Quasi-symmetric functions and Hecke algebras actions

Florent Hivert*

Abstract

We define new actions of the symmetric group and the degenerate Hecke algebra on polynomials for which quasi-symmetric functions are the invariants. We give an interpretation of these actions in terms of the representation theory of a degenerate quantum group. The extension of these actions to the generic Hecke algebra allows us to define quasi-symmetric and non-commutative analogs of Hall-Littlewood functions.

Résumé

Nous définissons deux nouvelles actions du groupe symétrique et de l'algèbre de Hecke dégénérée sur les polynômes pour lesquelles les fonctions quasi-symétriques sont les invariants. Nous donnons une interprétation de ces actions en termes de théorie des représentations d'un groupe quantique dégénéré. L'extension de ces actions à l'algèbre de Hecke générique nous permet de définir des analogues quasi-symétriques et non commutatifs des fonctions de Hall-Littlewood.

1 Introduction

Recently, two generalizations of symmetric functions have been introduced: noncommutative symmetric functions **Sym** [8] and quasi-symmetric functions QSym [9]. These two Hopf algebras are dual to each other [18]. They appear as character rings of various quantized algebras at $q = 0$: Hecke algebras, the quantum group of Dipper and Donkin [3, 6], and a degenerate version of $U_q(gl_N)$ called $\mathcal{U}_0(gl_N)$ [7].

In this paper, we define actions of the symmetric group and of the degenerate Hecke algebra on polynomials for which the quasi-symmetric functions are the invariants. We interpret these operators as Weyl and Demazure operators for the algebra $\mathcal{U}_0(gl_N)$. Extending then this action to the generic Hecke algebra, we get a q-analog of the Weyl symmetrizer. This allows us to define quasi-symmetric analogs of Hall-Littlewood functions, and by duality noncommutative ones. We describe then their expansion in the natural basis of **Sym** and QSym corresponding to simple and projective indecomposable modules for $H_N(0)$ together with some other properties which generalizes the classical ones.

The classical Hall-Littlewood functions arise in many contexts, such as the calculation of the character table of finite linear groups, the description of graded representations of the symmetric group in the cohomology of unipotent varieties,

*`hivert@univ-mlv.fr` Institut Gaspard Monge – Université de Marne-la-Vallée – 2, Rue de la Butte-Verte – 93166 Noisy-le-Grand Cedex

or in the representation theory of affine Lie algebras. It would be of interest to find similar interpretations for the quasi-symmetric and the noncommutative ones.

The author wishes to thank J.-Y. Thibon without the one this work were not possible, and D. Krob, A. Lascoux and J.-C. Novelli for their helpful comments on this work.

1.1 Background

A *composition* K *(resp. a pseudo-composition)* of n is a p-tuple of positive (*resp.* non-negative) integers whose sum is n. These integers are called the parts of the composition, p is called the length of K and is denoted by $\ell(K)$. A non-increasing composition is called a *partition*. Let $I = (i_1, \dots, i_q)$ and $J = (j_1, \dots, j_p)$ be two compositions. By (I, J) we mean the concatenation of the two compositions defined by $(I, J) = (i_1, \dots, i_q, j_1, \dots, j_p)$. We denote by $I \triangleright J$ the composition $(i_1, \dots, i_q + j_1, \dots, j_p)$. Subsets of $\{1, \dots, n-1\}$ are in one-to-one correspondence with compositions of n:

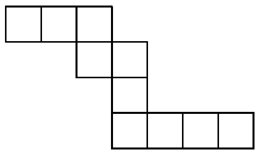
$$S = \{i_1 < i_2 < \cdots < i_p\} \longmapsto \mathrm{C}(S) = (i_1, i_2 - i_1, i_3 - i_2, \dots, n - i_p).$$

The inverse bijection (descent set of a composition) is given by:

$$K = (k_1, \dots, k_p) \longmapsto \mathrm{Des}(K) = \{k_1 + \cdots + k_j \ , \ j = 1 \dots p-1\}.$$

For instance, the composition $(3,1,2)$ of 6 corresponds to the subset $\{3,4\}$ of $\{1,2,3,4,5\}$.

A composition can be represented by a skew Young diagram called a *ribbon diagram* of shape I (see [17]). For example, the ribbon diagram of $I = (3,2,1,4)$ is



The *conjugate* composition $I^\sim$ of I is obtained by reading from left to right the heights of the columns of the ribbon diagram of I. On their descent set, the conjugate is the complement in $\{1 \dots n\}$. For example, the compositions $(3,2,1,4)^\sim = (1,1,2,3,1,1,1)$ correspond to descent sets $\{3,5,6\}$ and $\{1,2,4,7,8,9\}$.

Let I and J be two compositions of the same number n. We say that I is *finer* than J iff $\mathrm{Des}(I) \supseteq \mathrm{Des}(J)$. We will denote this by $I \succeq J$. This can be read on compositions in the following way: Let $J = (j_1, \dots, j_p)$. The composition I is finer than J iff there exist compositions I_1 of j_1 , I_2 of j_2, ... , I_p of j_p such that $I = (I_1, I_2, \dots, I_p)$ is the composition obtained by gluing $I_1, \dots I_p$ one after another. In this case we call the composition $\#(I,J) = (\ell(I_1), \dots, \ell(I_p))$ the *refining composition*.

Finally let $K = (k_1, \dots, k_m)$ be a composition. The *major index* of MacMahon [17] is defined: $\mathrm{Maj}(K) = \sum_{i \in \mathrm{Des}(K)} i$.

Example 1 Let $I = (2,2,1,2,1,1,1,2,1,1)$ and $J = (2,3,5,2,1,1)$. Then we can write $J = (2, 2+1, 2+1+1+1, 2, 1, 1)$, and so I is finer than J. Then $\#(I,J) = (1,2,4,1,1,1)$ and $\mathrm{Maj}(J) = 2 + 5 + 10 + 12 + 13 = 42$.

Let $(A,<)$ be a totally ordered alphabet. A *quasi-tableau* of ribbon shape I is an object obtained by filling a ribbon diagram r of shape I by letters of A in such a way that each row of r is non-decreasing from left to right and each column of r is strictly increasing from *top to bottom*. A word is said to be a *quasi-ribbon word* of shape I if it can be obtained by reading from *bottom to top* and from left to right the columns of a quasi-tableau of shape I. For example, the word $u = aacbabbac$ is not a quasi-ribbon word since the planar representation of u obtained by writing its decreasing factors as columns is not a quasi-tableau. On the other hand, the word $v = aacbacdcd$ is a quasi-ribbon word of shape $(3,1,3,2)$.

$$\begin{array}{|c|c|c|ccc} \hline a & a & a & & & \\ \hline & & b & & & \\ \cline{3-5} & & \mathbf{c} & \mathbf{b} & \mathbf{a} & \\ \cline{3-6} & & & & b & c \\ \cline{5-6} \end{array} \qquad \begin{array}{|c|c|c|ccc} \hline a & a & a & & & \\ \hline & & b & & & \\ \cline{3-5} & & c & c & c & \\ \cline{3-6} & & & & d & d \\ \cline{5-6} \end{array}$$

The algebra of *noncommutative symmetric functions* [8] is the free associative algebra $\mathbf{Sym} = \mathbb{C}\langle S_1, S_2, \dots \rangle$ generated by an infinite sequence of noncommutative indeterminates S_k, called *complete* symmetric functions. For a composition $I = (i_1, i_2, \dots, i_r)$, one sets $S^I = S_{i_1} S_{i_2} \dots S_{i_r}$. The family (S^I) is a linear basis of $\mathbf{Sym}$. A useful realization can be obtained by taking an infinite alphabet $A = \{ a_1, a_2, \dots \}$ and defining its complete homogeneous symmetric functions by the generating function

$$\sum_{n \geq 0} t^n S_n(A) = (1 - t\,a_1)^{-1} (1 - t\,a_2)^{-1} (1 - t\,a_3)^{-1} \dots \tag{1}$$

The noncommutative *ribbon Schur functions* R_I can be defined by

$$R_I = \sum_{J \preceq I} (-1)^{\ell(I) - \ell(J)} S^J \,. \tag{2}$$

The R_I form a basis of $\mathbf{Sym}$. In the realization of $\mathbf{Sym}$ given by equation (1), R_I reduces to the sum of all words of shape I [8].

The algebra of noncommutative symmetric functions is in natural duality with the algebra of quasi-symmetric functions introduced by Gessel in [9] (*cf.* [8, 18]).

Let $X = \{x_1 < x_2 < \cdots < x_n\}$ be a totally ordered set of commutative indeterminates. We denote $\mathcal{P}(X)$ (resp. $\mathcal{P}_k(X)$) the set of the subsets (resp. k-elements subsets) of X. Let m be the monomial $x_1^{k_1} \dots x_n^{k_n}$ where the k_i are possibly zero. For readability, we identify $m = X^K$ with the pseudo-composition denoted by $K = [k_1, k_2, \dots, k_n]$. We define the *support* of m as the subset $A \subset X$ of the x_i whose exponent is non-zero and the composition I of non-zero exponents written in the order of the variables. In the sequel we write A^I in place of the monomial m. For example if $X = \{x_1 < x_2 < x_3 < x_4\}$, we write $x_1^2 x_3 = [2,0,1,0] = \{x_1, x_3\}^{(2,1)}$ and $x_1^3 x_2^5 x_4 = [3,5,0,1] = \{x_1, x_2, x_4\}^{(3,5,1)}$.

An polynomial $f \in \mathbb{C}[X]$ is said to be *quasi-symmetric* iff for each composition $I = (i_1, \dots, i_r)$ the coefficient of the monomials A^I are independent of the set of variables $A \in \mathcal{P}_r(X)$. The quasi-symmetric polynomials form a subalgebra of $\mathbb{C}[X]$ denoted by QSym.

It is obvious to see that the family of *quasi-monomial functions* defined by

$$M_I = \sum_{A \in \mathcal{P}_r(X)} A^I = \sum_{j_1 < \cdots < j_r} x_{j_1}^{i_1} \ldots x_{j_r}^{i_r} \tag{3}$$

labeled by compositions $I = (i_1 \ldots , i_r)$ form a basis of QSym. For example $M_{(2,1)} = \{x_1, x_2\}^{(2,1)} + \{x_1, x_3\}^{(2,1)} + \{x_1, x_4\}^{(2,1)} + \{x_2, x_3\}^{(2,1)} + \{x_2, x_4\}^{(2,1)} + \{x_3, x_4\}^{(2,1)}$ and we write $M_{(2,1)} = [2,1,0,0] + [2,0,1,0] + [2,0,0,1] + [0,2,1,0] + [0,2,0,1] + [0,0,2,1]$ in side of $M_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_1^2 x_4 + x_2^2 x_3 + x_2^2 x_4 + x_3^2 x_4$

Another important basis of $Qsym$ is given by the *quasi-ribbon functions*

$$F_I = \sum_{I \preceq J} M_J \ , \tag{4}$$

e.g., $F_{122} = M_{122} + M_{1112} + M_{1211} + M_{11111}$. It is important to note that F_I is the commutative image of the sum of all quasi-ribbon words of shape I. The pairing $\langle \cdot, \cdot \rangle$ between QSym and **Sym** is defined by $\langle M_I, S^J \rangle = \delta_{IJ}$ or equivalently $\langle F_I, R_J \rangle = \delta_{IJ}$ (*cf.* [18, 8]). This duality can be interpreted as the canonical duality between the Grothendieck groups respectively associated with finite dimensional and projective modules over 0-Hecke algebras [5, 6].

2 Quasi-symmetrizing actions

The aim of this section is to show that there are actions of the symmetric group and of the degenerate Hecke Algebra on polynomials whose invariants are the quasi-symmetric functions. A representation-theoretical interpretation of these actions is provided by the 0-Hecke algebra and a degenerate quantum group studied by Krob and Thibon in [7]. The following construction provides Weyl and Demazure character formulas for this quantum group.

The set of permutations of the alphabet $X = \{x_1, \ldots , x_n\}$ will be identified with the symmetric group $\mathfrak{S}_n$. For $i = 1, 2, \ldots , n-1$, let σ_i denote the transposition that interchanges x_i and x_{i+1}, and fixes all other elements. We denote the usual action of $\mathfrak{S}_n$ on $\mathbb{Z}[X]$ by $\sigma \cdot m$.

Let q be a formal or complex parameter. The Hecke algebra $H_n(q)$ of type A_{n-1} is the algebra generated by the $(T_i)_{i=1,\ldots,n-1}$ with the relations:

$$\begin{aligned} T_i^2 &= (q-1)\ T_i + q && \text{for } 1 \le i \le n-1, \\ T_i\ T_j &= T_j\ T_i && \text{for } |i-j| > 1, \\ T_i\ T_{i+1}\ T_i &= T_{i+1}\ T_i\ T_{i+1} && \text{for } 1 \le i \le n-2. \end{aligned} \tag{5}$$

For generic q (different from 0 or a non-trivial root of unity) $H_n(q)$ is isomorphic to $C[\mathfrak{S}_n]$.

Let $\sigma = \sigma_{i_1} \cdots \sigma_{i_p}$ be a reduced word (*i.e.* a minimal length decomposition). The defining relations of $H_n(q)$ ensure that the element $T_\sigma = T_{i_1} \cdots T_{i_p}$ is independent of the reduced word for σ. The family $(T_\sigma)_{\sigma \in \mathfrak{S}_n}$ is a basis of the Hecke algebra.

Definition 1 *Let $0 < i < n$. And $m = x_1^{k_1} \dots x_n^{k_n} = [k_1, \dots, k_n]$ a monomial. the operator $\sigma_i \odot$ act on m by*

$$\sigma_i \odot [k_1, \dots, k_i, k_{i+1}, \dots, k_n] = \begin{cases} [k_1, \dots, k_{i+1}, k_i, \dots, k_n] & \text{if } k_i = 0 \text{ or } k_{i+1} = 0 \\ [k_1, \dots, k_i, k_{i+1}, \dots, k_n] & \text{if } k_i \neq 0 \text{ and } k_{i+1} \neq 0 \end{cases} \tag{6}$$

This define an action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{Z}[X]$ called the quasi-symmetrizing action.

Proof — It is easy to see that $\sigma \odot A^I = \{\sigma(x) \mid x \in A\}^I$. □

Example 2 $\sigma_1 \odot x_1^6 x_2 = \sigma_1 \odot [6, 2, 0] = [6, 2, 0] = x_1^6 x_2$ and $\sigma_1 \odot x_1^6 x_3^2 = \sigma_1 \odot [6, 0, 2] = [0, 6, 2] = x_2^6 x_3$. If σ is the permutation which exchanges 1 and 4 then $\sigma \odot x_1 x_3^2 = \sigma \odot [1, 0, 2, 0] = \sigma \odot \{x_1, x_3\}^{(1,2)} = \{x_3, x_4\}^{(1,2)} = [0, 0, 1, 2] = x_3 x_4^2$. This actions is different from the classical one, for example: $\sigma_1 \cdot x_1^2 x_2 = x_1 x_2^2$ instead of $\sigma_1 \odot x_1^2 x_2 = x_1^2 x_2$. It is important to see that the quasi-symmetrizing action is an action on the vector space of polynomials, with no relation with the algebra structure: $(\sigma_1 \odot x_1^2)(\sigma_1 \odot x_2) = x_1 x_2^2$ whereas $(\sigma_1 \odot x_1^2 x_2) = x_1^2 x_2$.

Proposition 1 *A polynomial f is quasi-symmetric iff $\sigma \odot f = f$ for all permutations σ.*

In [13], Lascoux and Schützenberger showed that there exist several families of operators acting on $\mathbb{K}[X]$ which satisfy the braids and Hecke relations. In particular the so-called isobaric divided differences, given by

$$\pi_i . f = \frac{x_i f - x_{i+1} \, \sigma_i . f}{x_i - x_{i+1}}$$

give an action of the degenerate Hecke algebra $H_n(0)$ (see, *e.g.,* [16, 2, 1]).

In the sequel, unless explicitly stated, we always use the quasi-symmetrizing action. We write σf instead of $\sigma \odot f$.

Definition 2 *Let f be a polynomial and $i < n$. The quasi-symmetrizing isobaric divided differences are defined by*

$$\pi_i \, f = \frac{x_i f - x_{i+1} \, \sigma_i \, f}{x_i - x_{i+1}} \quad \text{and} \quad \overline{\pi}_i = \pi_i - Id. \tag{7}$$

Proposition 2 *The action of π and $\overline{\pi}$ is given by*

$$\pi_i \, [k_1, \dots, k_i, k_{i+1}, \dots, k_n] = \begin{cases} \displaystyle\sum_{u=0}^{k_i} [k_1, \dots, k_i - u, u, \dots, k_n] & \text{if } k_i \neq 0 \text{ and } k_{i+1} = 0 \\ [k_1, \dots, k_n] & \text{if } k_i \neq 0 \text{ and } k_{i+1} \neq 0 \\ - \displaystyle\sum_{u=1}^{k_{i+1}-1} [k_1, \dots, u, k_{i+1} - u, \dots, k_n] & \text{if } k_i = 0 \text{ and } k_{i+1} > 1 \\ [k_1, \dots, k_n] & \text{if } k_i = 0 \text{ and } k_{i+1} = 0 \\ 0 & \text{if } k_i = 0 \text{ and } k_{i+1} = 1 \end{cases} \tag{8}$$

$$\overline{\pi}_i\,[k_1,\ldots,k_i,k_{i+1},\ldots,k_n] =$$

$$\begin{cases} \displaystyle\sum_{u=1}^{k_i} [k_1,\ldots,u,k_i-u,\ldots,k_n] & \textit{if } k_i\neq 0 \textit{ and } k_{i+1}=0 \\ \displaystyle -\sum_{u=1}^{k_{i+1}} [k_1,\ldots,u,k_{i+1}-u,\ldots,k_n] & \textit{if } k_i=0 \textit{ and } k_{i+1}\neq 0 \\ 0 & \textit{if } k_i=0 \textit{ and } k_{i+1}=0 \\ 0 & \textit{if } k_i\neq 0 \textit{ and } k_{i+1}\neq 0 \end{cases} \tag{9}$$

Proof — These formulas are obvious consequences of the identity:

$$x^n - y^n = (x-y)\sum_{u+v=n-1} x^u y^v.$$

For example : $\pi_1\,[1,2,3]=[1,2,3]$ and $\pi_2\,[1,0,3] = -\,[1,2,1]-[1,1,2]$. Using this we derive the following:

Theorem 1 *The π_i and $\overline{\pi}_i$ operators satisfy the braid relations, together with $\pi_i^2=\pi_i$ and $\overline{\pi}_i^2=-\overline{\pi}_i$.*

Let $\sigma=\sigma_{i_1},\ldots,\sigma_{i_p}$ be a reduced word. This ensures that the operators $\pi_\sigma=\pi_{i_1},\ldots,\pi_{i_p}$ and $\overline{\pi}_\sigma=\overline{\pi}_{i_1},\ldots,\overline{\pi}_{i_p}$ are independent of the reduced word for σ. For convenience, we state that $\pi_e = Id$ where e is the indentity of the symmetric group.

Corollary 1 *The mapping $T_\sigma\mapsto\overline{\pi}_\sigma$ defines an action of the Hecke algebra at $q=0$.*

Let ω be the maximal permutation $n,n-1,\ldots,1$. The operator $\pi_\omega = \sum_{\sigma\in\mathfrak{S}_n}\overline{\pi}_\sigma$ is called the maximal symmetrizer.

Proposition 3 *A polynomial f is quasi-symmetric iff $\pi_\omega f = f$.*

3 Weyl and Demazure formulas for $\mathcal{U}_0(gl_N)$

A representation-theoretical explanation of the previous construction is provided by the degenerate quantum group $\mathcal{U}_0(gl_N)$ studied by Krob and Thibon in [7]. It is the specialization $q=0$ of a non-standard analogue of the universal enveloping algebra of gl_N. They show that the characters of the irreducible polynomial modules over this algebra are the quasi-symmetric functions F_I.

The algebra $\mathcal{U}_0(gl_N)$ is generated by three kinds of elements called Chevalley generators: the raising $(e_i)_{1\le i\le N-1}$, the lowering $(f_i)_{1\le i\le N-1}$ and the diagonal $(k_i)_{1\le i\le N}$ ones. We denote by $\mathcal{U}_0(b_+)$ the subalgebra generated by e_i and k_i.

The irreducible polynomial modules of degree n are indexed by compositions K of n. The basis of the module $\boldsymbol{D}_K$ is indexed by the quasi-ribbon words of shape K over the alphabet $\{1,\ldots,N\}$. The evaluation of a word w (*i.e.* the pseudo-composition whose i-th part is the number of i's in w) is called the weight of the corresponding vector. We identify the weight of a word with its commutative images. The action of the Weyl group $\mathfrak{S}_N$ on weights is then the quasi-symmetrizing action.

Let us describe the action of the Chevalley generators on the quasi-ribbon basis of $\boldsymbol{D}_K$. Let w be a word, and i a integer. The diagonal generator k_i

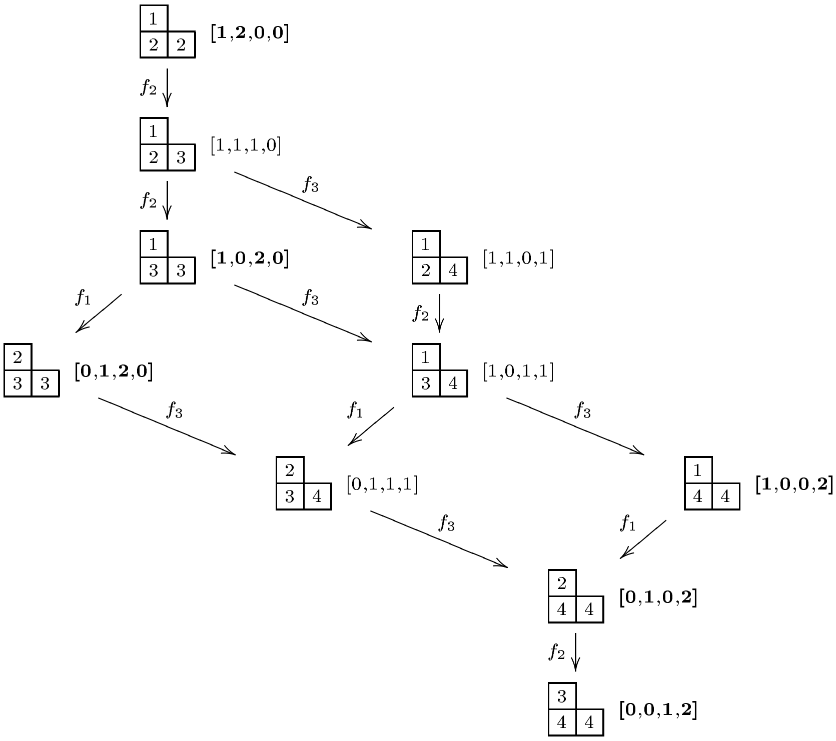


Figure 1: Quasi-crystal graph of the module $\boldsymbol{D}_{(1,2)}$ for $\mathcal{U}_0(gl_4)$.

sends all the words w which contain the letter i to 0, keeping the other words unchanged. Let w^+ (*resp.* w^-) be the word obtained by replacing the last i by an $i+1$ (*resp.* the last $i+1$ by a i). If there is no such letter w^+ is not defined. The raising operator e_i sends w to w^- if w^- exists and is a quasi-ribbon word of shape K, otherwise it sends w to 0. The lowering operator f_i send w to w^+ if w^+ exists and is a quasi-ribbon word of shape K, otherwise it sends w to 0. Figure (1) shows the structure of a $\mathcal{U}_0(gl_N)$-module. We call this graph a quasi-crystal graph. For simplification we only show the action of the f_i. The action of e_i reverses that of the f_i.

Example 3 In the module $\boldsymbol{D}_{(1,2)}$ for $\mathcal{U}_0(gl_4)$ the vector 212 is sent to 0 by f_1 because 222 is not a quasi-ribbon word of shape $(1,2)$. On the other hand f_2 sends it to 213.

Krob and Thibon showed that the character of this module $\boldsymbol{D}_K$ is the quasi ribbon function F_K. Recall that in the classical case (*i.e.* gl_N) the character is the Schur function s_λ which can been obtained by the action of the Weyl-Demazure symmetrizer : $s_\lambda = \pi_\omega x^\lambda$. So it is natural to ask whether, in the degenerate case, there is a symmetrization formula which gives the character. The answer is positive :

Theorem 2 *Let $K = (k_1, \ldots, k_p)$ be a composition. The character of the irreducible $\mathcal{U}_0(gl_N)$-module $\boldsymbol{D}_K$ is given by*

$$F_K = \pi_\omega X^K \tag{10}$$

where $X^K = x_1^{k_1} \cdots x_p^{k_p} = [k_1, \ldots, k_p, 0, \ldots, 0]$.

The second natural question is the following. Is the Demazure-Weyl symmetrizer refinable into Demazure partial symmetrizers as in [2] ? The answer is again positive. The *extremal weight vectors* are, by definition, the vectors of weight σX^K, the action of the Weyl group being the quasi-symmetrizing action. They appear in bold-type on the figure 1. The operators π_σ, called the Demazure operators, allow to compute the character of the Demazure modules; that is, the $\mathcal{U}_0(b_+)$-modules generated by the extremal weight vectors.

Theorem 3 *Let K be a composition and σ a permutation. The Demazure module $\mathcal{U}_0(b_+)v$ generated by the unique vector $v \in D_K$ of extremal weight $\sigma(X^K)$ is the space generated by the vectors of weight less or equal than $\sigma(X^K)$. Its character (generating function of the weights), is given by*

$$\chi\left(\mathcal{U}_0(b_+)v\right) = \pi_\sigma(X^K). \tag{11}$$

Example 4 For the algebra $\mathcal{U}_0(gl_4)$ the module $D_{(1,2)}$ is of dimension 10. Its basis is indexed by the words: 212, 213, 313, 214, 323, 314, 324, 414, 424, 434, of respective weights: $[1,2,0,0]$, $[1,1,1,0]$, $[1,0,2,0]$, $[1,1,0,1]$, $[0,1,2,0]$, $[1,0,1,1]$, $[0,1,1,1]$, $[1,0,0,2]$, $[0,1,0,2]$, $[0,0,1,2]$.
The extremal weights are the following:
$[1,2,0,0]$, $[1,0,2,0]$, $[0,1,2,0]$, $[1,0,0,2]$, $[0,1,0,2]$, $[0,0,1,2]$.
Fix $\sigma = (1423)$. The vector v of weight $[1,0,0,2] = \sigma \odot [1,2,0,0]$ generate a Demazure module of dimension 6 whose character is given by:
$\chi\left(\mathcal{U}_0(b_+)v\right) = \pi_\sigma[1,2,0,0] = [1,2,0,0] + [1,1,1,0] + [1,0,2,0] + [1,1,0,1] + [1,0,1,1] + [1,0,0,2]$.

4 Generic Hecke algebra and Hall-Littlewood functions

We first recall some facts of the classical theory. Our notations will be essentially those of [15], to which the reader is referred for more details.

Let $\Delta_n(q) = \prod_{i<j\le n}(qx_i - x_j)$. Then, on the one hand, the Hall-Littlewood polynomial $Q_\lambda(x_1, \ldots, x_n; q)$ indexed by a partition λ of length $\le n$ is defined by [14]

$$Q_\lambda = \frac{(1-q)^{\ell(\lambda)}}{[m_0]_q!} \sum_{\sigma \in \mathfrak{S}_n} \sigma\left(x^\lambda \frac{\Delta_n(q)}{\Delta_n(1)}\right) \tag{12}$$

where $m_0 = n - \ell(\lambda)$ and the q-integers are here defined by $[n]_q = (1-q^n)/(1-q)$.

On the other hand, $H_n(q)$ acts on $\mathbb{C}[X]$ by the formula $T_i = (q-1)\,\pi_i + \sigma_i$ and it is shown in [4] that if one defines the q-symmetrizing operator $\Box_\omega \in H_n(q)$

by $\square_\omega = \sum_{\sigma\in\mathfrak{S}_n} T_\sigma$, then

$$Q_\lambda(x_1,\ldots,x_n;q^{-1}) = q^{-\binom{N}{2}} \frac{(1-q^{-1})^{\ell(\lambda)}}{[m_0]_{q^{-1}!}} \square_\omega(x^\lambda) \ . \tag{13}$$

The normalization factor $1/[m_0]_q!$ is here to ensure stability with respect to the adjunction of variables, and if we denote by X the infinite set $X = \{x_1, x_2, \ldots, \}$ then $Q_\lambda(X;q) = \lim_{n\to\infty} Q_\lambda(x_1,\ldots,x_n;q)$.

We have the specializations: $Q_\lambda(X;0)$ is equal to the Schur function s_λ and $Q_\lambda(X;1)$ is equal to the monomial function m_λ,

The P-functions are defined by

$$P_\lambda(X;q) = \frac{1}{(1-q)^{\ell(\lambda)}[m_1]_q!\cdots[m_n]_q!} Q_\lambda(X;q)$$

where m_i is the multiplicity of the part i in λ.

We consider these functions as elements of the algebra $\mathbf{sym} = \mathbf{sym}(X)$ of symmetric functions with coefficients in $\mathbb{C}(q)$. There is a scalar product $\langle\,,\rangle$ on **sym**, for which the Schur functions s_λ form an orthonormal basis. We denote by $(Q'_\mu(X;q))$ the adjoint basis of $P_\lambda(X;q)$ for this scalar product. It is easy to see that $Q'_\mu(X;q)$ is the image of $Q_\mu(X;q)$ by the ring homomorphism $p_k \mapsto (1-q^k)^{-1}p_k$ (in λ-ring notation, $Q'_\mu(X;q) = Q(X/(1-q);q)$). In the Schur basis,

$$Q'_\mu(X;q) = \sum_\lambda K_{\lambda\mu}(q) s_\lambda(X) \tag{14}$$

where the $K_{\lambda\mu}(q)$ are the Kotska-Foulkes polynomials. The polynomial $K_{\lambda\mu}(q)$ is the generating function of a statistic c called *charge* on the set $\mathrm{Tab}(\lambda,\mu)$ of Young tableaux of shape λ and weight μ [15].

Motivated by these remarks, we may seek a quasi-symmetrizing action of the generic Hecke Algebra.

Theorem 4 *The operators T_i defined by*

$$T_i = (1-q)\,\overline{\pi}_i + q\,\sigma_i \tag{15}$$

verify the Hecke relations.

In the classical case, the divided differences operators commute with the multiplication by symmetric polynomials. So it is sufficient to check these identities on a basis of the module of the polynomials over the symmetric ones. The Grothendieck and Schubert polynomials are helpful in this case [16]. In our case, since the quasi-symmetrizing action do not commute with the product, the ring $\mathbb{C}[X]$ considered as a $Qsym$-module is not free. So the proof is done by a direct calculation, checking the braid relation for all the monomials over three variables.

This make it possible to define a q-symmetrizing operator by the formulas (see *e.g.*, [4])

$$\square_\omega = \sum_{\sigma\in\mathfrak{S}_n} T_\sigma = \big(1 + T_{n-1} + T_{n-2}\,T_{n-1} + \ldots + T_1\cdots T_{n-1}\big) \sum_{\sigma\in\mathfrak{S}_{n-1}} T_\sigma \, .$$

Proposition 4 *The q-symmetrizing operator $\square_\omega$ considered as acting on $\mathbb{Z}[q]$ has for image the space of quasi symmetric functions.*

Moreover, if we take coefficients in $\mathbb{C}(q)$, the operator $1/[n]_q!\,\square_\omega$ is a projector with range the space of quasi-symmetric functions.

For $q = 0$, this is nothing but the Weyl-Demazure symmetrizer π_ω. The proof involves the Yang-Baxter factorization of $\square_\omega$ using the fact that there is a reduced word for ω ending with any elementary transposition.

Definition 3 *let $I = (i_1, \dots, i_p)$ be a composition and $X^I = x_1^{i_1} \dots x_p^{i_p}$. The* quasi-symmetric Hall-Littlewood function G_I *is defined by*

$$G_I(x_1, \dots, x_n; q) = \frac{1}{[p]_q!\ [n-p]_q!}\,\square_\omega(X^I). \tag{16}$$

As in the symmetric case the factor is for compatibility with adjunction of variables. The functions $G_I(X;q)$ form a basis of quasi-symmetric functions with coefficient in $\mathbb{Z}[q]$.

We have the following specializations: $G_I(X;0)$ is the quasi-ribbon function F_I and $G_I(X;1)$ is the quasi-monomial function M_I.

For instance $G_{(2,1)}(x_1,x_2,x_3;q) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + (1-q)x_1x_2x_3$. On this example, we verify that at $q=0$, one has $G_{(2,1)}(x_1,x_2,x_3;0) = F_{(2,1)}$ and at $q=1$, on has $G_{(2,1)}(x_1,x_2,x_3;1) = M_{(2,1)}$.

Theorem 5 *The expansion of G_I in the quasi-ribbon basis is given by*

$$G_I = \sum_{J \succeq I} (-1)^{\ell(J)-\ell(I)} q^{s(I,J)} F_J \tag{17}$$

where $s(I,J)$ is defined as follows. Let $(k_1, \dots, k_p)$ be the refining composition $\#(J,I)$. Then $s(I,J) = (k_1-1) + 2(k_2-1) + \cdots + p(k_p-1)$.

Example 5 Let us compute $G_{(1,1,2,1)}$ on $X = \{x_1,x_2,x_3,x_4,x_5\}$. Start with $m = x_1x_2x_3^2x_4 = [1,1,2,1,0]$. Since m is symmetric for σ_1, σ_2, and σ_3, the operators T_1, T_2, and T_3 multiply m by q. We can also deduce that $\square_{\omega'}\,m = [4]_q!m$ because $[n]_q!$ is the generating series for the permutations of $\mathfrak{S}_n$ counted by their length (ω' denote the maximal permutation for $\mathfrak{S}_{n-1}$).

$$\begin{aligned}
T_4(m) &= [1,1,2,0,1] \\
T_3\,T_4(m) &= [1,1,0,2,1] + (1-q)([1,1,1,1,1]) \\
T_2\,T_3\,T_4(m) &= [1,0,2,1,1] + q(1-q)([1,1,1,1,1]) \\
T_1\,T_2\,T_3\,T_4(m) &= [0,1,1,2,1] + q^2(1-q)([1,1,1,1,1]).
\end{aligned}$$

From the factorization $\square_\omega = (1 + T_4 + T_3\,T_4 + T_2\,T_3\,T_4 + T_1\,T_2\,T_3\,T_4)\,\square_{\omega'}$, we get

$$\begin{aligned}
&\square_\omega([1,1,2,1,0]) = [4]_q!\big(\\
&\quad [1,1,2,1,0] + [1,1,2,0,1] + [1,1,0,2,1] + [1,0,1,2,1] + [0,1,1,2,1] \\
&\qquad\qquad\qquad\qquad\qquad\qquad (1-q)(1+q+q^2)\,[1,1,1,1,1]\big).
\end{aligned}$$

Using the basis of QSym

$$\begin{aligned}\Box_\omega([1,1,2,1,0]) &= [4]_q!(M_{(1,1,2,1)} + (1-q^3)M_{(1,1,1,1,1)}) \\ &= [4]_q!(F_{(1,1,2,1)} - q^3 F_{(1,1,1,1,1)}).\end{aligned}$$

This proves that $G_{(1,1,2,1)} = F_{(1,1,2,1)} - q^3 F_{(1,1,1,1,1)}$.

Similarly, one would find

$$\begin{aligned}G_{(3,2)} = F_{(3,2)} - q\,F_{(2,1,2)} - q\,F_{(1,2,2)} + q^2\,F_{(1,1,1,2)} - q^2\,F_{(3,1,1)} +{} \\ q^3\,F_{(2,1,1,1)} + q^3\,F_{(1,2,1,1)} - q^4\,F_{(1,1,1,1,1)}.\end{aligned}$$

The transition matrix is an upper unitriangular matrix (*i.e.* 1 on the diagonal, and $M_{I,J}$ is zero unless $I \succeq J$), corresponding to the inverse of the q-Kotska matrix. The analogue of the expression $s_\lambda = \sum K_{\lambda\mu}(q)P_\mu$ will be obtained by means of the dual basis, which lives in the space of noncommutative symmetric functions.

Indeed, we can now define the noncommutative Hall-Littlewood symmetric functions by duality. (G_I) is a basis of the algebra of quasi-symmetric functions with coefficients in $\mathbb{Z}[q]$. The dual basis is a basis of noncommutative symmetric functions. We denote this basis by H_I. The analogue of the expression $Q'_\mu = \sum K_{\lambda\mu}s_\lambda$ is the following formula.

Theorem 6 *The transition matrix whose rows are the H_J expanded in the R_I basis is a lower unitriangular matrix with positive coefficients. Moreover the expansion is given by:*

$$H_J(A;q) = \sum_{J \succeq I} q^{\mathrm{Maj}(\#(J,I)^\sim)} R_I. \tag{18}$$

We observe that the analogues of the Kotska-Foulkes polynomials reduce here to monomials. For instance $H_{(1^n)} = \sum_K q^{\mathrm{Maj}(K^\sim)} R_K$ where $K^\sim$ is the conjugate composition of K. Another example is: $H_{(3,2,1)} = R_{(3,2,1)} + q^2 R_{(5,1)} + qR_{(3,3)} + q^3 R_{(6)}$. As the $H_I(A,q)$ form a basis of **Sym**, we can express the product of two H_I.

Theorem 7 *Let I and J two compositions of lengths i and j. Then*

$$H_I H_J = \sum_{J \succeq K} q^{\mathrm{Maj}(\#(J,K)^\sim)} \left(c(j,k)H_{(I,K)} + c(j,k-1)H_{(I \triangleright K)}\right) \tag{19}$$

where k is the length of the composition K and $c(u,v) = q^{(v-u)}\frac{[u]!}{[v]!}$.

This is a consequence of the formula $R_I R_J = R_{(I,J)} + R_{(I \triangleright J)}$. For example

$$\begin{aligned}H_{(3,1,2)}H_{(1,2)} = &H_{(3,1,2,1,2)} + (1-q)(q+1)H_{(3,1,3,2)} \\ &+ q(1-q)(q+1)H_{(3,1,2,3)} + q(q-1)^2(q+1)H_{(3,1,5)}.\end{aligned}$$

We are now interested in the specializations of the Hall-Littlewood functions at roots of the unity. The noncommutative Hall-Littlewood functions have a factorization property similar the the one discovered by Leclerc, Lascoux and Thibon [10, 11].

Theorem 8 *Let k be a integer and ζ be a kth root of the unity. Suppose that $I = (i_1, \dots, i_p)$ is a composition. Write $p = ck + r$, we break the composition I in blocks of length k. There are c compositions $J_1, \dots, J_c$ of length k and a composition J_{c+1} of length r such that $I = (J_1, \dots, J_{c+1})$. Then the $H_I(A;\zeta)$ factorize in the following way:*

$$H_I(A;\zeta) = H_{J_1}(A;\zeta)H_{J_2}(A;\zeta)\cdots H_{J_{c+1}}(A;\zeta) \tag{20}$$

For instance, if ζ is a 3rd root of the unity, $H_{(3,2,4,1,5,3,2,1)}(A;\zeta) = H_{(3,2,4)}(A;\zeta)H_{(1,5,3)}(A;\zeta)H_{(2,1)}(A;\zeta)$.

References

[1] Berstein I.N., Gelfand I.M. and Gelfand S.I., 1973, *Schubert cells and the cohomology of the space G/P*, Russian Math. Surveys, **28** (1973), 1–26.

[2] Demazure M., *Une formule des caractères*, Bull. Sc. Math., **98** (1974), p. 163-172.

[3] Dipper R. and Donkin S., *Quantum GL_n*, Proc. London Math. Soc., **63** (1991), p. 165-211.

[4] Duchamp G., Krob D., Lascoux A., Leclerc B., Scharf T. and Thibon J.-Y., *Euler-Poincaré characteristic and polynomial representations of Iwahori-Hecke algebras*, Publ. RIMS, Kyoto Univ., **31** (1995), p. 179-201.

[5] Duchamp G., Krob D., Leclerc B. and Thibon J.-Y., *Fonctions quasi-symétriques, fonctions symétriques non commutatives et algèbres de Hecke à $q = 0$*, C.R. Acad. Sci. Paris, **322** (1996), 107–112.

[6] Krob D. and Thibon J.-Y., *Noncommutative symmetric functions IV: Quantum linear groups and Hecke algebras at $q = 0$*, J. Alg. Comb., **6** (1997), 339–376.

[7] Krob D. and Thibon J.-Y, *A Crystalizable version of $U_q(\mathcal{G}l_n)$*, Proc. 9ème conf. "Formal power series and algebraic combinatorics", Krattenthaler C. Ed., Vienne, (1997).

[8] Gelfand I.M., Krob D., Leclerc B., Lascoux A., Retakh V.S. and Thibon J.-Y., *Noncommutative symmetric functions*, Adv. in Math., **112**, 218–348, 1995.

[9] Gessel I., *Multipartite P-partitions and inner products of skew Schur functions*, [in "Combinatorics and algebra", C. Greene, Ed.], Contemporary Mathematics, **34**, 289–301, 1984.

[10] A. Lascoux, B. Leclerc and J.Y. Thibon, *Fonctions de Hall-Littlewood et polynômes de Kostka-Foulkes aux racines de l'unité*, C.R. Acad. Sci. Paris, **316** (1993), 1-6.

[11] A. Lascoux, B. Leclerc and J.Y. Thibon, *Green polynomials and Hall-Littlewood functions at roots of unity*, Europ. J. Combinatorics, **15** (1994), 173-180.

[12] A. Lascoux, B. Leclerc and J.Y. Thibon, *Polynômes de Kostka-Foulkes et graphes cristallins des groupes quantiques de type A_n*, C.R. Acad. Sci. Paris, **320** (1995), 131-134.

[13] A. Lascoux and M. P. Schützenberger, *Symmetrisation operators on polynomial rings*, Funct. Anal. Appl., **21** (1987), 77-78.

[14] D. E. Littlewood, *On certain symmetric functions*, Proc. London Math. Soc., **43** (1961), 485-498.

[15] Macdonald I.G., *Symmetric functions and Hall polynomials*, Clarendon Press, (1995).

[16] Macdonald I.G., *Note on Schubert Polynomials*, Publications du LACIM, Montreal, 1991.

[17] MacMahon P.-A., *Combinatorial analysis*, Cambridge University Press, (1915) Chelsea reprint 1960.

[18] Malvenuto C. and Reutenauer C., *Duality between quasi-symmetric functions and Solomon descent algebra*, J. Algebra, **177**, 967–982, 1995.