DEFORMATION OF SYMMETRIC FUNCTIONS AND THE RATIONAL STEENROD ALGEBRA

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ABSTRACT. In 1999, Reg Wood conjectured that the quotient of $\mathbb{Q}[x_1, \dots, x_n]$ by the action of the rational Steenrod algebra is a graded regular representation of the symmetric group $\mathfrak{S}_n$. As pointed out by Reg Wood, the analog of this statement is a well known result when the rational Steenrod algebra is replaced by the ring of symmetric functions; actually, much more is known about the structure of the quotient in this case.

We introduce a non-commutative q-deformation of the ring of symmetric functions, which specializes at $q = 1$ to the rational Steenrod algebra. We use this formalism to obtain some partial results. Finally, we describe several conjectures based on an extensive computer exploration. In particular, we extend Reg Wood's conjecture to q formal and to any $q \in \mathbb{C}$ not of the form $-a/b$, with $a \in \{1, \dots, n\}$ and $b \in \mathbb{N}$.

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1. Introduction

The *rational Steenrod algebra* is the subalgebra $\mathcal{A}$ of the Weyl algebra generated by the *rational Steenrod squares* $D_k := \sum_i x_i^{k+1}\partial_i$ [Woo97]. The Steenrod squares are derivations, and satisfy the Lie relations $[D_k, D_l] = (l-k)D_{k+l}$. In particular, the Steenrod algebra is generated by just D_1 and D_2.

Let $\mathbb{C}[X_n] := \mathbb{C}[x_1, \dots, x_n]$ be a ring of polynomials, and set $\pi := x_1 \dots x_n$. Consider the action of the Steenrod algebra on $\mathbb{C}[X_n]$ through the algebraic Thom map, so that for $f \in \mathcal{A}$ and $p \in \mathbb{C}[X_n]$,

$$\rho(f).p := \frac{1}{\pi} f\pi.p. \tag{1}$$

This action seems to share properties with the natural action of the ring of symmetric polynomials on polynomials by multiplication.

Conjecture 1 (Rational hit conjecture of Reg Wood [Woo98, Woo01])**.** The quotient $\mathbb{C}[X_n]_{/\mathcal{A}^+\mathbb{C}[X_n]}$ is a graded regular representation of the symmetric group $\mathfrak{S}_n$.

In this article, we present preliminary results from research in progress around this conjecture. We refer to [Woo97, Woo98, Woo01] for motivations.

It is actually easier to get rid of the algebraic Thom map, by conjugating once for all the Steenrod algebra by this map. So, we consider instead the isomorphic algebra denoted Steen generated by the Weyl operators

$$P_k := \frac{1}{\pi} D_k \pi = \sum_i x_i^k (1 + x_i \partial_i). \tag{2}$$

Note that those operators are no longer derivations.

Now, P_k appears clearly as a usual symmetric power-sum, with a deformation. Since the goal is to transfer properties of the ring of symmetric polynomials to the Steenrod algebra, this suggests that we interpolate continuously between the two. Hence, we introduce the *q-Steenrod algebra* denoted $\mathrm{Steen_q}$ which is generated by the Weyl operators

$$P_{q,k} := \sum_i x_i^k (1 + q x_i \partial_i). \tag{3}$$

The 0-Steenrod algebra is the ring of symmetric polynomials, while the 1-Steenrod algebra is the rational Steenrod algebra conjugated by the algebraic Thom map.

In the following, q can be either a given complex number q_0, or a formal parameter. In the later case we take $\mathbb{K} := \mathbb{C}(q)$ as base field, or $\mathbb{K} := \mathbb{C}[q]$, when we just need a base ring. Roughly speaking, all the statements we consider are of algebraic nature, and hold for q formal if, and only if, they hold for q generic (that is in most cases for all but a countable set).

1.1. Summary of the results.

In [GH94], a concrete realization Harm of the quotient $\mathbb{C}[X_n]_{/\,\mathrm{Sym}^+\mathbb{C}[X_n]}$ is constructed as a space of so-called *harmonics*. Following a similar approach, we construct a concrete realization $\mathrm{Harm_q}$ of the quotient $\mathbb{C}[X_n]_{/\,\mathrm{Steen_q}^+\mathbb{C}[X_n]}$.

Both our theoretical results, and our computer exploration leads us to extend Reg Wood's conjecture as follows:

Conjecture 2. Let q be formal, or a complex number not of the form $-a/b$, with $a \in \{1, \dots, n\}$, $b \in \mathbb{N}$ and $a \le b$. Then, for any n, $\mathrm{Harm_q}$ is isomorphic to Harm as a graded $\mathfrak{S}_n$-module. In particular, it is isomorphic to the regular representation of $\mathfrak{S}_n$.

The main results of this article are summarized in the following two propositions. Recall that a monomial $x^K = x_1^{k_1} \cdots x_n^{k_n}$ in $\mathbb{K}[X_n]$ is called *staircase monomial* if $k_i \leq n-i$, for all i.

Proposition 1. *The following are equivalent:*

(a) *Conjecture 2 holds for q formal;*
(b) *For any n, $\dim \mathrm{Harm_q} \geq \dim \mathrm{Harm}$;*
(c1) *For any n, and any polynomial $p \in \mathrm{Harm_q}$, all the variables have degree at most $n-1$ in p;*
(c2) *For any n, the staircase monomials are the leading terms of the polynomials in $\mathrm{Harm_q}$ for the lexicographic order.*
(d) *Any basis of $\mathbb{C}[X_n]_{/\,\mathrm{Sym}^+ \mathbb{C}[X_n]}$ is a basis of $\mathbb{K}[X_n]_{/\,\mathrm{Steen_q}^+ \mathbb{K}[X_n]}$.*

Proposition 2. *Assume that conjecture 2 holds for q generic. Then, for $q_0 \in \mathbb{C}$, the following are equivalent:*

(a) *Conjecture 2 holds for q_0;*
(b) *For any n, $\dim \mathrm{Harm_{q_0}} \leq \dim \mathrm{Harm}$;*
(c) *For any n, the non-staircase monomials are the leading terms of the polynomials in $\mathrm{Hit_q}$ for the lexicographic order.*

1.2. **Organization of the paper.** After a background section, we define the q-deformed Steenrod algebra $\mathrm{Steen_q}$ (Definition 5), together with q-hit and q-harmonic polynomials. We compute the dimension and the structure of $\mathrm{Steen_q}$ (Theorems 3.1 and 3.2).

In the section 4, we describe the relations between the generic setting (when q is a formal parameter), the complex setting (when q in a complex number) and the classical setting (when $q = 0$).

Then, we analyze the space of polynomials that can be hit from harmonics by successive application of at most n Steenrod operators. This provides information in small degrees. In particular we show that the submodules of degree smaller than n of $\mathbb{K}[X_n]/\,\mathrm{Sym}^+$ and $\mathbb{K}[X_n]/\,\mathrm{Steen_q}^+$ are isomorphic (Corollary 1).

In the next section, we investigate the relation between the q-harmonics in n variables and in $n+1$ variables. We construct a trick to go from n to $n+1$: the so-called *strings*. The main result of this part is that the conjecture is equivalent to the fact that the harmonics in n-variables are of degree smaller than n in each variable (Proposition 17). We also analyze completely the simple cases of 1 and 2 variables.

In the next-to-last section we describe some attempts to prove conjecture 2 by looking for q-analogs of elementary symmetric function (Subsection 7.1) and q-analogs of Schubert polynomials (Subsection 7.2), and we explain why they fail.

Finally, the last-section is devoted to a computer exploration which motivates our conjectures. In particular, we give, for small numbers of variables and small degrees, a tight superset of the values of q for which conjecture 2 does not hold. Most computations where realized with the open source library `MuPAD-Combinat` [HTa][1] for the Computer Algebra System `MuPAD` [The96]. The extra tools we developed at this occasion are available upon request, and will eventually be integrated into `MuPAD-Combinat`, after appropriate cleanup and documentation.

2. Preliminaries

The main topic of this paper is the study of the q-deformation of a module. Hence we will work in two different settings: either $q := q_0$ is a complex number, in which case the base field is $\mathbb{K} := \mathbb{C}$, or q is a formal complex number, and $\mathbb{K} := \mathbb{C}(q)$.

[1] `http://mupad-combinat.sf.net`

Many of the results are true in both settings; in this case, we use $\mathbb{K}$ as base field without precision.

For an alphabet X of commuting variables, we denote respectively by $\mathbb{K}[X]$, $\mathbb{K}(X)$, and $\mathrm{Sym}(X)$ the ring of polynomials, the field of fractions, and the ring of symmetric functions in the variables in X. We denote by X_n the alphabet of the n commuting variables $\{x_1, x_2, \dots, x_n\}$. Unless explicitly stated, $X := X_\infty$ is the alphabet in the countably many variables $\{x_1, \dots, x_n, \dots\}$, and we use the shorthand $\mathrm{Sym} = \mathrm{Sym}(X_\infty)$.

2.1. **Graded rings and Hilbert series.** In the following, most of the vector spaces we consider are graded, with finite dimensional homogeneous components, and comparing their dimensions will be one of our main tools. The *Hilbert series* of a graded vector space $A := \bigoplus_{d=0}^{\infty} A_d$ is the generating series of the dimensions of the homogeneous components of A:

$$\mathrm{Hilb}_t(A) := \sum_{d=0}^{\infty} \dim A_d t^d. \tag{4}$$

Given two formal series $H(t) := \sum_d h_d t^d$ and $L(t) := \sum_d l_d t^d$, we say that $H(t)$ is *dominated* by $L(t)$, and write $H(t) \leq L(t)$, whenever $h_d \leq l_d$ for all d. The relation of domination is preserved by multiplication by series with nonnegative coefficients, such as Hilbert series and generating functions.

Let us recall some basic Hilbert Series.

Proposition 3. *The Hilbert series of* $\mathbb{K}[X_n]$ *is*

$$\mathrm{Hilb}_t(\mathbb{K}[X_n]) = \frac{1}{(1-t)^n}\,. \tag{5}$$

The Hilbert series of the algebra $\mathrm{Sym} := \mathrm{Sym}(X_\infty)$ *of symmetric functions on a countably many variables is*

$$\mathrm{Hilb}_t(\mathrm{Sym}) = \prod_{i>0} \frac{1}{1-t^i}\,. \tag{6}$$

The Hilbert series of the algebra $\mathrm{Sym}(X_n)$ *of symmetric polynomials on n variables is*

$$\mathrm{Hilb}_t(\mathrm{Sym}(X_n)) = \prod_{i=1}^{n} \frac{1}{1-t^i}\,. \tag{7}$$

2.2. **Representations of the symmetric group on polynomials.** The representation theory of the symmetric groups is a well known combinatorial subject. The goal of this subsection is to recall several basic useful facts. For more details, the reader can refer to [FH96, GH94, Sag91] or, in French, to [Kro95].

The irreducible representations V_λ of the symmetric group $\mathfrak{S}_n$ are naturally indexed by partitions $\lambda := (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ of sum n (denoted by $\lambda \vdash n$). There is a particular occurrence of V_λ, called the Specht module, in the natural representation of $\mathfrak{S}_n$ on polynomials. This module is spanned by the so-called Specht polynomials. Recall that the partition λ is depicted (in the French way) as a diagram of boxes putting λ_i boxes in the i-th row, starting from the bottom. A filling of the boxes by the numbers $(1, 2 \dots, n)$ is called a *standard filling* F of shape λ. A standard filling which is increasing from left to right along rows and from bottom to top along columns is called a *standard tableau*.

For example here is the diagram of the partition $\lambda := (5,3,2)$ together with a standard filling and a standard tableau of shape λ.

$$\begin{array}{|c|c|}\hline & \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 8 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 6 & 9 \\ \hline \end{array}$$
$$\begin{array}{|c|c|c|}\hline & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 9 & 1 & 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 5 & 7 \\ \hline \end{array}$$
$$\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 10 & 4 & 3 & 7 & 6 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 8 & 10 \\ \hline \end{array}$$

Recall that the *Vandermonde determinant* of a finite alphabet of variables $\{y_1, \dots, y_k\}$ is defined by

$$\Delta(y_1, \dots, y_k) := \begin{vmatrix} 1 & y_1 & y_1^2 & \dots & y_1^{k-1} \\ 1 & y_2 & y_2^2 & \dots & y_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & y_k & y_k^2 & \dots & y_k^{k-1} \end{vmatrix} = \prod_{1 \le i < j \le k} (y_j - y_i). \tag{8}$$

For each standard filling F, the *Specht polynomial* Sp_F is defined as the product of the Vandermonde determinants of the variables indexed by the columns of F.

For example the Specht polynomial associated with the previous filling is

$$\begin{aligned} \mathrm{Sp}_F &:= \Delta(x_{10}, x_9, x_8) \cdot \Delta(x_4, x_1, x_2) \cdot \Delta(x_3, x_5) \cdot \Delta(x_7) \cdot \Delta(x_6) \\ &= (x_9 - x_{10})(x_8 - x_{10})(x_8 - x_9) \cdot (x_1 - x_4)(x_2 - x_4)(x_2 - x_1) \cdot (x_5 - x_3) \cdot 1 \cdot 1\,. \end{aligned}$$

It is easy to see that, for $\sigma \in \mathfrak{S}_n$, and F a filling, $\sigma\,\mathrm{Sp}_F = \mathrm{Sp}_{\sigma F}$. Hence, the span of $\{\mathrm{Sp}_F\}$, where F runs through the standard fillings of a given shape $\lambda \vdash n$ is a representation V_λ of $\mathfrak{S}_n$, which is called *Specht module* indexed by λ.

Proposition 4. *The Specht modules $(V_\lambda)_{\lambda \vdash n}$ form a complete family of irreducible representations of $\mathfrak{S}_n$.*

The Specht polynomials indexed by fillings are not linearly independent. A basis of the Specht module V_λ is given by $\{\mathrm{Sp}_F\}$, where F runs through the standard tableaux of shape λ.

Note that the polynomials Sp_F are homogeneous. Furthermore, each Specht module is the lowest degree occurrence of the corresponding irreducible representation; more precisely:

Proposition 5. *Let d be the degree of the Specht module V_λ. The multiplicity of the irreducible representation V_λ in $\mathbb{K}[X_n]$ in degree d is 1; Any occurrence of a representation isomorphic to V_λ in $\mathbb{K}[X_n]$ occurs in degree at least d.*

There is a construction for higher degree occurrences of representations isomorphic to V_λ by [TY93] using so-called *higher Specht polynomials*. They form an explicit basis for the quotient module $\mathbb{K}[X_n]_{/\,\mathrm{Sym}^+(X_n)\mathbb{K}[X_n]}$.

Let W a representation of $\mathfrak{S}_n$. Recall that the direct sum of all representations isomorphic to some V_λ is a sub-representation called the *isotypic component* indexed by λ of W and denoted W_λ. Then W decomposes canonically as $W = \bigoplus_{\lambda \vdash n} W_\lambda$. Moreover, in the group ring $\mathbb{K}\mathfrak{S}_n$ of $\mathfrak{S}_n$, there are special elements e_λ called *central orthogonal idempotents* such that $e_\lambda W = W_\lambda$ and $e_i e_j = \delta_{i,j}$.

Consider a non zero polynomial f in V_λ and a representation W isomorphic to V_λ. By Maschke's theorem, the set of the images of f by all Sym_n-morphisms from V_λ to W is a line. Moreover, for two linearly independent polynomials f_1 and f_2, these lines are in direct sum.

Definition 1. Let $f \in V_\lambda$ be a non zero polynomial, and W be a $\mathfrak{S}_n$-representation. The set of the images of f by all $\mathfrak{S}_n$-morphisms from V_λ to W is called the *Garnir component* of W associated to f denoted by $W_\lambda(f)$.

$W_\lambda(f)$ has naturally a structure of vector space isomorphic to $\hom(V_\lambda, W)$ of which it is a sort of concrete realization. Then we know that

Proposition 6. *Let W be a representation of $\mathfrak{S}_n$. Then*

$$W = \bigoplus_{\lambda \vdash n} W_\lambda = \bigoplus_{\lambda \vdash n} \left(\bigoplus_{Shape(T)=\lambda} W_\lambda(\mathrm{Sp}_T) \right) , \tag{9}$$

where the last sum is on standard tableaux of shape λ.

Note that this direct sum decomposition is dependent of the chosen basis of V_λ (here the Specht basis).

2.3. **The Weyl algebra.** The following brief introduction to the Weyl algebra is essentially taken from [Woo97]. For a variable x_i define the partial derivative

$$\partial_i := \frac{\partial}{\partial x_i}.$$

Following standard practice, we write abbreviated expressions for monomials:

$$x^K := \prod x_i^{k_i}, \quad \partial^L := \partial_i^{l_i},$$

where $K := (k_1, \dots, k_n, \dots)$ and $L := (l_1, \dots, l_n, \dots)$ are exponent vectors. The *degree* of x^K is $\deg(x^K) := \sum_i k_i$, and the *order* of ∂^L is $\mathrm{ord}(x^L) := \sum_i l_i$.

Weyl monomials are monomials in the variables x_i and the partial derivatives ∂_i. They act by multiplication and derivation on $\mathbb{K}[X_n]$, and form an algebra under addition and composition. For example,

$$\partial_1 . x_1^4 = 4x_1^3 , \qquad \partial_1 . x_2^2 = 0 , \qquad \partial_1\partial_2 . p = \partial_2\partial_1 . p ,$$
$$x_1x_2.(1 + x_2x_4) = x_2x_1(1 + x_2x_4) = x_1x_2 + 2x_1x_2^2x_4 ,$$
$$\partial_1 x_1 . p = \partial_1.(x_1p) = (\partial_1.x_1)p + x_1(\partial_1.p) = (1 + x_1\partial_1).p ,$$

This is formulated more precisely as follows:

Definition 2. The Weyl algebra Weyl := Weyl(X) is the associative algebra, with unit, generated by x_i, ∂_i, subject to the relations

$$[x_i, x_j] = 0, \qquad [\partial_i, \partial_j] = 0, \qquad [x_i, \partial_j] = \delta_{ij},$$

where square brackets denotes the Lie product and δ_{ij} the Kronecker symbol.

The Weyl algebras Weyl(X_n) are defined for each n in a similar way by restricting to the finite set of variables $x_1, \dots, x_n$ and the corresponding partial derivatives.

By repeated differentiation, any element of the Weyl algebra can be expressed as a linear combination of Weyl monomials where the polynomial part is on the left and the derivatives on the right. For example,

$$\partial_1 x_1 = x_1\partial_1 + 1 \tag{10}$$
$$\partial_1^k x_1 = x_1\partial_1^k + k\partial_1^{k-1} \tag{11}$$
$$\partial_1\partial_2 x_1 x_2 = x_1x_2\partial_1\partial_2 + x_1\partial_1 + x_2\partial_2 + 1 \tag{12}$$

We shall refer to such monomials as the *standard monomials*. As an Abelian group, Weyl is freely generated by the standard monomials $x^K\partial^L$, as K and L range over exponent vectors.

An element f in the Weyl algebra lies in *filtration* k if the maximum order of a term of f in standard form does not exceed k. Then, Weyl(X_n) and Weyl are filtered algebras in the sense that the composition product of two elements, in filtration m and k respectively lies in filtration $m+k$. The Weyl algebras are graded by assigning grading $\deg(x^K) - \mathrm{ord}(\partial^L)$ to $x^K\partial^L$.

Given a polynomial or a Weyl operator p, we denote by $\mathrm{o}(p)$ the *orbit sum* of p, that is the sum of all the polynomials in the orbit of m under the action of the symmetric group. Typically, if λ is a partition, then $m_\lambda := \mathrm{o}(x^\lambda)$ is the *monomial symmetric function* indexed by λ. Note that we have to be a little bit careful, since $\mathrm{o}(p)$ might be undefined for an infinite alphabet X, as for example for $\mathrm{o}(x_1 + x_2)$; however, for a monomial or Weyl monomial m, $\mathrm{o}(m)$ is always defined. We will consider differential operators such as

$$P_k := \sum_i x_i^k(1 + x_i\partial_i) := \mathrm{o}(x_1^k(1 + x_1\partial_1)) \tag{13}$$

which are obtained by such symmetrization over an infinite alphabet. To be precise, such operators do not belong to the Weyl algebra, because they involve infinite sums. They do not even necessarily act properly on $\mathbb{K}[X]$; for example $P_k.1 = x_1^k + \cdots + x_n^k + \ldots$ involves an infinite sum. In general, an infinite sum makes sense as an operator on $\mathbb{K}[X]$ providing that, for each n, all but a finite number of the monomials $x^K\partial^L$ annihilate $\mathbb{K}[X_n]$.

In practice, this is not an issue; such symmetric operators can be manipulated in the same classical manner as, for example, symmetric functions on an infinite number of variables. They restrict to elements of the Weyl algebra $\mathrm{Weyl}(X_n)$ for each n, and the notions of order, filtration and grading carry over properly. By extension, we also call them Weyl operators.

The *wedge product* (or *formal product*) of two differential operators is defined on standard monomials of the Weyl algebra by

$$x^K\partial^L \wedge x^M\partial^N := x^{K+L}\partial^{M+N}.$$

In other words, the partial derivatives are made to commute with all the variables, and Weyl becomes polynomial in both sets of variables x_i, ∂_i under the wedge product. In working out the composition product of two standard forms in the Weyl algebra, the wedge product is the term of top filtration. The wedge product is extended to infinite sums by linearity.

2.4. Duality in the Weyl algebra and scalar product on polynomials.

Definition 3. The *dual* f^* of a Weyl operator f is defined by sesquilinearity on the standard monomials:

$$(x^K\partial^L)^* := x^L\partial^K.$$

For example,

$$(2x_1x_2^2\partial_2^2\partial_3^3 + 3ix_2\partial_1^4)^* = 2x_2^2x_3^3\partial_1\partial_2^2 - 3ix_1^4\partial_2.$$

Note that in the literature, the dual of a polynomial p is often written $p(\partial)$ instead of p^* [GH94].

Remark 1. The dual is an anti-morphism with respect to composition:

$$(fg)^* = g^*f^* \tag{14}$$

Denote by $p(0)$ the constant term of a polynomial p. For two polynomials p and q, set

$$\langle p \,|\, q\rangle := (\overline{p}^*.q)(0). \tag{15}$$

It is easy to check that this defines a scalar product which makes the monomials $\{x^K\}$ into an orthogonal basis. In fact, we have

$$\langle x^K \,|\, x^L\rangle = \begin{cases} 0 & \text{if } K \neq L \\ K! & \text{if } K = L \end{cases}, \tag{16}$$

where $K! = k_1!k_2!\cdots k_n!$.

Proposition 7. *Let f be a Weyl operator, seen as a linear endomorphism of $\mathbb{K}[X]$. Then, the adjoint of f with respect to the scalar product $\langle . \,|\, . \rangle$ is its dual Weyl operator f^*.*

Proof. Note first that for a Weyl operator f and a polynomial p, we have

$$(f^*.p)(0) = ((f.1)^*.p)(0)\,; \tag{17}$$

indeed, $(f - f.1)$ is a pure differential operator, hence $(f - f.1)^*.p$ is a pure polynomial with no constant term. Then,

$$\begin{aligned} \langle f.p \,|\, q\rangle = ((f.p)^*.q)(0) = ((fp.1)^*.q)(0) = ((fp)^*.q)(0) &= (p^* f^*.q)(0) \\ &= (p^*.(f^*.q))(0) = \langle p \,|\, f^*.q\rangle\,, \end{aligned} \tag{18}$$

as desired. □

2.5. **Hit and harmonic polynomials.** In this subsection, we fix a set S of homogeneous Weyl operators, and denote by S^+ the operators in S of positive degree.

Definition 4. A polynomial is called S*-hit* if there exists some polynomials $p_1, \dots, p_k$ and operators $f_1, \dots, f_k \in S^+$ such that

$$p = f_1.p_1 + \cdots + f_k.p_k. \tag{19}$$

A polynomial p is called S*-harmonic* if it satisfies the differential equations

$$f^*.p = 0, \qquad \text{for all } f \in S^+. \tag{20}$$

The sub-spaces of S-hit and S-harmonic polynomials are denoted respectively by Hit and Harm.

For example, if $S = \{x_1^2 + \cdots + x_n^2\}$, the harmonics in $\mathbb{K}[X_n]$ are the polynomials p which satisfy

$$\partial_1^2 p + \cdots + \partial_n^2 p = 0;$$

hence the name *harmonic* [GH94]. If S consists only of polynomials, as in the previous example, then the space of S-hits is simply the ideal generated by S^+ in $\mathbb{K}[X_n]$.

Remark 2. Consider an operator $f \in S^+$, any Weyl operator g, and a polynomial p. Then, $fg.p = f.(g.p)$ is hit. Dually, if p is harmonic then $gf^*.p = g.(f^*.p) = g.0 = 0$: it is killed by any operator in the left ideal generated by S^{+*} in the Weyl algebra. Hence, S and the right ideal $S^+\,\mathrm{Weyl}(X_n)$ generated by S^+ in the Weyl algebra have the same hit and harmonic polynomials.

Furthermore, if A^+ is any subspace of the Weyl algebra such that $S^+ \subset A^+ \subset S^+\,\mathrm{Weyl}(X_n)$, then Hit is the image $A.\mathbb{K}[X_n]$ of $A^+ \otimes \mathbb{K}[X_n]$ by the bilinear evaluation map $(f, p) \mapsto f.p$.

Proposition 8. *A polynomial p is S-harmonic if, and only if, it is orthogonal to any S-hit polynomial q. That is, the space of S-harmonic polynomials is the orthogonal complement of the space of S-hit polynomials.*

Proof. This is a simple variation on the fact that the image of a linear morphism is the orthogonal complement of the kernel of its adjoint. Let p be a polynomial. By duality, for any polynomial q and operator $f \in S^+$, we have

$$\langle f^*.p \,|\, q\rangle = \langle p \,|\, f.q\rangle.$$

Assume that p is S-harmonic; then it is obviously orthogonal to any $f.q$ and by linearity to any S-hit. Reciprocally, assume that p is orthogonal to any hit polynomial, and consider an operator $f \in S^+$; then $f^*.p = 0$ because it is orthogonal to all polynomials. □

Proposition 9. *Let A be the graded algebra generated by S^+, acting on $\mathbb{K}[X_n]$. Then,* Harm *generates $\mathbb{K}[X_n]$ as an A-module. In particular,*

$$\mathrm{Hilb}_t(A).\,\mathrm{Hilb}_t(\mathrm{Harm}) \geq \mathrm{Hilb}_t(\mathbb{K}[X_n]), \tag{21}$$

with equality if, and only if, $\mathbb{K}[X_n]$ is a free A-module.

Proof. It is sufficient to verify, by induction on the degree, that any homogeneous polynomial can be written in the form $p = 1.p_0 + f_1.p_1 + \cdots + f_k.p_k$ where the p_i's are harmonic. Then, $\mathbb{K}[X_n] = A.\,\mathrm{Harm}$ is a graded quotient of $A \otimes \mathrm{Harm}$, which yields the desired inequality on the Hilbert series. □

Proposition 10. *Assume that the operators in S are symmetric. Then, any Specht polynomial Sp_F is S-harmonic. Hence, the space of S-harmonics contains at least one copy of each irreducible representation of the symmetric group $\mathfrak{S}_n$.*

Proof. Consider an operator $f \in S^+$. Its dual f^* is an $\mathfrak{S}_n$-morphism of negative degree. Hence, $f^*.\,\mathrm{Sp}_F$ lies in the same isotypic component as Sp_F, and $\deg(f^*.\,\mathrm{Sp}_F) < \deg(\mathrm{Sp}_F)$. By minimality of the degree of Sp_F, $f^*.\,\mathrm{Sp}_F = 0$. □

As a final example, take for S the n symmetric power-sums on n variables $(p_k := x_1^k + \cdots + x_n^k)_{k=1,\dots,n}$. Then, the space Hit of hit polynomials is the ideal generated by the symmetric polynomials. There exists an explicit Gröbner basis for the lexicographic term order, which shows that the leading terms of Hit are the non-staircase monomials (see e.g. [Stu93]).

The harmonics are the polynomials p which satisfy the generalized harmonic equations

$$p_k^*.p = \partial_1^k.p + \cdots + \partial_n^k.p = 0, \qquad \text{for } k = 1, \dots, n. \tag{22}$$

The Vandermonde $\Delta := \prod(x_j - x_i)$ is the Specht polynomial $\mathrm{Sp}_{(1,\dots,1)}$, and as such is harmonic. Moreover, any partial derivative ∂_i commute with the p_k^*, and it follows that it stabilizes the space Harm of harmonic polynomials. Actually, any harmonic is of the form $p^*.\Delta$, for some polynomial p, and if a set B of $n!$ polynomials is a basis of $\mathbb{K}[X_n]/\,\mathrm{Sym}\,.\mathbb{K}[X_n]$, then $(p^*.\Delta)_{p\in B}$ is a basis of Harm. Finally, $\mathbb{K}[X_n]$ is a free $\mathrm{Sym}(X_n)$-module, and the Hilbert series of $\mathrm{Harm}(X_n)$ is given by

$$\begin{aligned} \mathrm{Hilb}_t(\mathrm{Harm}(X_n)) &= \frac{\mathrm{Hilb}_t(\mathbb{K}[X_n])}{\mathrm{Hilb}_t(\mathrm{Sym}(X_n))} \\ &= \prod_{i=1}^{n} \frac{1-t^i}{1-t} = (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n-1}). \end{aligned} \tag{23}$$

For further details, we refer to [GH94].

2.6. **Specialization lemmas.** In this subsection, U is a $\mathbb{C}$-vector space with a distinguished basis B. The goal here is to deal with q-deformations of finite dimensional sub-spaces of $\mathbb{C}(q) \otimes U$ and their specializations.

Lemma 2.1. *Let $(u_1(q), \dots, u_k(q))$ be a family of vectors of $\mathbb{C}(q)\otimes U$. For all but a finite set of values $q_0 \in \mathbb{C}$, namely the roots of the denominators of the coefficients of the u_i's on the basis B, the specialization $q = q_0$ is well defined. In this case,*

$$\mathrm{Rank}_{\mathbb{C}}(u_1(q_0), \dots, u_k(q_0)) \leq \mathrm{Rank}_{\mathbb{C}(q)}(u_1(q), \dots, u_k(q))\,. \tag{24}$$

Moreover, there is only a finite number of values $q_0 \in C$ such that the inequality above is strict, namely the roots of the non-trivial principal minors.

Let $V(q)$ be a subspace of $\mathbb{C}(q) \otimes U$ of finite dimension. For any $q_0 \in \mathbb{C}$, let $V(q_0)$ be its specialization, that is the vector space of all meaningful specializations of elements of $V(q)$. Note that this definition is highly dependent on the basis B.

Lemma 2.2. *There exists a basis $B'(q) = (v_1(q), \dots, v_k(q))$ of $V(q)$ such that the coefficients of the elements of $B'(q)$ in the basis B are polynomials in q, and for any specialization $q = q_0$, the family $B'(q_0)$ is a basis of $V(q_0)$ (in particular $\dim_{\mathbb{C}} V(q_0) = \dim_{\mathbb{C}(q)} V(q)$).*

This is a classical result, but let us give a short effective proof.

Proof. First remark that, by the preceding lemma, the vector space $V(q_0)$ of all meaningful specializations of elements of $V(q)$ is of $\mathbb{C}$-dimension at most the $\mathbb{C}(q)$-dimension of $V(q)$. Consequently we just have to construct a basis of $V(q)$ without denominators such that all specializations are linearly independent. Let us construct such a basis.

Choose a $\mathbb{C}(q)$-basis $A(q)$ of $V(q)$ and expand the vectors of $A(q)$ over B. By a suitable multiplication, we can remove the denominators and then suppose that the coefficients of the expansion are in $\mathbb{C}[q]$. Now we apply the fraction-free Gram-Schmidt orthogonalization to $A(q)$, and get a basis $A'(q) = (a_1(q), \dots, a_n(q))$. By dividing each $a_i(q)$ by the greatest common divisor of its coefficients, (it is often called the *content* of $a_i(q)$) we get a new basis $B'(q) = (b_i(q))$ of $V(q)$.

This basis is orthogonal on $\mathbb{C}[q]$ and thus remains orthogonal for any specialization at $q = q_0$. Furthermore, an element $b_i(q)$ of $B'(q)$ is content-free, so $b_i(q_0)$ never vanishes. We conclude that, for any q_0, the family $B'(q_0)$ is a $\mathbb{C}$-basis for $V(q_0)$. □

Moreover, since the character table of $\mathfrak{S}_n$ is with integer coefficients, the action of central idempotents commutes with specializations. Thus if $V(q)$ is an $\mathfrak{S}_n$-module then

- $V(q)$ is isomorphic as $\mathfrak{S}_n$-modules to $\mathbb{C}(q) \otimes V(q_0)$ for all q_0;
- $\mathbb{C}(q_0)$ and $\mathbb{C}(q_1)$ are isomorphic as $\mathfrak{S}_n$-modules for all q_0, q_1.

3. The q-Steenrod algebra

The similarities between Sym and Steen leads us to define a new algebra to interpolate between the two.

Definition 5. Let $P_{q,k}$ be the following q-analogue of the P_k operator:

$$P_{q,k} := \sum_i x_i^k (1 + q x_i \partial_i).$$

The *q-Steenrod algebra* $\mathrm{Steen_q}$ is the algebra over $\mathbb{C}(q)$ generated by the $P_{q,k}$, $k > 0$.

Note that the specializations at $q = 0$ and $q = 1$ yield respectively the symmetric functions and the rational Steenrod algebra (conjugated by the Thom map):

- $P_{0,k}$ is the usual operator of multiplication by the symmetric power-sum p_k.
- $P_{1,k} = P_k$.

Remark 3. It would be tempting to actually introduce two parameters q and q':

$$P_{q,q',k} := \sum_i x_i^k (q' + q x_i \partial_i). \tag{25}$$

Now, the rational Steenrod algebra itself is also a special case of the (q, q')-Steenrod algebra, with $q' = 0$ and $q = 1$: working in projective space versus q removes the singularity at $q = \infty$. The price to pay is that $\mathbb{C}[q, q']$ is not anymore a principal ideal domain; this renders computations, and especially linear algebra, far less practical. Since we are mainly interested in the cases $q = 0$ and $q = 1$, we stick in the following to a single parameter.

3.1. **Structure and dimension of the q-Steenrod algebra.** The q-Steenrod squares satisfy the following commutation rule:

Proposition 11. *For $k, l \in \mathbb{N}$, and q complex or formal,*

$$[P_{q,k}, P_{q,l}] = q(l-k)P_{q,k+l}\,. \tag{26}$$

For a composition $\mu := (\mu_1, \dots, \mu_k)$, define $P_{q,\mu} := P_{q,\mu_1} \cdots P_{q,\mu_k}$. Using the commutation rules, it is obvious that any $P_{q,\mu}$ can be rewritten as a linear combination of $P_{q,\lambda}$'s indexed by partitions. For example,

$$P_{q,(1,2)} = qP_{q,(3)} + P_{q,(2,1)}\,.$$

Theorem 3.1. *Let q be formal or a complex number.*

(a) *The operators $P_{q,\lambda}$, where λ runs through all integer partitions form a vector space basis of* $\mathrm{Steen_q}$*;*
(b) *The Hilbert Series of the q-Steenrod algebra is given by*

$$\mathrm{Hilb}_t(\mathrm{Steen_q}) = \mathrm{Hilb}_t(\mathrm{Sym}) = \prod_{i>0} \frac{1}{1-t^i}\,; \tag{27}$$

(c) *The ideal of relations between the $P_{q,k}$'s is generated by the commutation relations*

$$[P_{q,k}, P_{q,l}] - q(l-k)P_{q,k+l}\,. \tag{28}$$

(d) $\mathrm{Steen_q}$ *is the enveloping algebra of the Lie algebra spanned by the $P_{q,k}$.*

This is a straightforward generalization of results in [Woo97] about the rational Steenrod algebra, and follows from an easy triangularity property with respect to monomial functions. Namely, consider the expansion of

$$\begin{aligned} P_{q,(2,1)} &= \mathrm{o}(x_1^2x_2) + (q+1)\,\mathrm{o}(x_1^3) \\ &\quad + q^2\,\mathrm{o}(x_1^5\partial_1^2) + q^2\,\mathrm{o}(x_1^3x_2^2\partial_1\partial_2) + q\,\mathrm{o}(x_1^3x_2\partial_1) + 2q(q+1)\,\mathrm{o}(x_1^4\partial_1) + q\,\mathrm{o}(x_1^2x_2^2\partial_1). \end{aligned}$$

Its polynomial part is a symmetric function, and can be expressed in terms of monomial symmetric function: $P_{q,(2,1)}.1 = m_{(2,1)} + (q+1)m_{(3)}$. Note that $m_{(2,1)}$ has coefficient 1, while m_3 is indexed by a shorter partition. This generalizes immediately:

Lemma 3.1. *The polynomial part $P_{q,\lambda}.1$ of $P_{q,\lambda}$ is a symmetric function of the form*

$$P_{q,\lambda}.1 = m_\lambda + \sum_\mu c_\mu m_\mu,$$

where μ runs through partitions of length $\ell(\mu) < \ell(\lambda)$.

It follows that, except for $q = 0$, all the q-Steenrod algebras on an infinite alphabet are isomorphic independently of q; only their action on polynomials differ:

Theorem 3.2. *Let q be formal (respectively a non zero complex number). Then, the q-Steenrod algebra* $\mathrm{Steen_q}$ *is isomorphic as a graded algebra to $\mathbb{C}(q) \otimes$* Steen *(respectively* Steen*).*

Proof. By renormalizing the q-Steenrod squares $\widetilde{P}_{q,k} := \frac{1}{q}P_{q,k}$, we obtain the commutation relations

$$[\widetilde{P}_{q,k}, \widetilde{P}_{q,l}] = (l-k)\widetilde{P}_{q,k+l}\,. \tag{29}$$

Hence the algebra $\mathrm{Steen_q}$ has the same presentation with generators and relations as $\mathbb{C}(q) \otimes \mathrm{Steen}$. □

We consider now the q-Steenrod algebra over a finite alphabet.

Proposition 12. *For any finite alphabet X_n,*

$$\prod_{i=1}^{n} \frac{1}{1-t^i} = \mathrm{Hilb}_t(\mathrm{Sym}(X_n)) \leq \mathrm{Hilb}_t(\mathrm{Steen_q}(X_n)) \leq \frac{1}{(1-t)^n} \prod_{i=1}^{n} \frac{1}{1-t^i}$$

Proof. The first inequality follows again from lemma 3.1. Note that, for $k \geq 1$, any Weyl monomial $x^K \partial^L$ appearing in $P_{q,k}$ satisfies $k_i \geq 2l_i$ for all i. This property is preserved by multiplication, and it follows that $P_{q,\lambda}$ can be written as a linear combination of orbit-sums $\mathrm{o}(x^{2K+L}\partial^L)$, where $\sum k_i + l_i = n$, and the k_i are decreasing. The second inequality follows. □

It appears from computations that the first inequality is strict, starting from degree $n+1$. Namely, for $n \leq 5$, we noticed that the polynomials $(P_{q,\lambda}(X_n).x_1)_{\lambda \vdash n+1}$ are always linearly independent (see Sections A.2 and 7.1), whereas there is a linear relation between the symmetric functions $(p_\lambda(X_n))_{\lambda \vdash n+1}$.

The second inequality is very coarse, and could certainly be refined. For example, one can easily check that

$$\mathrm{Hilb}_t(\mathrm{Steen_q}(X_1)) = \frac{1}{1-t} \ll \frac{1}{(1-t)^2}. \tag{30}$$

Still, this inequality implies that $\mathrm{Hilb}_t(\mathrm{Steen_q}(X_n)) < \mathrm{Hilb}_t(\mathrm{Steen_q}(X))$; indeed the dimension of $\mathrm{Steen_q}_d$ is counted by the number $p(d)$ of partitions whose asymptotic [HR18] is:

$$p(d) \approx \frac{1}{4\sqrt{3}\,n} \exp \pi \sqrt{\frac{2n}{3}}. \tag{31}$$

3.2. **Action on polynomials; hit and harmonic polynomials.** We consider now the natural action of $\mathrm{Steen_q}$ on $\mathbb{K}[X_n]$. Following subsection 2.5 with $S = \mathrm{Steen_q}^+$, we define the spaces $\mathrm{Hit_q}$ of *q-hit polynomials* and $\mathrm{Harm_q}$ of *q-harmonic* polynomials. For $q \neq 0$, the q-Steenrod algebra is generated by $P_{q,1}$ and $P_{q,2}$, and it follows that the q-harmonics can be described by just two differential equations:

$$P^*_{q,1}.p = 0 \qquad \text{and} \qquad P^*_{q,2}.p = 0\,. \tag{32}$$

Furthermore, it follows immediately from the triangularity property described in lemma 3.1 that any symmetric function is q-hit.

Proposition 13. *Let q be formal or a complex number, and X be a commutative alphabet. Then,* $\mathrm{Steen_q}(X).1 = \mathrm{Sym}(X)$.

4. Specializations

The action of the q-Steenrod algebra behaves properly with respect to specialization. That is, if a polynomial $p(q)$ specializes properly to $p(q_0)$, then $P_{q,k}.p(q)$ and $P^*_{q,k}.p(q)$ specializes properly to $P_{q_0,k}.p(q_0)$ and $P^*_{q_0,k}.p(q_0)$ respectively. Hence, we can apply the specialization lemmas to obtain some properties of the q-harmonics and q-hits with respect to specialization. In particular we obtain the implication $(b) \Rightarrow (a)$ of proposition 1:

Proposition 14. *Let $q_0 \in \mathbb{C}$. Then,*

(a) *Let $S(q)$ be a set of homogeneous polynomials in $\mathbb{C}(q)[X_n]$, which specializes properly at $q = q_0$. Then, the* $\mathrm{Steen_q}$*-module (resp.* $\mathrm{Steen_q}^+$*-module) $M(q)$ generated by S in $\mathbb{C}(q)[X_n]$ specializes to the* $\mathrm{Steen_{q_0}}$*-module (resp.* $\mathrm{Steen_{q_0}}^+$*-module) $M(q_0)$ generated by $S(q_0)$.*

(b) *The Hilbert series of $M(q)$ dominates the Hilbert series of $M(q_0)$;*

(c) *Let $p \in \mathbb{C}[X_n]$. The Hilbert series of the* $\mathrm{Steen_q}$*-module generated by p dominates the Hilbert series of the corresponding* $\mathrm{Sym}(X_n)$*-module;*

(d) *Let B be a* $\mathrm{Sym}(X_n)$*-module basis of $\mathbb{C}[X_n]$, e.g. the Schubert polynomials or the staircase monomials. Then B spans $\mathbb{C}(q)[X_n]$ as a* $\mathrm{Steen_q}$*-module; note that this module is not necessarily free.*

(e) $\mathrm{Hit_q}$ *specializes at $q = q_0$ to* $\mathrm{Hit_{q_0}}$*; in particular the Hilbert series of* $\mathrm{Hit_q}$ *dominates the Hilbert series of* $\mathrm{Hit_{q_0}}$*; furthermore, for all but a countable number of $q_0 \in \mathbb{C}$, there is equality;*

(f) $\mathrm{Hit_q} + \mathrm{Harm} = \mathbb{C}(q)[X_n]$.

Proof. (a) Fix a degree d. Extract from $S(q)$ a finite subset $A(q)$ which generates linearly all the other elements of $S(q)$ of degree less than d, and specializes properly at $q = q_0$. Consider the set $B(q)$ of all $P_{q,\lambda}.p(q)$, where p belongs to $A(q)$ and λ is a partition of $d - \deg(p)$. Then, $B(q)$ generates $M(q)_d$, and specializes to $B(q_0)$ which generates $M(q_0)_d$. Hence, $M(q)_d$ specializes to $M(q_0)_d$.

The others are direct applications of (a). □

Similarly, the following proposition provides the implication $(b) \Rightarrow (a)$ of proposition 2, and more:

Proposition 15. (a) *If $p(q)$ is q-harmonic and specializes properly at $q = q_0$, then $p(q_0)$ is harmonic;*

(b) *There exists a basis of* $\mathrm{Harm_q}$ *which specializes properly for any q_0;*

(c) $\mathrm{Harm_q}$ *specializes to a subspace* $\mathrm{Harm_q}(q_0)$ *of* $\mathrm{Harm_{q_0}}$

(d) *The Hilbert series of* $\mathrm{Harm_q}$ *is dominated by the Hilbert series of* Harm*; furthermore, for all but a countable number of $q_0 \in \mathbb{C}$, there is equality;*

(d) Hit *and* $\mathrm{Harm_q}$ *are in direct sum.*

(e) *The orthogonal projector $\mathbb{C}(q)[X_n] \mapsto \mathrm{Harm} \otimes \mathbb{C}(q)$ is injective on* $\mathrm{Harm_q}$.

Proof. (a) For any k, $P^*_{q,k}.p(q)$ specializes to $P^*_{q_0,k}.p(q)$ for all k; hence the latter is zero whenever the former is.

(b) Direct application of the specialization lemma 2.2.

(c) Consequence of (a) and (b).

(d) Take $f \in \mathrm{Hit} \cap \mathrm{Harm_q}$. Then, for all k, $\sum_i (1 + qx_i\partial_i)\partial_i^k f = 0$.

Write $f = q^{\mathrm{val}(f)}(f_0 + qf_1)$, with $0 \neq f_0 \in \mathbb{C}[X_n]$ and $f_1 \in \mathbb{C}(q)[X_n]$ such that f_1 has no $\frac{1}{q}$ coefficient. Then, $f_0 + qf_1$ is also q-harmonic, and it follows that f_0 is harmonic. This is in contradiction with f_0 being hit. □

5. The low degree and truncated cases

One of the main difficulties is that the q-Steenrod algebra on a finite alphabet X_n is much bigger than $\mathrm{Sym}(X_n)$. In this section, we get around this difficulty by truncating the q-Steenrod algebra to make it share the dimension of $\mathrm{Sym}(X_n)$. No truncation occurs in low degree, so all the results of this section also apply to the full q-Steenrod algebra in degree $d \leq n$.

The *truncated q-Steenrod algebra* $\mathrm{TSteen_q}(X_n)$ is the subspace of $\mathrm{Steen_q}(X_n)$ spanned by $(P_{q,\lambda})_{\ell(\lambda)\leq n}$. Note that $\mathrm{Steen_q}(X_n)$ and $\mathrm{TSteen_q}(X_n)$ coincide in degree $d \leq n$; however $\mathrm{TSteen_q}(X_n)$ is not an algebra.

Here also, applying lemma 3.1, yields $\mathrm{TSteen_q}(X_n).1 = \mathrm{Sym}$, and it follows that $\mathrm{Hilb}_t(\mathrm{TSteen_q}(X_n)) = \mathrm{Hilb}_t(\mathrm{Sym}(X_n))$, for q formal or a complex number.

Problem 1. For which polynomials p do $\mathrm{TSteen_q}\,.p$ and $\mathrm{Steen_q}\,.p$ coincide ?

For such a polynomial p, find a straightening algorithm for rewriting an element of $\mathrm{Steen_q}\,.p$ as element of $\mathrm{TSteen_q}$.

The proper analogue of $\mathrm{Hit}_{\mathrm{q}}(X_n)$ is the space of *truncated q-hit polynomials* $\mathrm{THit}_{\mathrm{q}}(X_n)$ which is spanned by the $f.p$, where $f \in {\mathrm{TSteen}_{\mathrm{q}}}^{+}(X_n)$, and $p \in \mathrm{Harm}(X_n)$. The polynomials h in the orthogonal $\mathrm{THarm}_{\mathrm{q}}(X_n) := \mathrm{THit}_{\mathrm{q}}(X_n)^{\perp}$ are called *truncated q-harmonics.* Note that they do not necessarily satisfy $P^*_{q,1}.h = 0$ and $P^*_{q,2}.h = 0$.

Reg Wood's conjecture holds for the truncated q-Steenrod algebra, and for q formal:

Theorem 5.1. *Let q be formal. Then,*
(a) $\mathrm{Hilb}_t(\mathrm{THarm}_{\mathrm{q}}(X_n)) = \mathrm{Hilb}_t(\mathrm{Harm}(X_n))$;
(b) $\mathrm{THarm}_{\mathrm{q}}(X_n)$ *is a graded regular representation of the symmetric group;*
(c) $\mathbb{K}[X_n] = \mathrm{THarm}_{\mathrm{q}}(X_n) \bigoplus \mathrm{Hit}(X_n) = \mathrm{Harm}(X_n) \bigoplus \mathrm{THit}_{\mathrm{q}}(X_n)$.

Furthermore, similar statements hold in the isotypic and Garnir components of $\mathbb{K}[X_n]$.

Proof. The specialization $q = 0$ of $\mathrm{TSteen}_{\mathrm{q}}$ is Sym. Then $\mathrm{THit}_{\mathrm{q}}$ specializes to Hit. Hence, by proposition 15, $\mathrm{Hilb}_t(\mathrm{THarm}_{\mathrm{q}}) \leq \mathrm{Hilb}_t(\mathrm{Harm})$.

Furthermore, $\mathbb{K}[X_n]$ is a free $\mathrm{Sym}(X_n)$ module, so applying proposition 9 yields

$$\begin{aligned} \mathrm{Hilb}_t(\mathbb{C}[X_n]) &\leq \mathrm{Hilb}_t(\mathrm{TSteen}_{\mathrm{q}}(X_n))\,\mathrm{Hilb}_t(\mathrm{THarm}_{\mathrm{q}}(X_n)) \\ &\leq \mathrm{Hilb}_t(\mathrm{Sym}(X_n))\,\mathrm{Hilb}_t(\mathrm{Harm}) = \mathrm{Hilb}_t(\mathbb{C}[X_n]). \end{aligned} \tag{33}$$

It follows that equality must hold, and that $\mathrm{Hilb}_t(\mathrm{THarm}_{\mathrm{q}}) = \mathrm{Hilb}_t(\mathrm{Harm})$. The same reasoning can be carried over in any isotypic or Garnir component, and all the statements of the theorem follow. □

Corollary 1. *Let $\mathrm{Hilb}_{t,\leq d}(A)$ denote the truncation of the Hilbert series of A up to degree d.*
- *If q is formal, then* $\mathrm{Hilb}_{t,\leq n}(\mathrm{Steen}_{\mathrm{q}}(X_n)) = \mathrm{Hilb}_{t,\leq n}(\mathrm{Sym}(X_n))$.
- *If q is a complex number, then* $\mathrm{Hilb}_{t,\leq n}(\mathrm{Steen}_{\mathrm{q}}(X_n)) \leq \mathrm{Hilb}_{t,\leq n}(\mathrm{Sym}(X_n))$.

Proof. This is a reformulation of the preceding theorem by the fact that for $d \leq n$ the algebra $\mathrm{TSteen}_{\mathrm{q}}$ and $\mathrm{Steen}_{\mathrm{q}}$ coincide. □

6. Changing the number of variables

In the preceding section, we proved that conjecture 2 holds when the number of the variable is greater than the degree. The goal here is to relate such q-harmonics on a large number of variable with q-harmonics on fewer variables, trying to take down as much information as possible. In particular we prove the equivalence $(a) \iff (c1)$ of proposition 1.

Proposition 16. *Let X be a set of variables and $y \notin X$ be another variable. Let us denote by $\mathrm{Harm}_{\mathrm{q}}(X,y)_{\deg(y)\leq d}$ the subspace of $\mathrm{Harm}_{\mathrm{q}}(X,y)$ of polynomials of degree at most d in the variable y. The isomorphism*

$$\pi_{y^d} \;:\; \begin{cases} \mathbb{K}(X,y)_{\deg(y)\leq d}/\mathbb{K}(X,y)_{\deg(y)\leq d-1} & \hookrightarrow\!\!\!\!\rightarrow \quad \mathbb{K}(X) \\ f & \longmapsto \quad \mathrm{Coeff}(f, y^d) \end{cases} \tag{34}$$

defines an injective mapping :

$$\tilde{\pi}_{y^d} \;:\; \mathrm{Harm}_{\mathrm{q}}(X,y)_{\deg(y)\leq d}/\,\mathrm{Harm}_{\mathrm{q}}(X,y)_{\deg(y)\leq d-1} \hookrightarrow \mathrm{Harm}_{\mathrm{q}}(X)\,. \tag{35}$$

Proof. The injectivity of $\tilde{\pi}_{y^d}$ follows from the definition. The only thing to prove is that the image $\tilde{\pi}_{y^d}$ of a q-harmonic polynomial of degree d in $\mathbb{K}[X,y]$ is in $\mathrm{Harm}_{\mathrm{q}}(X)$. It comes from the fact that the $P^*_{q,k}$ are primitive, that is

$$P^*_{q,k}(X,y) = P^*_{q,k}(X) + P^*_{q,k}(y)\,. \tag{36}$$

$$\left(\sum_{x\in X}(1+qx\partial_x)\partial_x^k + (1+qy\partial_y)\partial_y^k\right).f = 0\,. \tag{37}$$

Write $f = f_0 y^d + f_1$ with $\deg_y(f_1) < d$. The identification of the homogeneous components of degree d in y in the preceding equation proves that

$$\left(\sum_{x\in X}(1+qx\partial_x)\partial_x^k\right).f_0 = 0\,. \tag{38}$$

This ends the proof. □

The following proposition yields as corollary the equivalence $(a) \Rightarrow (c1)$ and by an easy induction the implication $(a) \Rightarrow (c2)$ of proposition 2.

Proposition 17. *Let q be generic. Then, the following are equivalent*

(a) *Conjecture 2 holds;*
(b) *For all n, any q-harmonic on n variables has degree at most $n-1$ on each variable;*
(c) *For all n, the mapping*

$$\tilde\pi_{y^d}\ :\ \mathrm{Harm_q}(X_n,y)_{\deg(y)\le d}/\,\mathrm{Harm_q}(X_n,y)_{\deg(y)\le d-1} \hookrightarrow \mathrm{Harm_q}(X_n)\,, \tag{39}$$

is an isomorphism for $d \le n$. For $d > n$, $\mathrm{Harm_q}(X_n,y)_{\deg(y)=d} = \{0\}$.

Proof. $(c) \Rightarrow (b)$ is obvious.

$(a) \Rightarrow (c)$: For any degree d, the injectivity of $\tilde\pi_{y^d}$ implies that

$$\begin{aligned}\mathrm{Hilb}_t(\mathrm{Harm_q}(X_n,y)_{\deg(y)\le d}/\,\mathrm{Harm_q}(X_n,y)_{\deg(y)\le d-1})\\ \le t^d\,\mathrm{Hilb}_t(\mathrm{Harm_q}(X_n)),\end{aligned} \tag{40}$$

with equality if, and only if, $\tilde\pi_{y^d}$ is an isomorphism. On the other hand, we assumed that conjecture 2 holds. Identifying y with x_{n+1}, it follows that

$$\mathrm{Hilb}_t(\mathrm{Harm_q}(X_{n+1})) = (1+t+\cdots+t^n)\,\mathrm{Hilb}_t(\mathrm{Harm_q}(X_n)). \tag{41}$$

If for some $d \le n-1$, the inequality in equation 40 above is strict, the terms of degree d in equation 41 will differ. Then, by a similar reasoning, the existence of a non-trivial component $\mathrm{Harm_q}(X_n,y)_{\deg(y)\le d}/\,\mathrm{Harm_q}(X_n,y)_{\deg(y)\le d-1})$ for $d \ge n$ would contradict equation 41.

$(b) \Rightarrow (a)$: By the assumption, and the injectivity of $\tilde\pi_{x_n^d}$, for any n,

$$\mathrm{Hilb}_t(\mathrm{Harm_q}(X_{n+1})) \le (1+t+\cdots+t^n)\,\mathrm{Hilb}_t(\mathrm{Harm_q}(X_n)). \tag{42}$$

By induction

$$\mathrm{Hilb}_t(\mathrm{Harm_q}(X_{n+1})) \le (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^n) = \mathrm{Hilb}_t(\mathrm{Harm}). \tag{43}$$

Suppose now that, for a given n_0, this inequality is strict:

$$\mathrm{Hilb}_t(\mathrm{Harm_q}(X_{n_0+1})) < (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n_0}). \tag{44}$$

Let d be the valuation of the difference. Applying again induction, we obtain that

$$\mathrm{Hilb}_t(\mathrm{Harm_q}(X_{d+1})) < (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^d), \tag{45}$$

with the inequality still appearing in degree d. However, the number of variables now exceeds d, and this is in contradiction with Corollary 1. □

6.1. **Strings.** We introduce a little bit more formalism to analyze when q-harmonic polynomials in $\mathbb{K}[X_{n+1}]$ have variables of degree at most n.

Proposition 18. *Consider $f \in \mathbb{K}[X, y]$, and write its expansion on the variable y as*

$$f(X, y) = \sum_{i \geq 0} \frac{1}{i!} f_i(X)\, y^i \,. \tag{46}$$

Then, f is q-harmonic if, and only if, for any $i \geq 0$ and $k \geq 1$,

$$P^*_{q,k} \cdot f_i = -(1 + qi)\, f_{i+k} \,. \tag{47}$$

Proof. Since f is q-harmonic, $P^*_{q,k} \cdot f = 0$. Hence,

$$\sum_{i \geq 0} \frac{1}{i!} (P^*_{q,k} \cdot f_i) y^i + \sum_{i \geq 0} \frac{1}{i!} f_i\, (P^*_{q,k} \cdot y^i) = 0 \,, \tag{48}$$

and then

$$\sum_{i \geq 0} \frac{1}{i!} (P^*_{q,k} \cdot f_i)\, y^i + \sum_{i \geq k} \frac{1}{(i-k)!} f_i\, (1 + q(i-k))\, y^{i-k} = 0 \,. \tag{49}$$

Setting $j := i - k$ in the second sum, it follows that

$$\sum_{i \geq 0} \frac{1}{i!} (P^*_{q,k} \cdot f_i)\, y^i + \sum_{j \geq 0} \frac{1}{j!} f_{j+k}\, (1 + qj)\, y^j = 0 \,. \tag{50}$$

Equating the coefficients of y yields the desired equality, for all i. □

A remarkable consequence of proposition 18 is that a q-harmonic polynomial f can be reconstructed from its tail f_0. This motivates the following definition:

Definition 6. Let $g \in \mathbb{K}(X)$ be a homogeneous polynomial. The *string* generated by g is the sequence $F = \mathrm{String}(g) := (f_i)_{i \geq 0}$ defined by

$$f_0 := g, \qquad \text{and for all } i > 0, \quad f_i := -P^*_{q,i} \cdot g \,. \tag{51}$$

The *length* of the string $F = (f_i)$ is defined by

$$\ell(F) := 1 + \max\{i \mid f_i \neq 0\}. \tag{52}$$

The *head* of the string is $f_{\ell(f)}$.

The length of a string is nothing but the degree in the variable y of the associated polynomial $f(X, y) := \sum \frac{1}{i!} f_i(X)\, y^i$. Obviously, if g is of degree d then $f_i = 0$ for all $i > d$, consequently $\ell(\mathrm{String}(g)) < \deg(g)$ and the length is well defined.

Definition 7. A string $F = (f_i)_i$ is called *harmonic* if

$$P^*_{q,k} \cdot f_i = -(1 + qi)\, f_{i+k}, \qquad \text{for all } i, k \,. \tag{53}$$

Thanks to the relations in the q-Steenrod algebra, it is sufficient in the above definition to check that $P^*_{q,1}$ acts properly:

Proposition 19. *Let $F = (f_i)_i$ a string. Then F is harmonic if, and only if,*

$$P^*_{q,1} \cdot f_i = -(1 + qi)\, f_{i+1}, \qquad \text{for all } i \,. \tag{54}$$

Proof. This is a simple consequence of the commutation relation. Namely, we have

$$\begin{aligned} P^*_{q,k} \cdot f_l = -P^*_{q,k} P^*_{q,l} \cdot f_0 &= q(l-k) P^*_{q,(k+l)} \cdot f_0 - P^*_{q,l} P^*_{q,k} \cdot f_0 \\ &= -q(l-k) f_{k+l} + P^*_{q,l} f_k \,. \end{aligned} \tag{55}$$

The proposition follows by an easy induction. □

For example,

$$P^*_{q,2}.f_1 = -P^*_{q,2}P^*_{q,1}.f_0 = -qP^*_{q,3}.f_0 - P^*_{q,1}P^*_{q,2}.f_0 = qf_3 - (1+2q).f_3 = -(1+q)f_3 .$$

Here are some basic properties of harmonic strings:

Proposition 20. *Let $F = (f_i)_i$ be an harmonic string. Then,*

(a) *The head $f_{\ell(F)}$ of the string is harmonic;*
(b) *$f_i \neq 0$ whenever $0 \leq i \leq \ell(F)$ and $q \neq -1/i$.*

Proof. These are obvious consequences of the definitions: for $k > 0$, one has $P^*_{q,k}(f_{\ell(F)}) = f_{\ell(F)+k} = 0$ and $P^*_{q,\ell(F)-i}(f_i) = -(1+qi)f_{\ell(F)} \neq 0$. □

6.2. **Length of the strings.** Proving that no q-harmonic polynomial in $\mathbb{K}[X_{n+1}]$ has variable degree $\leq n$ is obviously equivalent to proving that all strings on $\mathbb{K}[X_n]$ have length $\leq n-1$. The rest of this section aims at some partial results in this direction.

Proposition 21. (a) *Let $q = 0$, and F be a 0-harmonic string on $\mathbb{K}[X_n]$. Then, the length of F is at most n.*
(b) *Let q be formal, and F be a q-harmonic string on $\mathbb{K}[X_n]$ of length d, so that f_0 is of the form $f_d.p$, where p is symmetric. Then, the length of F is at most n.*

(a) is a classical result, and (b) is a little extension to certain q-harmonic strings using a similar proof. Note that a q-harmonic string F for q generic will specialize at q_0 to a string $F(q_0)$ of same or smaller length. In particular, the above proposition shows that the string $F(0)$ will be of length at most $n-1$, so all the $f_i(q), i \geq n$ are divisible by q.

Proof. (a) Let F be a 0-harmonic string on $\mathbb{K}[X_n]$ of length $k > n$ and degree $d = n+1$. On n variables, there is an explicit relation between symmetric polynomials:

$$\sum_{|\lambda|=d} (-1)^{\ell(\lambda)} c_\lambda P^*_{0,\lambda} = 0 \tag{56}$$

where the c_λ are positive constants (cf. [Mac95]). This comes from the fact that, on n variables, the n first power sums generate $\mathrm{Sym}(X_n)$ as a $\mathbb{Q}$-algebra. Applying $P^*_{0,\lambda}$ to f_{k-d} yields $(-1)^{\ell(\lambda)} d_\lambda f_k$ where d_λ are polynomials in k. Hence a contradiction.

(b) Let F be a string. Thanks to the string relations,

$$P^*_{q,\lambda}(f_{k-|\lambda|}) = (-1)^{\ell(\lambda)} R_\lambda(q) f_k \tag{57}$$

for some polynomial $R_\lambda(q)$ with positive coefficients and constant term 1.

Assume now that d is the length of F, and that f_0 is of the form $f_d p$ with p symmetric. Write the matrix $(P^*_{q,\lambda}(m_\mu f_d))_{|\lambda|=|\mu|=n+1, \ell(\mu)\leq n}$. This matrix has one more column than rows. Furthermore, for $q = 0$ it has maximal rank. Hence, for q generic it is also of maximal rank, and there exists a unique linear combination of the rows, up to a scalar coefficient in $\mathbb{C}(q)$. We pick up such a linear combination whose coefficients C_λ are polynomials in $\mathbb{C}[q]$ which are relatively prime in their set:

$$\forall \mu\,, \qquad \sum_\lambda (-1)^{\ell(\lambda)} C_\lambda(q) P^*_{q,\lambda}(m_\mu f_d) = 0. \tag{58}$$

At $q = 0$, this relation specializes, up to a coefficient $c \neq 0$, to the relation between the p_λ's described in equation 56. We cancel out this coefficient, so that $C_\lambda(0) = c_\lambda$.

Applying this relation on f_{k-d} yields

$$0 = \sum_{\lambda} (-1)^{\ell(\lambda)} C_\lambda P^*_{q,\lambda}(f_0) = \left(\sum_{\lambda} C(q)_\lambda R_\lambda(q) \right) f_d. \tag{59}$$

However,

$$\sum_{\lambda} C_\lambda(0) R_\lambda(0) = \sum_{\lambda} c_\lambda \neq 0. \tag{60}$$

Hence, $f_d = 0$, which is a contradiction. □

6.3. **Complete analysis of the 2 variables case.** We work over the alphabet $X = \{x\}$, and make use of the strings machinery. For simplicity, let us use the notation $x^{[d]} := x^d/d!$. The Steenrod operators act by the rule:

$$P^*_{q,k}.x^{[d]} = (1 + q(d-k))\, x^{[d-1]}\,. \tag{61}$$

Let (f_i) be a harmonic string. Then,

$$-P^*_{q,1}.f_k = -P^*_{q,1}.(-P^*_{q,k}.f_0) = -(1+kq) P^*_{q,k+1}.f_0\,. \tag{62}$$

Define A_k so that

$$A_k(f_0) = (P^*_{q,1}P^*_{q,k} + (1+kq)P^*_{q,k+1}).f_0 = 0\,, \tag{63}$$

and suppose that $f_0 = x^{[d]}$. Then, $A_k(f_0)$ is of the form $a_k x^{[d-k]}$, with

$$a_k = (1 + q(d-k-1))(1 + q(d-k) + 1 + kq)\,. \tag{64}$$

Hence, $F = \mathrm{String}(x^{[d]})$ is harmonic if, and only if,

$$\text{for all } 0 < k < d \qquad (2+qd)(1+q(d-k-1)) = 0\,. \tag{65}$$

So, there are two possible solutions, namely $d < 2$ and $q = -2/d$.

Going back from strings to harmonic polynomials, we get the proposition:

Proposition 22. *Let q be formal or a complex number. Then, the q-harmonic polynomials in $\mathbb{K}[x,y]$ are spanned by*

$$1 \quad \textit{and} \quad (x-y)\,. \tag{66}$$

if q is not of the form $q = -2/d$ for d integer, $d \geq 2$, and by

$$1, \quad (x-y), \quad \textit{and} \quad (x-y)(x+y)^{d-1}\,. \tag{67}$$

otherwise.

In particular, if $n \geq 2$ and $d > n(n-1)/2$ the polynomial $f := (x_1 - x_2)(x_1 + x_2)^{d-1}$ is q_0-harmonic for $q_0 := -2/d$. However, the highest degree of a harmonic polynomial in n variables is $n(n-1)/2$. This shows that conjecture 2 would not hold without condition on q.

Proposition 23. *Take $d > 2$, and $n >= 2$ such that $n(n-1)/2 < d$. Then, the statement of conjecture 2 would not hold for $q_0 := -2/d$.*

7. Why is the q case more difficult than the $q = 0$ case?

The difficulties when trying to prove conjecture 2 arises from the fact that the algebra $\mathrm{Steen_q}(X_n)$ is much bigger than $\mathrm{Sym}(X_n)$. We cite here some consequences of this; in this section we will expand more on the last two items.

- $\mathbb{K}[X_n]$ is not a free $\mathrm{Steen_q}(X_n)$ module;
- The $\mathrm{Steen_q}$-modules generated by two polynomials p_1 and p_2 may be non-isomorphic, even when p_1 and p_2 are of the same degree, or in the same isotypic or Garnir component;

- There are no proper q-analogs of elementary symmetric functions e_k which vanishes on X_n when $k > n$;
- Their are very few operators which commute with the action of $\mathrm{Steen_q}$.

With this in view, conjecture 2 almost comes as a surprise: why isn't the increased size of $\mathrm{Steen_q}(X_n)$ counterbalanced by fewer q-harmonics ?

7.1. **Elementary symmetric functions do not exist.** As mentioned in section 2.5, one way to treat the classical case $q = 0$ is to construct an explicit Gröbner basis of the ideal Hit. This construction relies on the existence of the elementary e_k and complete h_k symmetric functions which satisfy for any k

$$\sum_{i=1}^{k} e_i h_{k-i} = 0, \qquad e_k(X+Y) = \sum_{i=1}^{k} e_i(X) e_{k-i}(Y), \tag{68}$$

and, last but not least,

$$e_k(X) = 0, \qquad \text{whenever } k > |X|. \tag{69}$$

Finding q-analogs of the elementary and complete symmetric functions which satisfy the first two conditions is feasible, for example by considering $\mathrm{Steen_q}$ as an appropriate quotient of the ring of non commutative symmetric functions. On the other hand, the third condition does not transfer properly. Indeed, it appears from our computations that, for $n \leq 5$, the operators $(P_{q,\lambda}(X_n))_{\lambda \vdash n+1}$ are linearly independent (see Sections A.2). Hence, no linear combination of them vanishes on X_n, and can serve as a reasonable q-analog of e_{n+1}.

7.2. **Schubert Polynomials do not exist.** In the classical case, the quotient $\mathbb{K}[X]/\operatorname{Sym}^+(X)$ is not only isomorphic to the regular representation of $\mathfrak{S}_n$ but also to many deformations of it; among them one can find Hecke algebras. The more general construction has been described by Lascoux and Schützenberger in [LS87], with a 5-parameters deformation.

Recall that the symmetric group is generated by the elementary transpositions σ_i which exchange i and $i+1$ for $i < n$. A presentation is given by the relations

$$\begin{aligned} \sigma_i^2 &= 1, && \text{for } 1 \leq i \leq n-1, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, && \text{for } |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, && \text{for } 1 \leq i \leq n-2. \end{aligned} \tag{70}$$

The last two relations are called the *braid relations*, since they give a presentation of the braid group. A decomposition $\sigma = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}$ of minimal length is called a *reduced word* for σ.

The Hecke algebras $H_n(v_1, v_2)$ are a 2-parameters family of quotients of dimension $n!$ of the braid algebra, generated by T_i with the relations

$$\begin{aligned} (T_i - v_1)(T_i - v_2) &= 0, && \text{for } 1 \leq i \leq n-1, \\ T_i T_j &= T_j T_i, && \text{for } |i-j| > 1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, && \text{for } 1 \leq i \leq n-2. \end{aligned} \tag{71}$$

Note that, as for the (q,q')-Steenrod algebra, the structure of the algebra depends only on the quotient v_1/v_2.

Here, the useful special case is $v_1 = v_2 = 0$. Indeed, $H_n(0,0)$ acts on polynomials by the *divided differences operators*

$$d_i(f) := \frac{f - \sigma_i f}{x_i - x_{i+1}}\,. \tag{72}$$

To avoid confusion with the derivations we use d_i instead of the usual notation ∂_i. Note that the d_i's are homogeneous operators of degree -1 on polynomials.

Given a reduced word $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ for a permutation σ, define $d_\sigma := d_{i_1}d_{i_2}\dots d_{i_k}$. Thanks to the braid relations, d_σ is independent of the choice of the reduced word for σ. The family $(d_\sigma)_{\sigma\in\mathfrak{S}_n}$ happens to be a basis of $H_n(0,0)$.

The key point here is the following proposition:

Proposition 24. *The divided differences commute with the action of symmetric functions: for any $p \in \mathrm{Sym}(X_n)$, any $i < n$ and any polynomial f,*

$$d_i(pf) = p\,d_i(f)\,. \tag{73}$$

The quotient $\mathbb{K}[X_n]/\,\mathrm{Sym}^+(X_n)$ is isomorphic to the regular representation of the divided differences algebra $H_n(0,0)$.

This construction was first defined in geometry, where the quotient $\mathbb{K}[X_n]/\,\mathrm{Sym}^+(X_n)$ is interpreted as the cohomology ring $H^*(F,\mathbb{Z})$ of the flag manifold F [Dem74, BGG73].

Using the divided differences, one can construct a basis of $\mathbb{K}[X_n]/\,\mathrm{Sym}^+(X_n)$ composed of the so-called Schubert polynomials:

Definition 8. Let $\rho := (n-1, n-2, \dots, 1, 0)$, so that $X^\rho = x_1^{n-1}x_2^{n-2}\dots x_{n-1}$. For each permutation $\sigma \in \mathfrak{S}_n$ the *Schubert polynomial* S_σ is defined by

$$\mathrm{S}_\sigma := d_{\sigma\omega}(x^\rho)\,, \tag{74}$$

where, for technical reasons, ω has been chosen to be the maximal permutation of $\mathfrak{S}_n$.

For more details about Schubert polynomials, we refer to [Mac91]. Note that they are denoted by $\mathfrak{S}_\sigma$ in [Mac91], and by X_σ in Lascoux and Schützenberger work [LS82, LS87].

The main theorem is

Theorem 7.1 (Lascoux-Schützenberger, 1982)**.** *The family of Schubert polynomials $(\mathrm{S}_\sigma)_{\sigma\in\mathfrak{S}_n}$ is a basis of $\mathbb{K}[X_n]$ as a free* Sym*-module. In particular, the family of Schubert polynomial $(\mathrm{S}_\sigma)_{\sigma\in\mathfrak{S}_n}$ is a basis of $\mathbb{K}[X_n]/\,\mathrm{Sym}^+$.*

Moreover the divided difference machinery provides a very efficient algorithm to decompose a polynomial in this basis.

This machinery relies heavily on the commutation property of proposition 24. One strategy to prove the conjecture 2 is to search for a family of linear operators of degree -1 which commute with the action of $\mathrm{Steen_q}$, hoping that they will generate a q-analog of the divided differences algebra. Unfortunately, we obtained by a computation that no such operator exists for, for example, $n = 1, 2, 3, 4$.

Maybe this machinery could still be made to work with some weaker skew commutation condition. A precise statement of this might be:

Problem 2. Does there exist an operator T of degree -1 on polynomials such that for any $f \in \mathrm{Steen_q}$, there exists $g \in \mathrm{Steen_q}$ such that

$$f\,T = T\,g\,. \tag{75}$$

What is the structure of the algebra generated by such operators ?

For a similar reason, we could not generalize the fact that Harm is spanned by the derivatives of the Vandermonde. Indeed, there are no reasonable q-analogs of the derivations which commute with the operators $P^*_{q,k}$'s.

Appendix A. Tables

We conclude by a series of tables of computer results which we found interesting during our exploration.

A.1. **Tables of q-harmonics.** The following listing describes the q-harmonics for $n = 2, 3, 4, 5$. Inside each irreducible representation, we choose only one polynomial. For example, in 4 variables, degree 2 the q-harmonics are of dimension 5. There is an irreducible representation of type $(3, 1)$ generated by the polynomial

$$(x_1 - x_2)((q + 1)(x_3 + x_4) - x_2 - x_1)\,,$$

and an irreducible representation of type $(2, 2)$ generated by the polynomial

$$(x_1 - x_2)(x_3 - x_4)\,,$$

which is actually a Specht polynomial.

```
////////////////////////////////////////////////////////////////////////////////
// q-Harmonics on 2 variables
////////////////////////////////////////////////////////////////////////////////
qHarmonics(2, 0, q) := // Degree 0: dimension 1
 [
  // Isotypic component [2]: 1 copy of dimension 1
  1
 ]:
qHarmonics(2, 1, q) := // Degree 1: dimension 1
 [
  // Isotypic component [1, 1]: 1 copy of dimension 1
  x1 - x2
 ]:
////////////////////////////////////////////////////////////////////////////////
// q-Harmonics on 3 variables
////////////////////////////////////////////////////////////////////////////////
qHarmonics(3, 0, q) := // Degree 0: dimension 1
 [
  // Isotypic component [3]: 1 copy of dimension 1
  1
 ]:
qHarmonics(3, 1, q) := // Degree 1: dimension 2
 [
  // Isotypic component [2, 1]: 1 copy of dimension 2
  x1 - x2
 ]:
qHarmonics(3, 2, q) := // Degree 2: dimension 2
 [
  // Isotypic component [2, 1]: 1 copy of dimension 2
  (x1 - x2)*(x3*(2*q + 2) - x2 - x1)
 ]:
qHarmonics(3, 3, q) := // Degree 3: dimension 1
 [
  // Isotypic component [1, 1, 1]: 1 copy of dimension 1
  (x1 - x2)*(x1 - x3)*(x2 - x3)
 ]:
////////////////////////////////////////////////////////////////////////////////
// q-Harmonics on 4 variables
////////////////////////////////////////////////////////////////////////////////
qHarmonics(4, 0, q) := // Degree 0: dimension 1
 [
  // Isotypic component [4]: 1 copy of dimension 1
  1
 ]:
qHarmonics(4, 1, q) := // Degree 1: dimension 3
 [
```

```
  // Isotypic component [3, 1]: 1 copy of dimension 3
  x1 - x2
 ]:
qHarmonics(4, 2, q) := // Degree 2: dimension 5
 [
  // Isotypic component [3, 1]: 1 copy of dimension 3
  (x1 - x2)*((x3 + x4)*(q + 1) - x2 - x1),
  // Isotypic component [2, 2]: 1 copy of dimension 2
  (x1 - x2)*(x3 - x4)
 ]:
qHarmonics(4, 3, q) := // Degree 3: dimension 6
 [
  // Isotypic component [3, 1]: 1 copy of dimension 3
  (x1 - x2)*(8*x1^2 + 8*x2^2 + x1*x2*(8 - 6*q) +
     + (x1*x3 + x1*x4 + x2*x3 + x2*x4)*(- 21*q - 12)
     + 24*x3*x4*(q + 1)*(3*q + 2) + (- 15*q - 12)*(x3^2 + x4^2)),
  // Isotypic component [2, 1, 1]: 1 copy of dimension 3
  (x1 - x2)*(x1 - x3)*(x2 - x3)
 ]:
qHarmonics(4, 4, q) := // Degree 4: dimension 5
 [
  // Isotypic component [2, 2]: 1 copy of dimension 2
  (x1 - x2)*(x3 - x4)*((x1*x3 + x1*x4 + x2*x3 + x2*x4)*(9*q + 6)
     - 2*(x1*x2 + x3*x4)*(q + 1)*(3*q + 4)
     + (- 2*q - 2)*(x1^2 + x2^2 + x3^2 + x4^2)),
  // Isotypic component [2, 1, 1]: 1 copy of dimension 3
  (x1 - x2)*(x1 - x3)*(x2 - x3)*(x4*(6*q + 3) - x2 - x3 - x1)
 ]:
qHarmonics(4, 5, q) := // Degree 5: dimension 3
 [
  // Isotypic component [2, 1, 1]: 1 copy of dimension 3
  (x1 - x2)*(x1 - x3)*(x2 - x3)*((x1*x2 + x1*x3 + x2*x3)*(15*q + 15*q^2 + 4)
     - 4*(x1*x4 + x2*x4 + x3*x4)*(q + 1)^2*(5*q + 2)
     - q*(q + 1)*(x1^2 + x2^2 + x3^2) + 6*x4^2*(q + 1)*(2*q + 1)*(5*q + 2))
 ]:
qHarmonics(4, 6, q) := // Degree 6: dimension 1
 [
  // Isotypic component [1, 1, 1, 1]: 1 copy of dimension 1
  (x1 - x2)*(x1 - x3)*(x1 - x4)*(x2 - x3)*(x2 - x4)*(x3 - x4)
 ]:
///////////////////////////////////////////////////////////////////////////////
// q-Harmonics on 5 variables
///////////////////////////////////////////////////////////////////////////////
qHarmonics(5, 0, q) := // Degree 0: dimension 1
 [
  // Isotypic component [5]: 1 copy of dimension 1
  1
 ]:
qHarmonics(5, 1, q) := // Degree 1: dimension 4
 [
  // Isotypic component [4, 1]: 1 copy of dimension 4
  x1 - x2
 ]:
qHarmonics(5, 2, q) := // Degree 2: dimension 9
 [
  // Isotypic component [4, 1]: 1 copy of dimension 4
  (x1 - x2)*((x3 + x4 + x5)*(2*q + 2) - 3*x2 - 3*x1),
  // Isotypic component [3, 2]: 1 copy of dimension 5
  (x1 - x2)*(x3 - x4)
 ]:
qHarmonics(5, 3, q) := // Degree 3: dimension 15
 [
  // Isotypic component [4, 1]: 1 copy of dimension 4
```

```
  (x1 - x2)*(5*x1^2 + 5*x2^2 + x1*x2*(5 - 3*q)
    + (x1*x3 + x1*x4 + x2*x3 + x1*x5 + x2*x4 + x2*x5)*(- 9*q - 5)
    + 5*(x3*x4 + x3*x5 + x4*x5)*(q + 1)*(3*q + 2)
    + (- 6*q - 5)*(x3^2 + x4^2 + x5^2)),
  // Isotypic component [3, 2]: 1 copy of dimension 5
  (x1 - x2)*(x3 - x4)*(x5*(4*q + 4) - x2 - x3 - x4 - x1),
  // Isotypic component [3, 1, 1]: 1 copy of dimension 6
  (x1 - x2)*(x1 - x3)*(x2 - x3)
 ]:
qHarmonics(5, 4, q) := // Degree 4: dimension 20
 [
  // Isotypic component [4, 1]: 1 copy of dimension 4
  (x1 - x2)*((24*q - 15)*(x1*x2^2 + x1^2*x2) - 15*x2^3
    - 4*(x1*x2*x3 + x1*x2*x4 + x1*x2*x5)*(q - 1)*(12*q + 5) - 15*x1^3
    - 12*(x1*x3*x4 + x1*x3*x5 + x2*x3*x4 + x1*x4*x5 + x2*x3*x5 + x2*x4*x5)
       *(2*q + 1)*(8*q + 5)
    + 120*x3*x4*x5*(q + 1)*(2*q + 1)*(4*q + 3)
    + (28*q + 20)*(x3^3 + x4^3 + x5^3)
    + (52*q + 20)*(x1^2*x3 + x1^2*x4 + x2^2*x3 + x1^2*x5 + x2^2*x4 + x2^2*x5)
    + (60*q + 30)*(x1*x3^2 + x1*x4^2 + x2*x3^2 + x1*x5^2 + x2*x4^2 + x2*x5^2)
    - 12*(2*q + 1)
       *(6*q + 5)*(x3*x4^2 + x3^2*x4 + x3*x5^2 + x3^2*x5 + x4*x5^2 + x4^2*x5)),
  // Isotypic component [3, 2]: 1 copy of dimension 5
  (x1 - x2)*(x3 - x4)*((x1*x3 + x1*x4 + x2*x3 + x2*x4)*(18*q + 12)
    - 3*(x1*x5 + x2*x5 + x3*x5 + x4*x5)*(q + 1)^2 + 6*x5^2*(q + 1)^2
    - (x1*x2 + x3*x4)*(q + 1)*(9*q + 13)
    + (- 4*q - 4)*(x1^2 + x2^2 + x3^2 + x4^2)),
  // Isotypic component [3, 1, 1]: 1 copy of dimension 6
  (x1 - x2)*(x1 - x3)*(x2 - x3)*((x4 + x5)*(6*q + 3) - 2*x2 - 2*x3 - 2*x1),
  // Isotypic component [2, 2, 1]: 1 copy of dimension 5
  (x1 - x2)*(x1 - x3)*(x2 - x3)*(x4 - x5)
 ]:
```

A.2. **Tables of cone dimensions.** The following tables present, for various polynomials p, the graded dimension of the *cone* $\mathrm{Steen_q}.p$ generated by p, for q generic, as well as for all the specializations at singular q_0's which yield a cone of strictly smaller dimension. For example, table 3 shows that for $p := x_1 \in \mathbb{K}[X_2]$, and in degree $2 + \deg p$ the dimension of the cone generated by p is 1 for $q_0 = -1$, and 2 for all over values of q.

Some comments are in order:

- As expected, 0 is never singular in degree $d \leq n$;
- As expected, there is no singular value for the cone generated by 1;
- For the cone generated by x_1^k in $\mathbb{K}[X_n]$, the singular values seem to be $-1/k, 0, 1/k, \dots, (n-2)/k$, appearing respectively in degree $n, n+1, \dots, n$; at this degree, the dimension falls by exactly 1. Aside for the scaling of the singular values by $1/k$, the dimensions seems to be independent of k.
- For the cone generated by the VanderMonde of all the variables in $\mathbb{K}[X_n]$, $n \geq 2$, the singular values seem to be all the rational number of the form $-2/k$, with k even or $k > n$. The singular values for $k = 2n-2, 2n-1, \dots$ appear respectively in degree $1, 2, \dots$; at this degree, the dimension falls by exactly 1.
- For $q = 0$, the cones generated by any two polynomials are obviously always isomorphic as Sym-module. As is shown by tables 18 and 19, this does not carry over to $q \neq 0$, even for two polynomials of same degree lying in the same isotypic component. Some further computations actually show that, in a Garnir component indexed by the partition $[3, 1, 1]$, and in degree 5, the higher Specht polynomial indexed by the tableau $[[5], [2], [1, 3, 4]]$ is the unique polynomial p, up to a scalar coefficient, such that -1 is singular for the cone in degree $2 + \deg p$.

d	q
1	1
2	2
3	3
4	5
5	7
6	10
7	13
8	18

TABLE 1. Graded dimension of the cone generated by $p := 1 \in \mathbb{K}[X_5]$

d	q	-1
1	1	0
2	1	0
3	1	0
4	1	0
5	1	0
6	1	0
7	1	0
8	1	0
9	1	0
10	1	0

TABLE 2. Graded dimension of the cone generated by $p := x_1 \in \mathbb{K}[X_1]$

d	q	-1	0
1	1		
2	2	1	
3	3	1	2
4	4	1	3
5	5	1	3
6	6	1	4
7	7	1	4
8	8	1	5
9	9	1	5
10	10	1	6

TABLE 3. Graded dimension of the cone generated by $p := x_1 \in \mathbb{K}[X_2]$

d	q	-1	0	1
1	1			
2	2			
3	3	2		
4	5	3	4	
5	7	3	5	6
6	10	4	7	9
7	13	4	8	11
8	17	5	10	15
9	21	5	12	18

TABLE 4. Graded dimension of the cone generated by $p := x_1 \in \mathbb{K}[X_3]$

d	q	-1	0	1	2
1	1				
2	2				
3	3				
4	5	4			
5	7	5	6		
6	11	7	9	10	
7	15	8	11	13	14
8	21	10	15	18	20
9	28	12	18	23	26

TABLE 5. Graded dimension of the cone generated by $p := x_1 \in \mathbb{K}[X_4]$

d	q	-1	0	1	2
1	1				
2	2				
3	3				
4	5				
5	7	6			
6	11	9	10		
7	15	11	13	14	
8	22	15	18	20	21

TABLE 6. Graded dimension of the cone generated by $p := x_1 \in \mathbb{K}[X_5]$

d	q	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
1	1				
2	2				
3	3				
4	5	4			
5	7	5	6		
6	11	7	9	10	
7	15	8	11	13	14
8	21	10	15	18	20

TABLE 7. Graded dimension of the cone generated by $p := x_1^2 \in \mathbb{K}[X_4]$

d	q	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$
1	1				
2	2				
3	3				
4	5	4			
5	7	5	6		
6	11	7	9	10	
7	15	8	11	13	14
8	21	10	15	18	20

TABLE 8. Graded dimension of the cone generated by $p := x_1^3 \in \mathbb{K}[X_4]$

d	q	-1	$-\frac{2}{3}$	$-\frac{2}{4}$	$-\frac{2}{5}$	$-\frac{2}{6}$	$-\frac{2}{7}$	$-\frac{2}{8}$	$-\frac{2}{9}$	$-\frac{2}{10}$	$-\frac{2}{11}$
1	1	0									
2	2	1	1								
3	2	1		1							
4	3	1			2						
5	3	1				2					
6	4	1					3				
7	4	1						3			
8	5	1							4		
9	5	1								4	
10	6	1									5

TABLE 9. Graded dimension of the cone generated by $p := x_1 - x_2 \in \mathbb{K}[X_2]$

d	q	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	0	1
1	1					
2	2	1				
3	3	2				
4	5	3			4	
5	7	3	6		5	6
6	10	4			7	9
7	13	4			8	11
8	16	5		15	10	14
9	20	5			12	17

TABLE 10. Graded dimension of the cone generated by $p := x_1 - x_2 \in \mathbb{K}[X_3]$

d	q	-1	$-\frac{1}{2}$	0	1	2
1	1					
2	2					
3	3	2				
4	5	4				
5	7	5		6		
6	11	7		9	10	
7	15	8	14	11	13	14
8	21	10		15	18	20

TABLE 11. Graded dimension of the cone generated by $p := x_1 - x_2 \in \mathbb{K}[X_4]$

d	q	-1	0	1	2
1	1				
2	2				
3	3				
4	5	4			
5	7	6			
6	11	9	10		
7	15	11	13	14	
8	22	15	18	20	21

TABLE 12. Graded dimension of the cone generated by $p := x_1 - x_2 \in \mathbb{K}[X_5]$

d	q	-1	0
1	1		
2	2		
3	3		
4	5		
5	7	6	
6	11	10	
7	15	13	14

TABLE 13. Graded dimension of the cone generated by $p := x_1 - x_2 \in \mathbb{K}[X_6]$

d	q	-1	$-\frac{2}{4}$	$-\frac{2}{5}$	$-\frac{2}{6}$	$-\frac{2}{7}$	$-\frac{2}{8}$	$-\frac{2}{9}$	$-\frac{2}{10}$	$-\frac{2}{11}$
1	1		0							
2	2	1	1	1						
3	3	2	1	2	2					
4	4	2	2	3	3	3				
5	5	3	2	4	4	4	4			
6	7	3	3	6	6	6	6	6		
7	8	4	3	7	7	7	7	7	7	
8	10	4	4	9	9	9	9	9	9	9

TABLE 14. Graded dimension of the cone generated by $p := \Delta(x_1, x_2, x_3) \in \mathbb{K}[X_3]$

d	q	-1	$-\frac{2}{4}$	$-\frac{2}{5}$	$-\frac{2}{6}$	$-\frac{2}{7}$	$-\frac{2}{8}$	$-\frac{2}{9}$	$-\frac{2}{10}$	$-\frac{2}{11}$
1	1				0					
2	2		1		1	1				
3	3	2	2		1	2	2			
4	5	4	3	4	2	3	4	4		
5	6	4	4		2	4	4	5	5	
6	9	6	5		4	6	7	7	8	8

TABLE 15. Graded dimension of the cone generated by $p := \Delta(x_1, x_2, x_3, x_4) \in \mathbb{K}[X_4]$

d	q	-1	$-\frac{2}{4}$	$-\frac{2}{6}$	$-\frac{2}{7}$	$-\frac{2}{8}$	$-\frac{2}{9}$	$-\frac{2}{10}$	$-\frac{2}{11}$	$-\frac{2}{12}$
1	1					0				
2	2			1		1	1			
3	3		2	2		1	2	2		
4	5	4	4	3	4	2	3	4	4	
5	7	6	5	4	6	2	4	5	6	6

TABLE 16. Graded dimension of the cone generated by $p := \Delta(x_1, x_2, x_3, x_4, x_5) \in \mathbb{K}[X_5]$

d	q	-1	$-\frac{2}{4}$	$-\frac{2}{6}$	$-\frac{2}{8}$	$-\frac{2}{9}$	$-\frac{2}{10}$	$-\frac{2}{11}$	$-\frac{2}{12}$	$-\frac{2}{13}$	$-\frac{2}{14}$
1	1						0				
2	2				1		1	1			
3	3			2	2		1	2	2		
4	5		4	4	3	4	2	3	4	4	
5	7	6	6	5	4	6	2	4	5	6	6

TABLE 17. Graded dimension of the cone generated by $p := \Delta(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{K}[X_6]$

d	q	$-\frac{1}{2}$
1	1	
2	2	
3	3	2
4	5	4

TABLE 18. Graded dimension of the cone generated by the higher Specht polynomial indexed by the tableau $[[4],[3],[1,2,5]]$ in the canonical Garnir component, namely $p := 2(x_2-x_1)(x_3-x_1)(x_3-x_2)(x_4^2+x_5^2) \in \mathbb{K}[X_5]$

d	q	-1	$-\frac{1}{2}$
1	1		
2	2	1	
3	3	2	2
4	5	2	4

TABLE 19. Graded dimension of the cone generated by the higher Specht polynomial indexed by the tableau $[[5],[2],[1,3,4]]$ in the canonical Garnir component, namely $p := 2(x_2-x_1)(x_3-x_1)(x_3-x_2)x_5x_4 \in \mathbb{K}[X_5]$

A.3. **Tables of singular values of q.** Let $d > 0$ and λ be a partition. A complex number q_0 is said to be a *singular value* in the isotypic component V_λ in degree d if the dimension of the q_0-harmonic polynomials in V_λ in degree d exceeds the corresponding dimension for q-harmonic polynomials. The following tables list, for each degree and isotypic component, a set which contains all singular values.

They were computed as follows: Start from a generating family M of the Sym-module $\mathbb{K}[x_1 \ldots, x_n]$, for example the staircase monomials. In the classical case the family $(p_\lambda.m)_{\ell(\lambda)\leq n, m\in M}$ is a basis of $\mathbb{K}[x_1 \ldots, x_n]$. The idea is to compute the determinant $D_{q,n,d}$ of the q-deformation $(P_{q,\lambda}.m)_{\ell(\lambda)\leq n, m\in M}$ in a given degree d. If, for a given $q = q_0$, the determinant does not vanish, then

$$\dim(\mathrm{Hit_{q_0,n,d}}) \geq \dim(\mathrm{Hit_{n,d}})\,.$$

From the computations of the tables A.1, we also know that 0 is not singular; hence:

$$\dim(\mathrm{Hit_{q_0,n,d}}) \geq \dim(\mathrm{Hit_{n,d}}) = \dim(\mathrm{Hit_{q,n,d}})\,.$$

Note that the determinant may vanish even for some non-singular q_0's, because for practical reasons we only used a subset of the Steenrod algebra $(P_{q,\lambda})_{\ell(\lambda)\leq n}$. The complete family $(P_{q,\lambda}.m)_{\lambda, m\in M}$ is not free any more for q generic, and finding the values q_0 which reduces the rank of this family is more complicated than computing a single determinant. Actually, what we did here was to consider only one of the maximal minors, instead of all of them.

We can refine these computation by working inside a given isotypic component of the representation $\mathfrak{S}_n$ or better inside one of its Garnir sub-component. This also drastically reduces the size of the considered families and determinants. Here, instead of computing the determinant of the family, we actually compute the determinant of the Gram matrix. The following tables show for λ and d the vanishing points of the determinant for the basis of the Garnir component composed of Higher Specht Polynomials (see [TY93]).

Note that the homogeneous component of lowest degree of a Garnir component V_λ is generated by its Specht polynomial S_λ; this polynomial is always q_0-harmonic, and q_0 is never singular at this degree. Consider for example the Garnir component $V_{(1,1,1)}$. The Specht polynomial $S_{(1,1,1)}$ is the Vandermonde $(x_1-x_2)(x_1-x_3)(x_2-x_3)$ of degree 3, and there are no polynomials in $V_{(1,1,1)}$ of lower degree. The first singular value is $-1/2$ in degree 4; it corresponds to the polynomial $(x_1+x_2+x_3)(x_1-x_2)(x_1-x_3)(x_2-x_3)$.

Consider now the Garnir component $V_{(3,1)}$ in degree 7. The determinant factorizes (upto a huge integer constant) in linear terms:

$$\begin{aligned} D_{q,(3,1),7} = (q+1)^{34}\,(2q+1)^6\,(3q+1)^2\,(3q+2)^6\,(4q+3)^2\,(5q+2)^4 \\ (5q+3)^2\,(7q+2)^2\,(7q+3)^2\,(7q+4)^2\,. \end{aligned}$$

As far as we have computed, all the determinants factored similarly.

λ	d	{Singular values} $\subset$
(3)	$1\dots\infty$	$\emptyset$
(2, 1)	1	$\emptyset$
(2, 1)	2	$\{-1\}$
(2, 1)	3	$\{-1, -2/3\}$
(2, 1)	4	$\{-1, -3/4, -2/3, -1/2\}$
(2, 1)	5	$\{-1, -2/3, -3/5, -1/2, -2/5\}$
(2, 1)	6	$\{-1, -2/3, -1/2, -2/5, -1/3\}$
(2, 1)	7	$\{-1, -2/3, -1/2, -3/7, -2/5, -1/3, -2/7\}$
(2, 1)	8	$\{-1, -2/3, -1/2, -2/5, -3/8, -1/3, -2/7, -1/4\}$
(2, 1)	9	$\{-1, -2/3, -1/2, -2/5, -1/3, -2/7, -1/4, -2/9\}$
(2, 1)	10	$\{-1, -2/3, -1/2, -2/5, -1/3, -3/10, -2/7, -1/4, -2/9, -1/5\}$
(1, 1, 1)	3	$\emptyset$
(1, 1, 1)	4	$\{-1/2\}$
(1, 1, 1)	5	$\{-1, -1/2, -2/5\}$
(1, 1, 1)	6	$\{-1, -1/2, -2/5, -1/3\}$
(1, 1, 1)	7	$\{-1, -1/2, -2/5, -1/3, -2/7\}$
(1, 1, 1)	8	$\{-1, -1/2, -2/5, -1/3, -2/7, -1/4\}$
(1, 1, 1)	9	$\{-1, -1/2, -2/5, -1/3, -2/7, -1/4, -2/9\}$
(1, 1, 1)	10	$\{-1, -1/2, -2/5, -1/3, -2/7, -1/4, -2/9, -1/5\}$

TABLE 20. Singular q_0's for $n = 3$, $d \leq 10$

λ	d	{Singular q_0's} $\subset$
(4)	$1\dots\infty$	$\emptyset$
(3, 1)	1	$\emptyset$
(3, 1)	2	$\{-1\}$
(3, 1)	3	$\{-1, -2/3\}$
(3, 1)	4	$\{-1, -3/4, -2/3, -1/2\}$
(3, 1)	5	$\{-1, -4/5, -3/4, -2/3, -3/5, -1/2, -2/5\}$
(3, 1)	6	$\{-1, -3/4, -2/3, -3/5, -1/2, -2/5, -1/3\}$
(3, 1)	7	$\{-1, -3/4, -2/3, -3/5, -4/7, -1/2, -3/7, -2/5, -1/3, -2/7\}$
(3, 1)	8	$\{-1, -3/4, -2/3, -3/5, -1/2, -3/7, -2/5, -3/8, -1/3, -2/7, -1/4\}$
(3, 1)	9	$\{-1, -3/4, -2/3, -3/5, -1/2, -4/9, -3/7, -2/5, -3/8, -1/3, -2/7, -1/4, -2/9\}$
(3, 1)	10	$\{-1, -3/4, -2/3, -3/5, -1/2, -3/7, -2/5, -3/8, -1/3, -3/10, -2/7, -1/4, -2/9, -1/5\}$
(2, 2)	2	$\emptyset$
(2, 2)	3	$\{-1\}$
(2, 2)	4	$\{-1\}$
(2, 2)	5	$\{-1, -3/5, -1/2\}$
(2, 2)	6	$\{-1, -2/3, -3/5, -1/2, -2/5\}$
(2, 2)	7	$\{-1, -3/5, -1/2, -3/7, -2/5, -1/3\}$
(2, 2)	8	$\{-1, -3/5, -1/2, -3/7, -2/5, -3/8, -1/3, -2/7\}$
(2, 2)	9	$\{-1, -3/5, -1/2, -3/7, -2/5, -3/8, -1/3, -2/7, -1/4\}$
(2, 2)	10	$\{-1, -3/5, -1/2, -3/7, -2/5, -3/8, -1/3, -3/10, -2/7, -1/4, -2/9\}$
(2, 1, 1)	3	$\emptyset$
(2, 1, 1)	4	$\{-1/2\}$
(2, 1, 1)	5	$\{-1, -1/2, -2/5\}$
(2, 1, 1)	6	$\{-1, -1/2, -2/5, -1/3\}$
(2, 1, 1)	7	$\{-1, -4/7, -1/2, -3/7, -2/5, -1/3, -2/7\}$
(2, 1, 1)	8	$\{-1, -1/2, -3/7, -2/5, -3/8, -1/3, -2/7, -1/4\}$
(2, 1, 1)	9	$\{-1, -1/2, -4/9, -3/7, -2/5, -3/8, -1/3, -2/7, -1/4, -2/9\}$
(2, 1, 1)	10	$\{-1, -1/2, -3/7, -2/5, -3/8, -1/3, -3/10, -2/7, -1/4, -2/9, -1/5\}$
(1, 1, 1, 1)	6	$\emptyset$
(1, 1, 1, 1)	7	$\{-1/3\}$
(1, 1, 1, 1)	8	$\{-1/2, -1/3, -2/7\}$
(1, 1, 1, 1)	9	$\{-1, -1/2, -1/3, -2/7, -1/4\}$
(1, 1, 1, 1)	10	$\{-1, -1/2, -2/5, -1/3, -2/7, -1/4, -2/9\}$

TABLE 21. Singular q_0's for $n = 4$, $d \leq 9$

λ	d	$\{$Singular q_0's$\}$ $\subset$
(5)	$1\dots\infty$	$\emptyset$
(4, 1)	1	$\emptyset$
(4, 1)	2	$\{-1\}$
(4, 1)	3	$\{-1, -2/3\}$
(4, 1)	4	$\{-1, -3/4, -2/3, -1/2\}$
(4, 1)	5	$\{-1, -4/5, -3/4, -2/3, -3/5, -1/2, -2/5\}$
(4, 1)	6	$\{-1, -5/6, -4/5, -3/4, -2/3, -3/5, -1/2, -2/5, -1/3\}$
(4, 1)	7	$\{-1, -4/5, -3/4, -5/7, -2/3, -3/5, -4/7, -1/2, -3/7, -2/5, -1/3, -2/7\}$
(4, 1)	8	$\{-1, -4/5, -3/4, -2/3, -5/8, -3/5, -4/7, -1/2, -3/7, -2/5, -3/8, -1/3, -2/7, -1/4\}$
(3, 2)	2	$\emptyset$
(3, 2)	3	$\{-1\}$
(3, 2)	4	$\{-1\}$
(3, 2)	5	$\{-1, -3/5, -1/2\}$
(3, 2)	6	$\{-1, -2/3, -3/5, -1/2, -2/5\}$
(3, 2)	7	$\{-1, -5/7, -2/3, -3/5, -4/7, -1/2, -3/7, -2/5, -1/3\}$
(3, 2)	8	$\{-1, -2/3, -5/8, -3/5, -4/7, -1/2, -3/7, -2/5, -3/8, -1/3, -2/7\}$
(3, 1, 1)	3	$\emptyset$
(3, 1, 1)	4	$\{-1/2\}$
(3, 1, 1)	5	$\{-1, -1/2, -2/5\}$
(3, 1, 1)	6	$\{-1, -1/2, -2/5, -1/3\}$
(3, 1, 1)	7	$\{-1, -4/7, -1/2, -3/7, -2/5, -1/3, -2/7\}$
(3, 1, 1)	8	$\{-1, -5/8, -4/7, -1/2, -3/7, -2/5, -3/8, -1/3, -2/7, -1/4\}$
(2, 2, 1)	4	$\emptyset$
(2, 2, 1)	5	$\{-1/2\}$
(2, 2, 1)	6	$\{-1, -1/2, -2/5\}$
(2, 2, 1)	7	$\{-1, -1/2, -3/7, -2/5, -1/3\}$
(2, 2, 1)	8	$\{-1, -1/2, -3/7, -2/5, -3/8, -1/3, -2/7\}$
(2, 1, 1, 1)	6	$\emptyset$
(2, 1, 1, 1)	7	$\{-1/3\}$
(2, 1, 1, 1)	8	$\{-1/2, -1/3, -2/7\}$

TABLE 22. Singular q_0's for $n = 5$, $d \leq 8$

λ	d	$\{$Singular q_0's$\}$ $\subset$
(6)	$1\dots\infty$	$\emptyset$
(5, 1)	1	$\emptyset$
(5, 1)	2	$\{-1\}$
(5, 1)	3	$\{-1, -2/3\}$
(5, 1)	4	$\{-1, -3/4, -2/3, -1/2\}$
(5, 1)	5	$\{-1, -4/5, -3/4, -2/3, -3/5, -1/2, -2/5\}$
(5, 1)	6	$\{-1, -5/6, -4/5, -3/4, -2/3, -3/5, -1/2, -2/5, -1/3\}$
(5, 1)	7	$\{-1, -6/7, -5/6, -4/5, -3/4, -5/7, -2/3, -3/5, -4/7, -1/2, -3/7, -2/5, -1/3, -2/7\}$
(5, 1)	8	$\{-1, -5/6, -4/5, -3/4, -5/7, -2/3, -5/8, -3/5, -4/7, -1/2, -3/7, -2/5, -3/8, -1/3, -2/7, -1/4\}$
(4, 2)	2	$\emptyset$
(4, 2)	3	$\{-1\}$
(4, 2)	4	$\{-1\}$
(4, 2)	5	$\{-1, -3/5, -1/2\}$
(4, 2)	6	$\{-1, -2/3, -3/5, -1/2, -2/5\}$
(4, 2)	7	$\{-1, -5/7, -2/3, -3/5, -4/7, -1/2, -3/7, -2/5, -1/3\}$
(4, 2)	8	$\{-1, -3/4, -5/7, -2/3, -5/8, -3/5, -4/7, -1/2, -3/7, -2/5, -3/8, -1/3, -2/7\}$
(4, 1, 1)	3	$\emptyset$
(4, 1, 1)	4	$\{-1/2\}$
(4, 1, 1)	5	$\{-1, -1/2, -2/5\}$
(4, 1, 1)	6	$\{-1, -1/2, -2/5, -1/3\}$
(4, 1, 1)	7	$\{-1, -4/7, -1/2, -3/7, -2/5, -1/3, -2/7\}$
(4, 1, 1)	8	$\{-1, -5/8, -4/7, -1/2, -3/7, -2/5, -3/8, -1/3, -2/7, -1/4\}$
(3, 3)	3	$\emptyset$
(3, 3)	4	$\{-1\}$
(3, 3)	5	$\{-1\}$
(3, 3)	6	$\{-1, -1/2\}$
(3, 3)	7	$\{-1, -4/7, -1/2, -2/5\}$
(3, 3)	8	$\{-1, -5/8, -4/7, -1/2, -3/7, -2/5, -1/3\}$
(3, 2, 1)	4	$\emptyset$
(3, 2, 1)	5	$\{-1/2\}$
(3, 2, 1)	6	$\{-1, -1/2, -2/5\}$
(3, 2, 1)	7	$\{-1, -1/2, -3/7, -2/5, -1/3\}$
(3, 2, 1)	8	$\{-1, -1/2, -3/7, -2/5, -3/8, -1/3, -2/7\}$
(3, 1, 1, 1)	6	$\emptyset$
(3, 1, 1, 1)	7	$\{-1/3\}$
(3, 1, 1, 1)	8	$\{-1/2, -1/3, -2/7\}$
(2, 2, 2)	6	$\emptyset$
(2, 2, 2)	7	$\{-1/2\}$
(2, 2, 2)	8	$\{-1/2\}$
(2, 2, 1, 1)	7	$\emptyset$
(2, 2, 1, 1)	8	$\{-1/3\}$

TABLE 23. Singular q_0's for $n = 6$, $d \leq 8$

λ	d	{Singular q_0's} $\subset$
(7)	$1\dots\infty$	$\emptyset$
(6, 1)	1	$\emptyset$
(6, 1)	2	$\{-1\}$
(6, 1)	3	$\{-1, -2/3\}$
(6, 1)	4	$\{-1, -3/4, -2/3, -1/2\}$
(6, 1)	5	$\{-1, -4/5, -3/4, -2/3, -3/5, -1/2, -2/5\}$
(6, 1)	6	$\{-1, -5/6, -4/5, -3/4, -2/3, -3/5, -1/2, -2/5, -1/3\}$
(6, 1)	7	$\{-1, -6/7, -5/6, -4/5, -3/4, -5/7, -2/3, -3/5, -4/7, -1/2, -3/7, -2/5, -1/3, -2/7\}$
(5, 2)	2	$\emptyset$
(5, 2)	3	$\{-1\}$
(5, 2)	4	$\{-1\}$
(5, 2)	5	$\{-1, -3/5, -1/2\}$
(5, 2)	6	$\{-1, -2/3, -3/5, -1/2, -2/5\}$
(5, 2)	7	$\{-1, -5/7, -2/3, -3/5, -4/7, -1/2, -3/7, -2/5, -1/3\}$
(5, 1, 1)	3	$\emptyset$
(5, 1, 1)	4	$\{-1/2\}$
(5, 1, 1)	5	$\{-1, -1/2, -2/5\}$
(5, 1, 1)	6	$\{-1, -1/2, -2/5, -1/3\}$
(5, 1, 1)	7	$\{-1, -4/7, -1/2, -3/7, -2/5, -1/3, -2/7\}$
(4, 3)	3	$\emptyset$
(4, 3)	4	$\{-1\}$
(4, 3)	5	$\{-1\}$
(4, 3)	6	$\{-1, -1/2\}$
(4, 3)	7	$\{-1, -4/7, -1/2, -2/5\}$
(4, 2, 1)	4	$\emptyset$
(4, 2, 1)	5	$\{-1/2\}$
(4, 2, 1)	6	$\{-1, -1/2, -2/5\}$
(4, 2, 1)	7	$\{-1, -1/2, -3/7, -2/5, -1/3\}$
(4, 1, 1, 1)	6	$\emptyset$
(4, 1, 1, 1)	7	$\{-1/3\}$
(3, 3, 1)	5	$\emptyset$
(3, 3, 1)	6	$\{-1/2\}$
(3, 3, 1)	7	$\{-1, -1/2, -2/5\}$
(3, 2, 2)	6	$\emptyset$
(3, 2, 2)	7	$\{-1/2\}$
(3, 2, 1, 1)	7	$\emptyset$

TABLE 24. Singular q_0's for $n = 7$, $d \leq 7$

λ	d	{Singular q_0's} $\subset$
(8)	$1\ldots\infty$	$\emptyset$
(7, 1)	1	$\emptyset$
(7, 1)	2	$\{-1\}$
(7, 1)	3	$\{-1, -2/3\}$
(7, 1)	4	$\{-1, -3/4, -2/3, -1/2\}$
(7, 1)	5	$\{-1, -4/5, -3/4, -2/3, -3/5, -1/2, -2/5\}$
(7, 1)	6	$\{-1, -5/6, -4/5, -3/4, -2/3, -3/5, -1/2, -2/5, -1/3\}$
(7, 1)	7	$\{-1, -6/7, -5/6, -4/5, -3/4, -5/7, -2/3, -3/5, -4/7, -1/2, -3/7, -2/5, -1/3, -2/7\}$
(6, 2)	2	$\emptyset$
(6, 2)	3	$\{-1\}$
(6, 2)	4	$\{-1\}$
(6, 2)	5	$\{-1, -3/5, -1/2\}$
(6, 2)	6	$\{-1, -2/3, -3/5, -1/2, -2/5\}$
(6, 2)	7	$\{-1, -5/7, -2/3, -3/5, -4/7, -1/2, -3/7, -2/5, -1/3\}$
(6, 1, 1)	3	$\emptyset$
(6, 1, 1)	4	$\{-1/2\}$
(6, 1, 1)	5	$\{-1, -1/2, -2/5\}$
(6, 1, 1)	6	$\{-1, -1/2, -2/5, -1/3\}$
(6, 1, 1)	7	$\{-1, -4/7, -1/2, -3/7, -2/5, -1/3, -2/7\}$
(5, 3)	3	$\emptyset$
(5, 3)	4	$\{-1\}$
(5, 3)	5	$\{-1\}$
(5, 3)	6	$\{-1, -1/2\}$
(5, 3)	7	$\{-1, -4/7, -1/2, -2/5\}$
(5, 2, 1)	4	$\emptyset$
(5, 2, 1)	5	$\{-1/2\}$
(5, 2, 1)	6	$\{-1, -1/2, -2/5\}$
(5, 2, 1)	7	$\{-1, -1/2, -3/7, -2/5, -1/3\}$
(5, 1, 1, 1)	6	$\emptyset$
(5, 1, 1, 1)	7	$\{-1/3\}$
(4, 4)	4	$\emptyset$
(4, 4)	5	$\{-1\}$
(4, 4)	6	$\{-1\}$
(4, 4)	7	$\{-1, -1/2\}$
(4, 3, 1)	5	$\emptyset$
(4, 3, 1)	6	$\{-1/2\}$
(4, 3, 1)	7	$\{-1, -1/2, -2/5\}$
(4, 2, 2)	6	$\emptyset$
(4, 2, 2)	7	$\{-1/2\}$
(4, 2, 1, 1)	7	$\emptyset$
(3, 3, 2)	7	$\emptyset$

TABLE 25. Singular q_0's for $n = 8$, $d \leq 7$

λ	d	{Singular q_0's} $\subset$
(9)	$1\ldots\infty$	$\emptyset$
(8, 1)	1	$\emptyset$
(8, 1)	2	$\{-1\}$
(8, 1)	3	$\{-1, -2/3\}$
(8, 1)	4	$\{-1, -3/4, -2/3, -1/2\}$
(8, 1)	5	$\{-1, -4/5, -3/4, -2/3, -3/5, -1/2, -2/5\}$
(8, 1)	6	$\{-1, -5/6, -4/5, -3/4, -2/3, -3/5, -1/2, -2/5, -1/3\}$
(7, 2)	2	$\emptyset$
(7, 2)	3	$\{-1\}$
(7, 2)	4	$\{-1\}$
(7, 2)	5	$\{-1, -3/5, -1/2\}$
(7, 2)	6	$\{-1, -2/3, -3/5, -1/2, -2/5\}$
(7, 1, 1)	3	$\emptyset$
(7, 1, 1)	4	$\{-1/2\}$
(7, 1, 1)	5	$\{-1, -1/2, -2/5\}$
(7, 1, 1)	6	$\{-1, -1/2, -2/5, -1/3\}$
(6, 3)	3	$\emptyset$
(6, 3)	4	$\{-1\}$
(6, 3)	5	$\{-1\}$
(6, 3)	6	$\{-1, -1/2\}$
(6, 2, 1)	4	$\emptyset$
(6, 2, 1)	5	$\{-1/2\}$
(6, 2, 1)	6	$\{-1, -1/2, -2/5\}$
(6, 1, 1, 1)	6	$\emptyset$
(5, 4)	4	$\emptyset$
(5, 4)	5	$\{-1\}$
(5, 4)	6	$\{-1\}$
(5, 3, 1)	5	$\emptyset$
(5, 3, 1)	6	$\{-1/2\}$
(5, 2, 2)	6	$\emptyset$
(4, 4, 1)	6	$\emptyset$

TABLE 26. Singular q_0's for $n = 9$, $d \leq 6$

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