1} \boldsymbol{T}_p m)_{/x_{n+1}=0} = 0$. Finally, we have proved $(\square_\omega^{(p)} m)_{/x_{p+1}=0} = \square_\omega^{(p)} m$, which is the result for $n = p$.

Let us prove it for any integer $n > p$. Let $f' = \square_\omega^{(n)} m$ together with $f = (\square_\omega^{(n+1)} m)_{/x_{n+1}=0}$. The factorization of $\square_\omega^{(n+1)}$ gives

$$f = ((1 + \boldsymbol{T}_n + \boldsymbol{T}_{n-1} \boldsymbol{T}_n + \boldsymbol{T}_1 \boldsymbol{T}_2 \dots \boldsymbol{T}_n) f')_{/x_{n+1}=0}. \tag{110}$$

Now suppose that g is a monomial of the form $[g_1, \ldots, g_n, 0]$ with $g_n \neq 0$. Then, all the monomials appearing in $\overline{\boldsymbol{\pi}}_n\, g$ have a non-zero final part, and consequently $(\overline{\boldsymbol{\pi}}_n\, g)_{/x_{n+1}=0} = 0$. It follows

$$(\boldsymbol{T}_n\, f)_{/x_{n+1}=0} = q\, \boldsymbol{\sigma}_n \left(f'_{/x_n=0} \right) = q\ f'_{/x_n=0},$$

so that equation (110) reads

$$f = f' + q\ (1 + \boldsymbol{T}_{n-1} + \boldsymbol{T}_{n-1}\, \boldsymbol{T}_{n-2} + \boldsymbol{T}_1\, \boldsymbol{T}_2 \ldots \boldsymbol{T}_{n-1}) \left(f'_{/x_n=0} \right).$$

Now, if we suppose that the result is true for $n-1$ we get that

$$f'_{/x_n=0} = [n-p]_q\, \Box_\omega^{(n-2)}\, X^K.$$

Putting the last two equations together we have

$$f = f' + q[n-p]_q\, \Box_\omega^{(n-1)}\, X^K = (1 + q[n-p]_q) f'.$$

Since $[n-p+1]_q = 1 + q[n-p]_q$ we get the property for $n+1$, and the proof follows by induction. $\Box$

This makes it possible to take the limits when $n \to +\infty$. As usual we call them quasi-symmetric Hall-Littlewood functions. Due to the expansion formula, the family of quasi-symmetric Hall-Littlewood functions is a basis of the algebra quasi-symmetric functions.

Moreover, it is possible to give an explicit expansion of the Hall-Littlewood functions in the classical bases. The nicest formula is in the quasi-ribbon basis.

Theorem 6.1. *The expansion of G_I in the quasi-ribbon basis is given by*

$$G_I = \sum_{J \succeq I} (-1)^{\ell(J)-\ell(I)} q^{s(I,J)} F_J \tag{111}$$

where $s(I, J)$ is defined as follows. Let $(k_1, \ldots, k_p)$ be the refining composition $\#(J, I)$. Then $s(I, J) = (k_1 - 1) + 2(k_2 - 1) + \cdots + p(k_p - 1)$.

Example 6.1. Let us compute $G_{(1,1,2,1)}$ on $X = \{x_1, x_2, x_3, x_4, x_5\}$. Start with $m = x_1 x_2 x_3^2 x_4 = [1, 1, 2, 1, 0]$. Since m is symmetric for $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\boldsymbol{\sigma}_3$, the operators T_1, T_2, and T_3 multiply m by q. We can also deduce that $\Box_{\omega'}\, m = [4]_q! m$ because $[n]_q!$ is the generating series for the permutations of $\mathfrak{S}_n$ counted by length (as usual ω' denote the maximal permutation for $\mathfrak{S}_{n-1}$).

$$\begin{array}{rcl} \boldsymbol{T}_4(m) &=& [1,1,2,0,1] \\ \boldsymbol{T}_3\boldsymbol{T}_4(m) &=& [1,1,0,2,1]+(1-q)[1,1,1,1,1] \\ \boldsymbol{T}_2\boldsymbol{T}_3\boldsymbol{T}_4(m) &=& [1,0,2,1,1]+q(1-q)[1,1,1,1,1] \\ \boldsymbol{T}_1\boldsymbol{T}_2\boldsymbol{T}_3\boldsymbol{T}_4(m) &=& [0,1,1,2,1]+q^2(1-q)[1,1,1,1,1]. \end{array}$$

From the factorization

$$\square_\omega = (1+\boldsymbol{T}_4+\boldsymbol{T}_3\boldsymbol{T}_4+\boldsymbol{T}_2\boldsymbol{T}_3\boldsymbol{T}_4+\boldsymbol{T}_1\boldsymbol{T}_2\boldsymbol{T}_3\boldsymbol{T}_4)\,\square_{\omega'},$$

we get

$$\begin{array}{l} \square_\omega([1,1,2,1,0]) = [4]_q!\big(\\ \quad [1,1,2,1,0]+[1,1,2,0,1]+[1,1,0,2,1]+[1,0,1,2,1]+[0,1,1,2,1] \\ \qquad\qquad\qquad\qquad\qquad\qquad (1-q)(1+q+q^2)[1,1,1,1,1]\big). \end{array}$$

Or, using the quasi-monomial basis of **Qsym**

$$\begin{array}{rcl} \square_\omega([1,1,2,1,0]) &=& [4]_q!(M_{(1,1,2,1)}+(1-q^3)M_{(1,1,1,1,1)}) \\ &=& [4]_q!(F_{(1,1,2,1)}-q^3F_{(1,1,1,1,1)}), \end{array}$$

so that $G_{(1,1,2,1)} = F_{(1,1,2,1)} - q^3F_{(1,1,1,1,1)}$.

Similarly, one would find

$$\begin{array}{l} G_{(3,2)} = F_{(3,2)} - q\,F_{(2,1,2)} - q\,F_{(1,2,2)} + q^2\,F_{(1,1,1,2)} - q^2\,F_{(3,1,1)} + \\ \qquad\qquad\qquad\qquad q^3\,F_{(2,1,1,1)} + q^3\,F_{(1,2,1,1)} - q^4\,F_{(1,1,1,1,1)}. \end{array}$$

Lemma 6.1. *Let $I=(i_1,\ldots,i_r)\succeq J=(j_1,\ldots,j_p)$ be two compositions. Let $K=\#(I,J)\vDash r$. The coefficient of $F_{(1^r)}$ in G_K is the same as the one of F_I in G_J. That is, the coefficient of F_I in G_J depends only on $\#(I,J)$.*

Proof — Let ϕ_I denote the map which takes a pseudo composition $m=[m_1,\ldots,m_n]$ of r and sends it to $\phi_I(m)=[m'_1,\ldots,m'_n]$ defined by

$$m'_i = \begin{cases} i_{m_1+\cdots+m_{i-1}+1}+\cdots+i_{m_1+\cdots+m_i}, & \text{if } m_i\neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For example, with $I=(1,2,5,4,1,2)$, we have

$$\phi_I[1,0,0,2,0,1,2]=[1,0,0,7,0,4,3] \quad \text{and} \quad \phi_I[3,0,0,1,2]=[8,0,0,4,3]$$

Remark that $\phi_I([1^r, 0, \dots, 0]) = X^I$, and that I is finer than any exponent composition appearing in $\phi_I(m)$. For a given m, we have that

$$\boldsymbol{T}_u\, \phi_I(m) = \phi_I(\boldsymbol{T}_u\, m) + \text{ some monomials not smaller than } I.$$

Moreover, if l is a monomial not smaller than I, then $\boldsymbol{T}_u\, l$ contains no monomials smaller that I. So, by induction,

$$\boldsymbol{T}_\sigma\, \phi_I(m) = \phi_I(\boldsymbol{T}_\sigma\, m) + \text{ some monomials not smaller than } I.$$

for any permutation σ. Finally, we find that

$$\Box_\omega\, \phi_I(m) = \phi_I(\Box_\omega\, m) + \text{ monomials not smaller than } I.$$

Back from monomials to quasi-ribbon function, the proof is done, since F_I is the sum of all monomials smaller than I. □

Example 6.2. Let $m = [3, 0]$. We have

$$\boldsymbol{T}_1\, m = [0, 3] + (1-q)[1, 2] + (1-q)[2, 1]$$

Let now $I = (3, 1, 2)$, so that $\phi_I(m) = [6, 0]$. Thus

$$\boldsymbol{T}_1\, \phi_I(m) = [0, 6] + (1-q)\left(\sum_{u=1}^{5} [u, 6-u]\right)$$

We verify that the only monomials of the form $[u, 6-u]$ smaller than I are $[3, 3] = \phi_I([1, 2])$ and $[4, 2] = \phi_I([2, 1])$.

Proof of Theorem 6.1 — Let $K = (k_1, \dots, k_p) \vDash r$. By virtue of the preceding lemma, it is sufficient to prove that the coefficient of $[1^r]$ in G_K is equal to

$$\begin{aligned}&(1-q)^{k_1-1}(1-q^2)^{k_2-1}(1-q^3)^{k_3-1}\dots(1-q^p)^{k_p-1} =\\ &\quad(1-q)^{r-p}1^{k_1-1}(1+q)^{k_2-1}(1+q+q^2)^{k_3-1}\dots(1+q+\dots+q^{p-1})^{k_p-1}\end{aligned}$$

We will prove this in two steps.

Lemma 6.2. *The coefficient of the monomial $[1^{r+1}]$ in the polynomial $G_{(K,1)}$ is equal to the one of $[1^r]$ in G_K.*

Lemma 6.3. *The coefficient of $[1^{r+1}]$ in $G_{(k_1,\dots,k_p+1)}$ is equal to the coefficient of $[1^r]$ in G_K, multiplied by $(1-q^p) = (1-q)(1+q+\dots q^{p-1})$.*

These two lemmas will prove the theorem by induction. □

Let us now prove the lemmas.

Proof of Lemma 6.2 — By proposition 6.1, we have

$$G_{(K,1)} = \sum_{\sigma \in \mathfrak{S}_{r+1/p+1}} \boldsymbol{T}_\sigma X^{(K,1)}.$$

Recall that for $\sigma \in \mathfrak{S}_{r+1/p+1}$, the operator $\boldsymbol{T}_\sigma$ has a reduced word of the form

$$(\boldsymbol{T}_{\sigma(1)-1} \cdots \boldsymbol{T}_2 \boldsymbol{T}_1)(\boldsymbol{T}_{\sigma(2)-1} \cdots \boldsymbol{T}_3 \boldsymbol{T}_2) \ \ldots \ (\boldsymbol{T}_{\sigma(p+1)-1} \cdots \boldsymbol{T}_{p+2} \boldsymbol{T}_{p+1})$$

with $\sigma(1) < \sigma(2) < \cdots < \sigma(p) < \sigma(p+1)$. But the $r+1$-th variable will have a zero exponent under $\boldsymbol{T}_\sigma X^K$ unless $\sigma(p+1) = r+1$. Now

$$\boldsymbol{T}_r \cdots \boldsymbol{T}_{p+2} \boldsymbol{T}_{p+1} X^K = [k_1 \ldots k_p, 0, \ldots, 0, 1]$$

Keeping the last variable aside, it just remains to compute the image of X^K under $\sum_{\sigma \in \mathfrak{S}_{r/p}} \boldsymbol{T}_\sigma$, which is G_K as well. And so the lemma is proved. □

Proof of Lemma 6.3 — Let us denote $K \triangleright 1$ the composition $(k_1, \ldots, k_p + 1)$ and $\hat{K} = (k_1, \ldots, k_{p-1})$. Let ω denote the maximal permutation of $\mathfrak{S}_{r+1}$ and ω' the maximal permutation of $\mathfrak{S}_r$. We want to compute the coefficient of $[1^{r+1}]$ in $\square_\omega[\hat{K}, k_p + 1, 0, \ldots, 0]$.

We will use the following factorization of the $\square_\omega$ operator:

$$\square_\omega = \square_{\omega'} \left(1 + \boldsymbol{T}_r + \boldsymbol{T}_r \boldsymbol{T}_{r-1} + \cdots + \boldsymbol{T}_r \ldots \boldsymbol{T}_1\right).$$

Let us compute the image of $X^{K \triangleright 1}$ under $\boldsymbol{T}_r \ldots \boldsymbol{T}_s$. There are two cases. If $s > p$, since all the exponents of $x^{p+1}, \ldots x^{r+1}$ are zero in $X^{K \triangleright 1}$ the action of the successive $\boldsymbol{T}_i$ is only to multiply it by q. Now if $s \leq p$, the action of the $\boldsymbol{T}_i$ for $i < p$ is again to multiply by q. Now since

$$\boldsymbol{T}_p[\hat{K}, k_p + 1, 0] = [\hat{K}, 0, k_p + 1] + (1-q) \sum_{u+v=k_p+1,\ u \neq 0,\ v \neq 0} [\hat{K}, u, v]$$

we get the following result:

$$\boldsymbol{T}_r \ldots \boldsymbol{T}_s[\hat{K}, k_p + 1, 0, \ldots, 0] =$$
$$\begin{cases} q^{p-s} \displaystyle\sum_{\substack{u_p + \cdots + u_{r+1} = k_p + 1 \\ u_{r+1} \neq 0}} (1-q)^{\#(u_p, \ldots, u_r)} [\hat{K}, u_p, \ldots, u_{r+1}] & \text{if } s \leq p, \\ q^{r-s}[\hat{K}, k_p + 1, 0, \ldots, 0] & \text{else,} \end{cases}$$

where $\#(u_p, \dots, u_r)$ is the number of non-zero u_i.

Since $\square_{\omega'}$ will never act on the last variable, we are only interested in the part of these sums such that $u_{r+1} = 1$. Let S be the part of $(1 + \boldsymbol{T}_r + \boldsymbol{T}_r \boldsymbol{T}_{r-1} + \cdots + \boldsymbol{T}_r \dots \boldsymbol{T}_1)[\hat{K}, k_P + 1]$ such that the exponent of the last variable is 1. The result of these computations is:

$$S = (1 + q + \cdots + q^{p-1}) \sum_{u_p + \cdots + u_r = k_p} (1-q)^{\#(u_p,\dots,u_r)} [\hat{K}, u_p, \dots, u_r, 1]$$

By analogy, we recognize the following sum:

$$S = (1 - q^p)\,(1 + \boldsymbol{T}_p + \boldsymbol{T}_{p+1}\boldsymbol{T}_p + \cdots + \boldsymbol{T}_{r-1} \dots \boldsymbol{T}_p)\,[K, 0, \dots, 0, 1]$$

Now using the facts that $\square_i \boldsymbol{T}_i = q\,\square_i$ and the factorization $\square_{\omega'} = \square_{\sigma'}\,\square_i$ of proposition 5.3, we find that $\square_{\omega'}\,\boldsymbol{T}_i = q\,\square_{\omega'}$, for any i. Hence

$$\square_{\omega'}\,S = (1 - q^p)(1 + q + \cdots + q^{r-p})\,\square_{\omega'}[K, 0, \dots, 0, 1]$$

The coefficient $(1 + q + \cdots + q^{r-p})$ simplifies with the normalization factor of $G_{K \triangleright 1}$, so we have proved the lemma. $\square$

As a consequence of the proof we get the expansions of the Hall-Littlewood basis in the quasi-monomial basis:

$$G_I = \sum_{J \succeq I} a_{I,J} M_J \tag{112}$$

where $a_{I,J}$ is defined as follows. Let $(k_1, \dots, k_p)$ be the refining composition $\#(J, I)$. Then

$$a_{I,J} = (1-q)^{k_1 - 1}(1-q^2)^{k_2 - 1} \cdots (1-q^p)^{k_p - 1}. \tag{113}$$

The transition matrix is an upper unitriangular matrix (*i.e.* 1 on the diagonal, and $M_{I,J}$ is zero unless $I \succeq J$), corresponding to the inverse of the q-Kotska matrix. The analogue of the expression $s_\lambda = \sum K_{\lambda\mu}(q) P_\mu$ will be obtained by means of the dual basis, which lives in the space of noncommutative symmetric functions. We will prove that the coefficient of the expansion are polynomials in q with integer coefficients.

6.3 Noncommutative Hall-Littlewood functions

In the former subsection we have proved that the family of quasi-symmetric Hall-Littlewood functions is a basis of **Qsym**. We can now define the noncommutative Hall-Littlewood symmetric functions by duality.

Definition 6.2. *The element of the dual basis (H_K) of (G_I) are called non-commutative symmetric Hall-Littlewood functions.*

The analogue of the expression $Q'_\mu = \sum K_{\lambda\mu} s_\lambda$ is the following formula.

Theorem 6.2. *The transition matrix whose rows are the H_J expanded in the R_I basis is a lower unitriangular matrix with positive coefficients. Moreover the expansion is given by:*

$$H_K(A;q) = \sum_{K \succeq J} q^{t(K,J)} R_J, \tag{114}$$

where $t(K,J) = \mathrm{Maj}\big(\#(K,J)\tilde{}\big)$

Before giving a proof, we need to give a more explicit way to compute the coefficient $\mathrm{Maj}\big(\#(K,J)\tilde{}\big)$ for $K \succeq J$. Let us suppose that $K = (k_1, \dots, k_p)$ and $J = (j_1, \dots, j_q)$. Then by definition of the refinement order, there exist $0 < u_1 < u_2 < \cdots < u_q = p$ such that

$$J = (k_1 + k_2 + \cdots + k_{u_1}, k_{u_1+1} + \cdots + k_{u_1}, \quad \dots \quad , k_{u_{p-1}+1} + \cdots + k_p).$$

Remark that the descent set of the composition $\#(K,J)$ is nothing but the set $U = \{u_1 < u_2 < \cdots < u_{q-1}\}$. Hence the decent set of the composition $\#(K,J)\tilde{}$ is the complementary of U in $\{1, \dots, p-1\}$. Since the major index of a composition is the sum of its descents, the coefficient $t(K,J)$ is the sum of the positions of the "+" in the former expression of J. For example if $K = (1,3,2,1,1,3,1,4,1)$ and $J = (4,7,1,5)$, we write

$$\begin{array}{rl} & J = (1{+}3\,,2{+}1{+}1{+}3\,,1\,,4{+}1) \\ \text{positions:} & \quad\;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7 \;\; 8 \end{array}$$

The "+" appear in position $1,3,4,5$ and 8 so that $t(K,J) = 21$. We verify that $\#(K,J)\tilde{} = (2,4,1,2)\tilde{} = (1,2,1,1,3,1)$ and thus its major index is $5*1+4*2+3*1+2*1+3 = 21$ as well.

Proof — The ribbon basis (R_J) is the dual basis of the quasi-ribbon basis (F_J). Hence the duality pairing of G_I and H_K is

$$\begin{aligned} \langle G_I, H_K\rangle &= \left\langle \sum_{J \succeq I} (-1)^{\ell(J)-\ell(I)} q^{s(I,J)} F_J, \sum_{K \succeq L} q^{t(K,L)} R_L \right\rangle \\ &= \sum_{J \succeq I} \sum_{K \succeq L} (-1)^{\ell(J)-\ell(I)} q^{s(I,J)+t(K,L)} \langle F_J, R_L\rangle \\ &= \sum_{K \succeq J \succeq I} (-1)^{\ell(J)-\ell(I)} q^{s(I,J)+t(K,J)}. \end{aligned} \tag{115}$$

Set $z(K,J,I) = (-1)^{\ell(J)-\ell(I)} q^{s(I,J)+t(K,J)}$. It is obvious that if $K = I$ the sum (115) reduces to 1. Hence, we only need to prove that the last sum is equal to zero for $K \succ I$.

Now suppose $I = (i_1, i_2, \ldots, i_p)$. Let l be the index of the first part of I which is broken in K, so that $K = (k_1, k_2, \ldots, k_q)$ with $i_u = k_u$ for $i < l$. Now the set $\{J \mid K \succeq J \succeq I\}$ breaks into two subsets, whether k_l appears alone as part of J or is added with k_{l+1}. Define the involution $\Psi_{K,I}$ of $\{J \mid K \succeq J \succeq I\}$ which exchanges the two preceding subsets as follows:

$$\Psi_{K,I}(k_1, \ldots, k_{l-1}, j_l, \ldots, j_r) = \begin{cases} (k_1, \ldots, k_{l-1}, k_l + j_{l+1}, \ldots, j_r) & \text{if } j_l = k_l \\ (k_1, \ldots, k_{l-1}, k_l, j_l - k_l, \ldots, j_r) & \text{if } j_l > k_l \end{cases}$$

For example, in the case $K = (2,1,1,3,2,1,1,3)$ and $I = (2,1,6,1,2,3)$, one has $l = 3$. Figure 3 shows the Hasse diagram of the set of composition J between K and I, together with the involution $\Psi_{K,I}$.

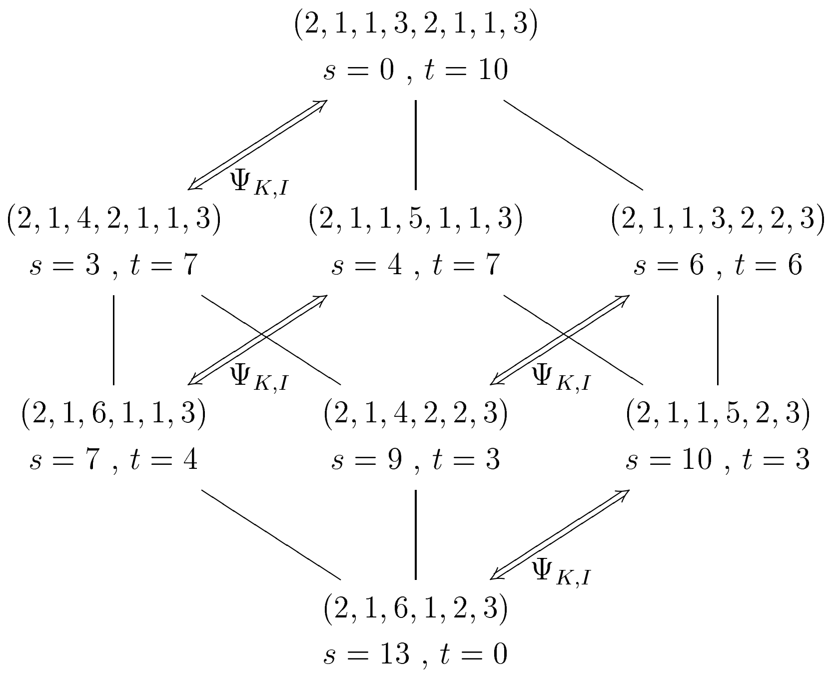


Figure 3: The involution $\Psi_{K,I}$.

Now $\#(J,I)$ is of the form $(1^{l-1}, a, \ldots)$. Thus, if J is of the first kind ($j_l =$

k_l), the composition $\#(\Psi_{K,I}(J), I)$ is equal to $(1^{l-1}, a-1, \dots)$. Therefore

$$s(I, \Psi_{K,I}(J)) = s(I, J) - q^l.$$

Similarly, we find that:

$$t(I, \Psi_{K,I}(J)) = t(I, J) + q^l.$$

Hence

$$z(K, J, I) + z(K, \Psi_{K,I}(J), I) = 0$$

One verifies on Figure 3 that the sum $s + t$ is constant along the double arrows corresponding to $\Psi_{K,I}$.

Since $\Psi_{K,I}$ is an involution, the sum (115) vanishes, and the proof is done. □

We observe that the analogues of the Kotska-Foulkes polynomials reduce here to monomials. For instance $H_{(1^n)} = \sum_K q^{\mathrm{Maj}(K^{\sim})} R_K$ where $K^{\sim}$ is the conjugate composition of K. Another example is: $H_{(3,2,1)} = R_{(3,2,1)} + qR_{(3,3)} + q^2 R_{(5,1)} + q^3 R_{(6)}$.

Corollary 6.3. *The H_I form a basis of noncommutative symmetric functions with coefficients in $\mathbb{Z}[q]$.*

The family $(G_K)_{\ell(K) \leq n}$ is a basis of the space of quasi-symmetric polynomials with coefficient in $\mathbb{Z}[q]$.

The family $(G_K)_K$ is a basis of the space of quasi-symmetric functions with coefficients in $\mathbb{Z}[q]$.

Thus, we can express the product of two H_K.

Theorem 6.3. *Let I and J two compositions of lengths r and s. Then*

$$H_I H_J = \sum_{J \succeq K} q^{t(J,K)} \left(c(r, s-p) H_{I \cdot K} + c(r, s-p+1) H_{I \triangleright K} \right) \tag{116}$$

where p is the length of the composition K and

$$c(r, v) = (1 - q^r)(1 - q^{r-1}) \dots (1 - q^{r-v+1}) \tag{117}$$

with the convention that $c(r, 0) = 1$ and $c(r, v) = 0$ if $v > i$.

This is a consequence of the formula $R_I R_J = R_{I \cdot J} + R_{I \triangleright J}$. For example

$$\begin{aligned} H_{(3,1,2)} H_{(1,2)} = & H_{(3,1,2,1,2)} + (1 - q^3) H_{(3,1,3,2)} \\ & + q(1 - q^3) H_{(3,1,2,3)} + q(1 - q^3)(1 - q^2) H_{(3,1,5)}. \end{aligned}$$

and

$$H_{(1,1,1)}H_{(1,1)} = H_{(1,1,1,1,1)} + (1-q^3)H_{(1,1,2,1)} + q(1-q^3)H_{(1,1,1,2)} + q(1-q^3)(1-q^2)H_{(1,1,3)}.$$

Proof — On the two preceding examples, one can remark that the two coefficients are the same. This is a general fact, which gives the starting point of the proof.

The coefficients depends only on the lengths of the compositions I, J and on $\#(J,K)$, but not on the parts. Let us write this formally,

Suppose that $I' = (1^r)$ and $J' = (1^s)$. Let $U = (u_1, \ldots, u_{r+s}) = I \cdot J$ and $L = (l_1, \ldots, l_t)$ be a composition of $r+s$. To L we associate a composition $\phi_{(I,J)}(L)$ of $|I| + |J|$ obtained by summing the first l_1 parts of U, the next l_2 parts, and so one. That is, $\phi_{(I,J)}(L)$ is given by

$$(u_1 + u_2 + \cdots + u_{l_1-1}, u_{l_1} + \cdots + u_{l_1+l_2-1}, \cdots, u_{l_1+\ldots} + \cdots + u_{r+s}) \tag{118}$$

Then, $\phi_{(I,J)}(L)$ is a composition of the same length as L, and U is finer than $\phi_{(I,J)}(L)$. For example, with the previous notations, $I = (3,1,2)$, $J = (1,2)$ so that $U = (3,1,2,1,2)$. Then $\phi_{(I,J)}(1,2,2) = (3,3,3)$ and $\phi_{(I,J)}(1,1,2,1) = (3,1,3,2)$.

Is is known that the product rule of ribbon functions is

$$R_U R_V = R_{U\cdot V} + R_{U \triangleright V}. \tag{119}$$

It appears to verify the equality

$$R_{\phi_{(I,J)}(U)} R_{\phi_{(I,J)}(V)} = R_{\phi_{(I,J)}(U\cdot V)} + R_{\phi_{(I,J)}(U \triangleright V)} \tag{120}$$

which says that the product of ribbon functions is "invariant under ϕ". Now, by definition of t, the coefficient of $H_{(1^r)}H_{(1^s)}$ on R_L is the same as the one of $H_I H_J$ on $R_{\phi_{(I,J)}(L)}$. Moreover, one has that $\#((1^{r+s}), L) = \#(U, \phi_{(I,J)}(L))$. It allows us to go back to the basis of Hall-Littlewood functions. Hence, we have proven the lemma:

Lemma 6.4. *The coefficient of $H_{(1^r)}H_{(1^s)}$ on H_L is the same as the one of $H_I H_J$ on $H_{\phi_{(I,J)}(L)}$.*

The coefficient of $H_I H_J$ on H_K where K is not of the form $\phi_{(I,J)}(L)$ is zero.

Hence it is enough to prove the case $H_{(1^r)}H_{(1^s)}$ for all r and s. Now suppose that K is a composition. Let us compute $H_{1^r \cdot K}$ and $H_{1^r \triangleright K}$. Recall that the coefficient $t(I,J)$, is obtained by adding the position of the "+" when we write the parts of J as a sum of parts of I. There are two kinds of compositions L greater than $1^r \cdot K$:

• The ones for which r is a descent, that is the compositions of the form $J \cdot K'$, where J is a composition of r and $K \succeq K'$. In this case the position of the "+" smaller than r are the same in L and J and the positions greater than r in L are of the form $r+s$ where s is the position of the corresponding "+" in K. Hence we get that

$$t\,(1^r \cdot K, J \cdot K') = t(1^r, J) + t(K, K') + (\ell(K) - \ell(K'))r. \tag{121}$$

For example, let $r = 4$ and $K = (2,3,1,4,1)$. Then the composition $L = (2,1,1,5,1,5)$ is of the form (J, K'), with $J = (2,1,1) \vDash 4$ and $K \succeq K' = (5,1,5)$. Then one writes

$$\begin{array}{rl} J = & (1{+}1\,,1\,,1\;\;,\;\;2{+}3\,,1\,,4{+}1) \\ \text{positions:} & \;\;1\;\;2\;\;3\;\;\;4\;\;\;5\;\;6\;\;7\;\;8 \end{array}$$

The positions of the "+" are 1 corresponding to $J = (1+1,1,1)$ and $5,8$ which correspond to $1,4$ when writing $K' = (5,1,5) = (2+3,1,4+1)$. One can verify that $1+5+8 = 1+(1+5)+(5-3)\cdot 4$.

• The other compositions are the ones of the form $(J \triangleright K')$, where J is a composition of r and $K \succeq K'$. The only difference with the preceding case is that there is a "+" in the r-th position. Hence

$$t\,(1^r \cdot K, J \triangleright K') = t(1^r, J) + t(K, K') + (1 + \ell(K) - \ell(K'))r \tag{122}$$

For example, let $r = 4$ and $K = (2,3,1,4,1)$. Then the composition $L = (2,1,6,1,5)$ is of the form $(J \triangleright K')$, with $J = (2,1,1) \vDash 4$ and $K \succeq K' = (5,1,5)$. Then one writes

$$\begin{array}{rl} J = & (1{+}1\,,1\,,1\;+\;2{+}3\,,1\,,4{+}1) \\ \text{positions:} & \;\;1\;\;2\;\;3\;\;\;4\;\;\;5\;\;6\;\;7\;\;8 \end{array}$$

One can verify that $1+4+5+8 = 1+(1+5)+(5-3+1)*4$.

The result of these computations is that :

$$H_{1^r \cdot K} = \sum_{J \vDash r} q^{t(1^r, J)} \sum_{K \succeq K'} q^{t(K,K') + (\ell(K) - \ell(K'))r} \left(R_{J \cdot K'} + q^r R_{J \triangleright K'}\right). \tag{123}$$

Similarly, one would find

$$H_{1^r \triangleright K} = \sum_{J \vDash r} q^{t(1^r, J)} \sum_{K \succeq K'} q^{t(K,K') + (\ell(K) - \ell(K'))(r-1)} \left(R_{J \triangleright K'}\right). \tag{124}$$

But, from the expansion of H_I and the product rule of R_I we get

$$H_{(1^r)}H_{(1^s)} = \sum_{J\vDash r} q^{t(1^r,J)} \sum_{K'\vDash s} q^{t(1^s,K')} \left(R_{J\cdot K'} + R_{J\triangleright K'}\right). \tag{125}$$

We are trying to prove that

$$H_{(1^r)}H_{(1^s)} = \sum_{K\vDash s} q^{t(1^s,K)}\big(c(r,s-\ell(K))H_{1^r\cdot K} + (1-q^r)c(r-1,s-\ell(K))H_{1^r\triangleright K}\big). \tag{126}$$

Hence we only need to prove the following formula corresponding to the coefficient of $R_{1^r\cdot K'}$.

$$q^{t(1^s,K')} = \sum_{(1^s)\succeq K\succeq K'} c(r,s-\ell(K))q^{(\ell(K)-\ell(K'))r}q^{t(1^s,K)+t(K,K')} \tag{127}$$

The coefficient of $R_{1^r\triangleright K'}$ follows immediately, replacing r by $r-1$. Let us prove this property by induction on K'. First define

$$z(1^s,K',K) = c(r,s-\ell(K))q^{(\ell(K)-\ell(K'))r}q^{t(1^s,K)+t(K,K')} \tag{128}$$

If $K'=(1)$, and thus $s=1$, one has $z(1,1,1)=1$ and the result is true. Moreover, if this is true for K' this is again true for $K'\cdot 1$, since $t(U\cdot 1, V\cdot 1)) = t(U,V)$ and thus

$$z(1^{s+1},K'\cdot 1,K\cdot 1) = z(1^s,K',K) \tag{129}$$

Thus it remains to show that the property for K implies the property for the composition $K'\triangleright 1$. As usual the sum

$$\sum_{(1^{s+1})\succeq K'\succeq K\triangleright 1} z(1^{s+1},K',K\triangleright 1) \tag{130}$$

breaks into two parts depending on the fact that $K=L\cdot 1$ for some $L\vDash s$ or not ($K=L\triangleright 1$). Let us prove the following lemma

Lemma 6.5. *Let* $(1^s)\succeq L\succeq K$, *then one has:*

$$\begin{aligned} z(1^{s+1},L\cdot 1,K\triangleright 1) &= z(1^s,L,K)q^{r+\ell(L)} \\ z(1^{s+1},L\triangleright 1,K\triangleright 1) &= z(1^s,L,K)q^s(1-q^{r-s+\ell(L)}) \end{aligned} \tag{131}$$

In the first case $K = L \cdot 1$, one has $\ell(L) = \ell(K) + 1$, which gives the q^r. Equation (129) and the fact that

$$t(L \cdot 1, K' \triangleright 1) = t(L, K) + \ell(L) \tag{132}$$

give the required result. In the second case, one gets q^s from

$$t(1^{s+1}, L \triangleright 1) = t(1^s, L) + s \tag{133}$$

and the $(1 - q^{r-s+\ell(L)})$ comes from $c(r, s + 1 - \ell(L))$. This proves the lemma.

Finally one concludes that

$$\sum_{(1^{s+1}) \succeq K' \succeq K \triangleright 1} z(1^{s+1}, K', K \triangleright 1) = q^s \sum_{(1^s) \succeq K' \succeq K} z(1^s, K', K). \tag{134}$$

By induction the right hand side is equal to $q^s q^{t(1^s, K')}$, and the fact that

$$q^{t(1^{s+1}, K' \triangleright 1)} = q^s q^{t(1^s, K')} \tag{135}$$

gives the desired equality. The proof follows by induction. □

As a corollary we get the following striking property:

Corollary 6.4. *Let $H'_K = (1-q)^{\ell(K)} H_K$. Then the structure constants of* **Sym** *in the H'_K basis are polynomials in q with non-negative integer coefficients.*

Indeed, the preceding theorem shows that if I and J two compositions of lengths r and s, then

$$H'_I H'_J = \sum_{J \succeq K} q^{t(J,K)} \left(d(r, s-p) H'_{I \cdot K} + d(r, s-p+1) H'_{I \triangleright K} \right) \tag{136}$$

where p is the length of the composition K and

$$d(r, v) = [r]_q [r-1]_q \ldots [r-v+1]_q \tag{137}$$

For example

$$\begin{aligned} H'_{(3,1,2)} H'_{(1,2)} =& H'_{(3,1,2,1,2)} + (1+q+q^2) H'_{(3,1,3,2)} \\ &+ q(1+q+q^2) H'_{(3,1,2,3)} + q(1+q+q^2)(1+q) H'_{(3,1,5)}. \end{aligned}$$

This raise the question of an interpretation of these functions in the representation theory.

We are now interested in the specializations of the Hall-Littlewood functions at roots of unity. The noncommutative Hall-Littlewood functions have a factorization property similar the the one discovered by Lascoux, Leclerc and Thibon [24, 25] (see also [38, 2]).

Corollary 6.5. *Let k be a integer and ζ be a k-th root of the unity. Suppose that $I = (i_1, \ldots, i_p)$ is a composition. Writing $p = ck + r$, we break the composition I into blocks of length k. There are c compositions $J_1, \ldots, J_c$ of length k and a composition J_{c+1} of length r such that $I = J_1 \cdot J_1 \cdots J_{c+1})$. Then the function $H_I(A;\zeta)$ factorizes in the following way:*

$$H_I(A;\zeta) = H_{J_1}(A;\zeta)H_{J_2}(A;\zeta)\cdots H_{J_{c+1}}(A;\zeta) \tag{138}$$

For instance, if ζ is a 3rd root of the unity,

$$H_{(3,2,4,1,5,3,2,1)}(A;\zeta) = H_{(3,2,4)}(A;\zeta)H_{(1,5,3)}(A;\zeta)H_{(2,1)}(A;\zeta).$$

Proof — This is an easy consequence of Theorem 6.3. By induction on c is is sufficient to prove that if J is a composition of length ck and K is a composition of any length one has $H_J(A;\zeta)H_K(A;\zeta) = H_{J,K}(A;\zeta)$, provided ζ is a k-th root of the unity. But Theorem 6.3 reads

$$H_J(A;\zeta)H_K(A;\zeta) = H_{J\cdot K}(A;\zeta) + (1-\zeta^k)\left(\sum \text{Other } H_L\right). \tag{139}$$

since $c(i,v)$ is a multiple of $(1-\zeta^k)$ unless $v = 0$. So the proof is done. □

Note that this property remains true for the modified function H'_K.

Appendix A - Tables

We give the transition matrices between the new Hall-Littlewood functions and the generalized Schur functions. The matrices are to be read as follows: The coefficient of the Hall-Littlewood function G_I (*resp.* H_I) on the function F_J (*resp.* R_J) is at the intersection of the row I and column J. A "." means a zero coefficient. An integer i means q^i, with the convention that overlined numbers encode negative coefficients. For example on the row indexed by 31 one reads that $G_{(31)} = F_{(31)} - qF_{(211)} - qF_{(121)} + q^2F_{(1111)}$.

$$\begin{array}{cc} & \begin{matrix} 3 & 21 & 12 & 111 \end{matrix} \\ \begin{matrix} 3 \\ 21 \\ 12 \\ 111 \end{matrix} & \begin{bmatrix} 0 & \bar{1} & \bar{1} & 2 \\ . & 0 & . & \bar{1} \\ . & . & 0 & \bar{2} \\ . & . & . & 0 \end{bmatrix} \end{array}$$

G_I on F_J, degree 3.

$$\begin{array}{cc} & \begin{matrix} 3 & 21 & 12 & 111 \end{matrix} \\ \begin{matrix} 3 \\ 21 \\ 12 \\ 111 \end{matrix} & \begin{bmatrix} 0 & . & . & . \\ 1 & 0 & . & . \\ 1 & . & 0 & . \\ 3 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

H_I on R_J, degree 3.

$$\begin{array}{cc} & \begin{matrix} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{matrix} \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{matrix} & \begin{bmatrix} 0 & \bar{1} & \bar{1} & 2 & \bar{1} & 2 & 2 & \bar{3} \\ . & 0 & . & \bar{1} & . & \bar{1} & . & 2 \\ . & . & 0 & \bar{2} & . & . & \bar{1} & 3 \\ . & . & . & 0 & . & . & . & \bar{1} \\ . & . & . & . & 0 & \bar{2} & \bar{2} & 4 \\ . & . & . & . & . & 0 & . & \bar{2} \\ . & . & . & . & . & . & 0 & \bar{3} \\ . & . & . & . & . & . & . & 0 \end{bmatrix} \end{array}$$

G_I on F_J, degree 4.

$$\begin{array}{cc} & \begin{matrix} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{matrix} \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{matrix} & \begin{bmatrix} 0 & . & . & . & . & . & . & . \\ 1 & 0 & . & . & . & . & . & . \\ 1 & . & 0 & . & . & . & . & . \\ 3 & 1 & 2 & 0 & . & . & . & . \\ 1 & . & . & . & 0 & . & . & . \\ 3 & 1 & . & . & 2 & 0 & . & . \\ 3 & . & 1 & . & 2 & . & 0 & . \\ 6 & 3 & 4 & 1 & 5 & 2 & 3 & 0 \end{bmatrix} \end{array}$$

H_I on R_J, degree 4.

	5	41	32	311	23	221	212	2111	14	131	122	1211	113	1121	1112	11111
5	0	$\bar{1}$	$\bar{1}$	2	$\bar{1}$	2	2	$\bar{3}$	$\bar{1}$	2	2	$\bar{3}$	2	$\bar{3}$	$\bar{3}$	4
41	.	0	.	$\bar{1}$	.	$\bar{1}$	.	2	.	$\bar{1}$	.	2	.	2	.	$\bar{3}$
32	.	.	0	$\bar{2}$	.	.	$\bar{1}$	3	.	.	$\bar{1}$	3	.	.	2	$\bar{4}$
311	.	.	.	0	.	.	.	$\bar{1}$	.	.	.	$\bar{1}$	.	.	.	2
23	.	.	.	.	0	$\bar{2}$	$\bar{2}$	4	.	.	.	.	$\bar{1}$	3	3	$\bar{5}$
221	.	.	.	.	.	0	.	$\bar{2}$	.	.	.	.	.	$\bar{1}$	.	3
212	.	.	.	.	.	.	0	$\bar{3}$	.	.	.	.	.	.	$\bar{1}$	4
2111	.	.	.	.	.	.	.	0	.	.	.	.	.	.	.	$\bar{1}$
14	.	.	.	.	.	.	.	.	0	$\bar{2}$	$\bar{2}$	4	$\bar{2}$	4	4	$\bar{6}$
131	.	.	.	.	.	.	.	.	.	0	.	$\bar{2}$	.	$\bar{2}$	.	4
122	.	.	.	.	.	.	.	.	.	.	0	$\bar{3}$	.	.	$\bar{2}$	5
1211	.	.	.	.	.	.	.	.	.	.	.	0	.	.	.	$\bar{2}$
113	.	.	.	.	.	.	.	.	.	.	.	.	0	$\bar{3}$	$\bar{3}$	6
1121	.	.	.	.	.	.	.	.	.	.	.	.	.	0	.	$\bar{3}$
1112	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	$\bar{4}$
11111	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0

G_I on F_J, degree 5.

	5	41	32	311	23	221	212	2111	14	131	122	1211	113	1121	1112	11111
5	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
41	1	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.
32	1	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.
311	3	1	2	0	.	.	.	.	.	.	.	.	.	.	.	.
23	1	.	.	.	0	.	.	.	.	.	.	.	.	.	.	.
221	3	1	.	.	2	0	.	.	.	.	.	.	.	.	.	.
212	3	.	1	.	2	.	0	.	.	.	.	.	.	.	.	.
2111	6	3	4	1	5	2	3	0	.	.	.	.	.	.	.	.
14	1	.	.	.	.	.	.	.	0	.	.	.	.	.	.	.
131	3	1	.	.	.	.	.	.	2	0	.	.	.	.	.	.
122	3	.	1	.	.	.	.	.	2	.	0	.	.	.	.	.
1211	6	3	4	1	.	.	.	.	5	2	3	0	.	.	.	.
113	3	.	.	.	1	.	.	.	2	.	.	.	0	.	.	.
1121	6	3	.	.	4	1	.	.	5	2	.	.	3	0	.	.
1112	6	.	3	.	4	.	1	.	5	.	2	.	3	.	0	.
11111	10	6	7	3	8	4	5	1	9	5	6	2	7	3	4	0

H_I on R_J, degree 5.

	6	51	42	411	33	321	312	3111	24	231	222	2211	213	2121	2112	21111	15	141	132	1311	123	1221	1212	12111	114	1131	1122	11211	1113	11121	11112	111111
6	0	$\bar{1}$	$\bar{1}$	2	$\bar{1}$	2	2	$\bar{3}$	$\bar{1}$	2	2	$\bar{3}$	2	$\bar{3}$	$\bar{3}$	4	$\bar{1}$	2	2	$\bar{3}$	2	$\bar{3}$	$\bar{3}$	4	2	$\bar{3}$	$\bar{3}$	4	$\bar{3}$	4	4	$\bar{5}$
51	.	0	.	$\bar{1}$	.	$\bar{1}$	.	2	.	$\bar{1}$	.	2	.	2	.	$\bar{3}$	.	$\bar{1}$	.	2	.	2	.	$\bar{3}$	.	2	.	$\bar{3}$	.	$\bar{3}$	.	4
42	.	.	0	$\bar{2}$	.	.	$\bar{1}$	3	.	.	$\bar{1}$	3	.	.	2	$\bar{4}$	.	.	$\bar{1}$	3	.	.	2	$\bar{4}$	.	.	2	$\bar{4}$	.	.	$\bar{3}$	5
411	.	.	.	0	.	.	.	$\bar{1}$	.	.	.	$\bar{1}$	.	.	.	2	.	.	.	$\bar{1}$	.	.	.	2	.	.	.	2	.	.	.	$\bar{3}$
33	.	.	.	.	0	$\bar{2}$	$\bar{2}$	4	.	.	.	.	$\bar{1}$	3	3	$\bar{5}$	.	.	.	.	$\bar{1}$	3	3	$\bar{5}$	.	.	.	.	2	$\bar{4}$	$\bar{4}$	6
321	.	.	.	.	.	0	.	$\bar{2}$	.	.	.	.	.	$\bar{1}$	.	3	.	.	.	.	.	$\bar{1}$	.	3	.	.	.	.	.	2	.	$\bar{4}$
312	.	.	.	.	.	.	0	$\bar{3}$	.	.	.	.	.	.	$\bar{1}$	4	.	.	.	.	.	.	$\bar{1}$	4	.	.	.	.	.	.	2	$\bar{5}$
3111	.	.	.	.	.	.	.	0	.	.	.	.	.	.	.	$\bar{1}$	.	.	.	.	.	.	.	$\bar{1}$	.	.	.	.	.	.	.	2
24	.	.	.	.	.	.	.	.	0	$\bar{2}$	$\bar{2}$	4	$\bar{2}$	4	4	$\bar{6}$	.	.	.	.	.	.	.	.	$\bar{1}$	3	3	$\bar{5}$	3	$\bar{5}$	$\bar{5}$	7
231	.	.	.	.	.	.	.	.	.	0	.	$\bar{2}$	.	$\bar{2}$	.	4	.	.	.	.	.	.	.	.	.	$\bar{1}$	.	3	.	3	.	$\bar{5}$
222	.	.	.	.	.	.	.	.	.	.	0	$\bar{3}$	.	.	$\bar{2}$	5	.	.	.	.	.	.	.	.	.	.	$\bar{1}$	4	.	.	3	$\bar{6}$
2211	.	.	.	.	.	.	.	.	.	.	.	0	.	.	.	$\bar{2}$	.	.	.	.	.	.	.	.	.	.	.	$\bar{1}$	.	.	.	3
213	.	.	.	.	.	.	.	.	.	.	.	.	0	$\bar{3}$	$\bar{3}$	6	.	.	.	.	.	.	.	.	.	.	.	.	$\bar{1}$	4	4	$\bar{7}$
2121	.	.	.	.	.	.	.	.	.	.	.	.	.	0	.	$\bar{3}$	.	.	.	.	.	.	.	.	.	.	.	.	.	$\bar{1}$	.	4
2112	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	$\bar{4}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$\bar{1}$	5
21111	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$\bar{1}$
15	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	$\bar{2}$	$\bar{2}$	4	$\bar{2}$	4	4	$\bar{6}$	$\bar{2}$	4	4	$\bar{6}$	4	$\bar{6}$	$\bar{6}$	8
141	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	.	$\bar{2}$	.	$\bar{2}$	.	4	.	$\bar{2}$	.	4	.	4	.	$\bar{6}$
132	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	$\bar{3}$	.	.	$\bar{2}$	5	.	.	$\bar{2}$	5	.	.	4	$\bar{7}$
1311	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	.	.	.	$\bar{2}$	.	.	.	$\bar{2}$	.	.	.	4
123	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	$\bar{3}$	$\bar{3}$	6	.	.	.	.	$\bar{2}$	5	5	$\bar{8}$
1221	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	.	$\bar{3}$	.	.	.	.	.	$\bar{2}$	.	5
1212	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	$\bar{4}$	.	.	.	.	.	.	$\bar{2}$	6
12111	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	.	.	.	.	.	.	.	$\bar{2}$
114	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	$\bar{3}$	$\bar{3}$	6	$\bar{3}$	6	6	$\bar{9}$
1131	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	.	$\bar{3}$	.	$\bar{3}$	.	6
1122	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	$\bar{4}$	.	.	$\bar{3}$	7
11211	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	.	.	.	$\bar{3}$
1113	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	$\bar{4}$	$\bar{4}$	8
11121	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	.	$\bar{4}$
11112	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	$\bar{5}$
111111	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0

G_I on F_J, degree 6.

	6	51	42	411	33	321	312	3111	24	231	222	2211	213	2121	2112	21111	15	141	132	1311	123	1221	1212	12111	114	1131	1122	11211	1113	11121	11112	111111
6	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
51	1	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
42	1	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
411	3	1	2	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
33	1	.	.	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
321	3	1	.	.	2	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
312	3	.	1	.	2	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3111	6	3	4	1	5	2	3	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
24	1	.	.	.	.	.	.	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
231	3	1	.	.	.	.	.	.	2	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
222	3	.	1	.	.	.	.	.	2	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2211	6	3	4	1	.	.	.	.	5	2	3	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
213	3	.	.	.	1	.	.	.	2	.	.	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2121	6	3	.	.	4	1	.	.	5	2	.	.	3	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2112	6	.	3	.	4	.	1	.	5	.	2	.	3	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
21111	10	6	7	3	8	4	5	1	9	5	6	2	7	3	4	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
15	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
141	3	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.
132	3	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.
1311	6	3	4	1	.	.	.	.	.	.	.	.	.	.	.	.	5	2	3	0	.	.	.	.	.	.	.	.	.	.	.	.
123	3	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	0	.	.	.	.	.	.	.	.	.	.	.
1221	6	3	.	.	4	1	.	.	.	.	.	.	.	.	.	.	5	2	.	.	3	0	.	.	.	.	.	.	.	.	.	.
1212	6	.	3	.	4	.	1	.	.	.	.	.	.	.	.	.	5	.	2	.	3	.	0	.	.	.	.	.	.	.	.	.
12111	10	6	7	3	8	4	5	1	.	.	.	.	.	.	.	.	9	5	6	2	7	3	4	0	.	.	.	.	.	.	.	.
114	3	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	2	.	.	.	.	.	.	.	0	.	.	.	.	.	.	.
1131	6	3	.	.	.	.	.	.	4	1	.	.	.	.	.	.	5	2	.	.	.	.	.	.	3	0	.	.	.	.	.	.
1122	6	.	3	.	.	.	.	.	4	.	1	.	.	.	.	.	5	.	2	.	.	.	.	.	3	.	0	.	.	.	.	.
11211	10	6	7	3	.	.	.	.	8	4	5	1	.	.	.	.	9	5	6	2	.	.	.	.	7	3	4	0	.	.	.	.
1113	6	.	.	.	3	.	.	.	4	.	.	.	1	.	.	.	5	.	.	.	2	.	.	.	3	.	.	.	0	.	.	.
11121	10	6	.	.	7	3	.	.	8	4	.	.	5	1	.	.	9	5	.	.	6	2	.	.	7	3	.	.	4	0	.	.
11112	10	.	6	.	7	.	3	.	8	.	4	.	5	.	1	.	9	.	5	.	6	.	2	.	7	.	3	.	4	.	0	.
111111	15	10	11	6	12	7	8	3	13	8	9	4	10	5	6	1	14	9	10	5	11	6	7	2	12	7	8	3	9	4	5	0

H_I on R_J, degree 6.

References

[1] I. N. BERNSTEIN, I. M. GELFAND, AND S. I. GELFAND, Schubert cells and the cohomology of the space G/P, *Russian Math. Surveys* **28** (1973), 1–26.

[2] P. BOUWKNEGT, AND K. PILCH, The deformed Virasoro algebra at roots of unity, *Commun. Math. Phys* **196** (1998), 249-288.

[3] R. W. CARTER, Representation theory of the 0-Hecke algebra, *J. Algebra* **15** (1986), 89–103.

[4] I. CHEREDNIK A unification of Knizhnik-Zamolodchikov and Dunkl operators via affine Hecke algebras, *Invent. Math.* **106**(2) (1991), 411-432.

[5] M. DEMAZURE, Une formule des caractères, *Bull. Sc. Math.* **98** (1974), 163–172.

[6] R. DIPPER, AND S. DONKIN, Quantum GL_n, *Proc. London Math. Soc.* **63** (1991), 165–211.

[7] G. DUCHAMP, D. KROB, A. LASCOUX, B. LECLERC, T. SCHARF, AND J.-Y. THIBON, Euler-Poincaré characteristic and polynomial representations of Iwahori-Hecke algebras, *Publ. RIMS, Kyoto Univ.* **31** (1995), 179–201.

[8] G. DUCHAMP, D. KROB, B. LECLERC, AND J.-Y. THIBON, Fonctions quasi-symétriques, fonctions symétriques non commutatives et algèbres de Hecke à $q = 0$, *C.R. Acad. Sci. Paris* **322** (1996), 107–112.

[9] H. O. FOULKES, A survey of some combinatorial aspects of symmetric functions, *in* "Permutations", Gauthier-Villars, Paris.

[10] A. M. GARSIA, M. HAIMAN A graded representation model for Macdonald's polynomials Proc. Nat. Acad. Sci. **90** (1993), 3607–3610.

[11] I. M. GELFAND, D. KROB, B. LECLERC, A. LASCOUX, V. S. RETAKH, AND J.-Y. THIBON, Noncommutative symmetric functions, *Adv. in Math.* **112** (1995), 218–348.

[12] I. GESSEL, Multipartite P-partitions and inner products of skew Schur functions, [in "Combinatorics and algebra", C. Greene, Ed.], *Contemporary Mathematics* **34** (1984), 289–301.

[13] J. A. GREEN, The characters of the finite general linear groups, *Trans. Am. Math. Soc.* **80** (1955), 402–449.

[14] P. HALL, The algebra of partitions *in* "combinatorics" P .A. MACMAHON, M.I.T. press (Cambridge), 70–84.

[15] F. HIVERT, Affine Hecke algebra and quasi-symmetric functions, preprint (1998).

[16] R. HOTTA, AND T. A. SPRINGER, A specialization theorem for certain Weyl group representations and application to the Green polynomials of unitary groups, *Inv. Math.* **41** (1977), 113–140.

[17] K. JOHNSEN, On a forgotten note by Ernst Steinitz on the theory of Abelian groups, *Bull. London Math. Soc.* **14** (1982), 353–355.

[18] A. N. KIRILLOV, AND N. YU. RESHETIKHIN, Bethe ansatz and the combinatorics of Young tableaux, *J. Sov. Math.* **41** (1988), 925–955.

[19] A. A. Klyachko, Lie element in the tensor algebra, *Siberian Math. J.* **15** (1974), 1296–1304.

[20] D. Krob, and J.-Y. Thibon, Noncommutative symmetric functions IV: Quantum linear groups and Hecke algebras at $q = 0$, *J. Alg. Comb.* **6** (1997), 339–376.

[21] D. Krob, and J.-Y. Thibon, A Crystalizable version of $U_q(\mathcal{G}l_n)$, *Proc. 9ème conf. "Formal power series and algebraic combinatorics", Krattenthaler C. Ed.*, Vienne, (1997).

[22] D. Krob, and J.-Y. Thibon, Noncommutative symmetric functions V: A degenerate version of $U_q(gl_N)$, preprint, (1997).

[23] A. Lascoux, Cyclic permutations on words, tableaux and harmonic polynomials, *Proc. Conf. on Algebraic Groups (Hyderabad, 1989)*, Manoj Prakashar, Madras (1991), 323-347

[24] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Fonctions de Hall-Littlewood et polynômes de Kostka-Foulkes aux racines de l'unité, *C.R. Acad. Sci. Paris* **316** (1993), 1–6.

[25] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Green polynomials and Hall-Littlewood functions at roots of unity, *Europ. J. Combinatorics* **15** (1994), 173–180.

[26] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Polynômes de Kostka-Foulkes et graphes cristallins des groupes quantiques de type A_n, *C.R. Acad. Sci. Paris* **320** (1995), 131–134.

[27] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties, *J. Math. Phys.* **38** (1997), 1041–1068.

[28] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, *Commun. Math. Phys.* **181** (1997), 205–263.

[29] A. Lascoux, and M. P. Schützenberger, Le Monoïde plaxique, *Quad. del. Ric. Sci.* **109** (1981), 129–156.

[30] A. Lascoux, and M. P. Schützenberger, Symmetry and flag manifold *Invariant Theory Springer L.N.* **996** (1983), P. 118-144.

[31] A. Lascoux, and M. P. Schützenberger, Key and standard bases *in* "Invariant Theory and Tableaux", IMA vol in Maths. and Appl. **19** (1990), 125–144

[32] A. Lascoux, and M. P. Schützenberger, Symmetrization operators on polynomial rings, *Funct. Anal. Appl.* **21** (1987), 77–78.

[33] B. Leclerc, and J.-Y. Thibon, Canonical bases of q-deformed Fock spaces, *Int. Math. Res. Notices* **9** (1996), 447–456.

[34] P. Littelman Crystal Graphs and Young tableaux *J. of Algebra* (to appear).

[35] D. E. Littlewood, On certain symmetric functions, *Proc. London Math. Soc.* **43** (1961), 485–498.

[36] G. Lusztig, Green polynomials and singularities of unipotent classes, *Advances in Math.* **42** (1981), 169–257.

[37] G. Lusztig, Equivariant K-theory and representations of Hecke algebras, *Proc Amer. Math. Soc.* **94** (1985), 337–342.

[38] I. G. Macdonald, Symmetric functions and Hall polynomials, *Clarendon Press*, (1995).

[39] I. G. Macdonald, Note on Schubert Polynomials, *Publications du LACIM*, Montreal, (1991).

[40] P. A. MacMahon, Combinatorial analysis, *Cambridge University Press*, (1915) Chelsea reprint 1960.

[41] C. Malvenuto, and C. Reutenauer, Duality between quasi-symmetric functions and Solomon descent algebra, *J. Algebra* **177** (1995), 967–982.

[42] A. Nakayashiki, and Y. Yamada Kotska polynomials and energy functions in solvable lattice models, *Selecta Mathematica*, **3** (1997), 547–599.

[43] J.-C. Novelli On the hypoplactic monoid *Proc. 9ème conf. "Formal power series and algebraic combinatorics", C. Krattenthaler Ed., Vienne*, (1997).

[44] I. Schur, Über die Darstellung der symmetrischen und der alternieren den Gruppe durch gebrochene lineare Substitutionen, *Crelle's Journal* **139** (1911), 155–250 (Ges. Abhandlugen, **1**, 346–441).

[45] E. Steinberg, A geometric approach to the representations of the full linear group over a Galois field, *Trans. Amer. Math. Sac.* **71** (1951), 274–282.

[46] E. Steinitz, Zur Theorie der Abel'schen Gruppen, *Jarhesbericht der DMV* **9** (1901), 80–85.

[47] M. Takeuchi, A two-parameter quantization of $GL(n)$, *Proc. Japan Acad.* **66** Ser. A (1990), 112–114 .

[48] B. C. V. Ung NCSF, a maple package for noncommutative symmetric functions, *Maple Technical Newsletter* **3-3** (1996), 24-29.

[49] B. C. V. Ung and S. Veigneau, ACE une environnement En combinatoire Algébrique, Proc. of the 7-th Conf. Formal Power Series And Algebraic Combinatorics, (1995).

[50] C. N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, *Phys. Rev. Lett.* **19** (1967), 1312–1325.