On the Center of the Quantum Pseudoplactic Algebra

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Abstract. This paper deals with the study of the center of a q-analog of the plactic algebra, the quantum pseudoplactic algebra defined by Krob and Thibon in [3]. We prove that the center of this algebra is reduced to the algebra generated by one polynomial, the so-called quantum elementary symmetric function of maximal weight.

1 Introduction

This paper is devoted to the study of the center of the so-called quantum pseudoplactic algebra, that is a q-analog of the plactic algebra. Since the plactic algebra is a very central tool in algebraic combinatorics, the idea of finding a good q-analog of it was a long non solved problem that goes back to Schützenberger (see [7]). Such an analog was found by Krob and Thibon in [3] and called by them the quantum pseudoplactic algebra. A challenging new research direction is therefore to try to generalize to the quantum context as many properties of the plactic algebra as we can.

Let us recall that the main open problem in this area is to prove that the quantum pseudoplactic algebra is equal to the diagonal algebra of $A_q(n)$, a quantum algebra that is Dipper-Donkin version of the quantized polynomial function algebra over $n \times n$ matrices (see [1]).

In this paper, we generalize a well-known property of the plactic algebra (see [5]): the center of the plactic algebra is the algebra generated by the maximal decreasing word (*i.e.*, the elementary symmetric function of maximum weight). We prove an analog of this property in our case: the center of the quantum pseudoplactic algebra is the algebra generated by the maximal decreasing q-bracket (*i.e.*, the quantum elementary symmetric function of maximum weight).

This property is clearly not fundamental in the usual case even if it has some combinatorial interest (see [5] for more details). However, we think that the combinatorial lemmas that we had to develop for this study are probably a first step in a better understanding of the quantum pseudoplactic algebra which combinatorics seems rather difficult to penetrate.

This paper is structured as follows. We first recall the basic definitions concerning the algebra $A_q(n)$ and $\Delta_n(q)$ as one can find them in [3] (Section 2). We then define the quantum elementary symmetric functions (Section 3.1) and present our main theorem (Section 3.2). The next part of the paper is devoted to the proof of this result. In Section 3.3, we prove that Λ_n belongs to the center of the quantum pseudoplactic algebra. We prove the reciprocal proposition in Section 3.4.

2 Background

In this section, we provide the reader the basic definitions as one could find them in [3].

2.1 The Hopf algebra $A_{\mathrm{q}}(N)$

The quantum group $A_q(n)$ is the $\mathbb{C}(q)$-algebra generated by the n^2 elements $(x_{ij})_{1\le i,j\le n}$ subject to the defining relations :

$$\begin{cases} x_{jk}\,x_{il} = q\,x_{il}\,x_{jk} & \text{for } i<j,\ k<l, \\ x_{ik}\,x_{il} = x_{il}\,x_{ik} & \text{for every } i,k,l, \\ x_{jl}\,x_{ik} - x_{ik}\,x_{jl} = (q-1)\,x_{il}\,x_{jk} & \text{for } i<j, k<l. \end{cases} \tag{1}$$

This algebra is a quantization of the bialgebra of polynomial functions on the variety of $n\times n$ matrices over $\mathbb{C}$ introduced by Dipper and Donkin in [1]. Up to the symmetry $x_{ij}\leftrightarrow x_{ji}$, it is also the specialization $GL_{q,1}(n)$ of Takeuchi's algebra [8].

The quantum determinant is a very important element of this Hopf algebra (see [1, 2]). The basic observation is the following lemma.

Lemma 1 ([3]). *The quantum determinant of $A_q(n)$ can be expressed by means of q-commutators as follows:*

$$\begin{aligned} \begin{vmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix}_q &\overset{\text{def}}{=} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\,x_{1\sigma(1)}\,\dots\,x_{n\sigma(n)} \\ &= \frac{1}{(1-q)^{n-1}}\,[x_{nn},[\dots,[x_{22},x_{11}]_q\dots]_q]_q \end{aligned} \tag{2}$$

where $[P,Q]_q = P\,Q - q\,Q\,P$.

2.2 The quantum pseudoplactic algebra

The quantum diagonal algebra $\Delta_q(n)$ is the subalgebra of $A_q(n)$ generated by $x_{11}, \ldots, x_{nn}$, *i.e.*,

$$\Delta_q(n) = \mathbb{C} < x_{11}, x_{22}, \ldots, x_{nn} > .$$

Let A be a totally ordered alphabet. The quantum pseudoplactic algebra $PPl_q(A)$ is the quotient of $\mathbb{C}(q){<}A{>}$ by the relations:

$$\begin{cases} q\,aab - (q+1)\,aba + baa = 0 & \text{if } a < b, \\ q\,abb - (q+1)\,baa + bba = 0 & \text{if } a < b, \\ \quad cab - acb - bca + bac \;\; = 0 & \text{if } a{<}b{<}c. \end{cases} \tag{3}$$

The motivation for the introduction of $PPl_q(A)$ comes from the following conjecture. Note that the conjecture may only be true for generic values of q since it is not the case for $q = 1$ and $q = 0$ (see [3, 6]).

Conjecture 1. Let $A = \{a_1, \ldots, a_n\}$ be a totally ordered alphabet of order n. For generic q, the mapping $\phi : a_i \mapsto x_{ii}$ induces an isomorphism between $PPl_q(A)$ and the diagonal algebra $\Delta_q(n)$.

3 Center of the quantum plactic algebra

3.1 Quantum elementary symmetric functions

Let $V = \oplus_{i=1}^{n} \mathbb{C}\zeta_i$ where the ζ_i verify the relations $\zeta_i\zeta_j = -\zeta_j\zeta_i$ for all i, j. We define the mapping

$$\Delta : V \to A_q(n) \otimes V$$

$$\zeta_i \mapsto \sum_{j=1}^{n} x_{ij} \otimes \zeta_j$$

Let us consider $\Lambda_k(V) = \oplus_{i_1<i_2<\cdots<i_k} \mathbb{C}\zeta_{i_1}\,\zeta_{i_2}\cdots\zeta_{i_k}$.

The following formula holds.

$$\begin{aligned} \chi(\Lambda_k(V)) &= \sum_{i_1<\cdots<i_k} \begin{vmatrix} x_{i_1i_1} & \ldots & x_{i_1i_k} \\ \vdots & \ddots & \vdots \\ x_{i_ki_1} & \ldots & x_{i_ki_k} \end{vmatrix}_q \\ &= \frac{1}{(1-q)^{k-1}} \sum_{i_1<\cdots<i_k} [x_{i_ki_k}, [\cdots[x_{i_2i_2}, x_{i_1i_1}]_q \cdots]_q \overset{\text{def}}{=} \Lambda_k(q), \end{aligned} \tag{4}$$

according to Lemma 1. This last element of the diagonal algebra will be called the *quantum elementary symmetric function* of weight k.

3.2 Center of the algebra

The aim of this paper is to prove the next theorem:

Theorem 1. *The center of the quantum pseudoplactic algebra is* $\mathbb{C}<\Lambda_n>$.

We will make the proof in two parts. First, we prove that Λ_n commutes with every letter, so that Λ_n belongs to the center of the pseudoplactic algebra. Next, we will prove that if an element L of the pseudoplactic algebra commutes with every letter, then it is a polynomial in Λ_n with coefficients in $\mathbb{C}$.

3.3 Λ_{n} belongs to the center of $\Delta_{\mathrm{q}}(n)$

The main proposition In this subsection, we prove the following result:

Proposition 1. *For every i, one has*

$$\Lambda_n a_i = a_i \Lambda_n,$$

except in the case $i = n - 1 = 2$ and $q = -1$.

A last notation and Schützenberger's involution The proof of this proposition decomposes as several lemmas we will prove further. Let us introduce two useful tools.

Notation 1 *Let $1 \leq i \leq j \leq n$ be two integers. We define $\Lambda_{j,i}$ as the following formula:*

$$\Lambda_{j,i} = [a_j, [a_{j-1}, \ldots, [a_{i+1}, a_i]_q \ldots]_q.$$

Since we will make use of Schützenberger's involution in the proof of Lemma 5, let us recall its definition. The Schützenberger's involution is an involution defined on words as follows:

$$\begin{array}{lcl} \# : PPl_q(A, <) & \rightarrow & PPl_q(A, >) \\ \quad A = \{a_1 < \cdots < a_n\} & & A = \{a_1 > \cdots > a_n\} \\ \quad x_1 \cdots x_k & \mapsto & x_k \cdots x_1 \end{array} \tag{5}$$

One can easily see that this involution exchanges the first and the second defining equations of the quantum pseudoplactic algebra and transforms the third one in itself. Moreover, $\#^2 = Id$, so that $\#$ is an involution that transforms quantum relations in other quantum relations.

Schützenberger's involution acts on $PPl_q(A)$ as if we had the relation $\#(a_k) = a_{n-k+1}$. We will make the proof with this convention. Notice also that we have the relation:

$$\#([a, b]_q) = \#(ab - qba) = \#(b)\#(a) - q\#(a)\#(b) = [\#b, \#a]_q. \tag{6}$$

This relation is also true for $q = 1$, that is, for the usual bracket.

Proof of the proposition Before giving the proof of the proposition, let us give our five technical lemmas.

Lemma 2. *Let B be a subset of the alphabet A. Let B_q be the set of all q-brackets with letters taken in B (that is, B_q is the transitive closure of B under the q-bracket). Let a and c be two letters such that $a < b < c$ for all $b \in B$. Then, if u belongs to B_q, one has:*

$$[u,[a,c]] = 0.$$

Lemma 3. *For all $i \leq j$, one has:*

$$\Lambda_{j,i} = [\cdots[a_j, a_{j-1}]_q, \ldots, a_i]_q.$$

Lemma 4. *For all $i \leq j < n$, one has:*

$$[a_n, [a_n, \lambda_{j,i}]_q] = 0.$$

Lemma 5. *For all $1 < i \leq j$, one has:*

$$[[\Lambda_{j,i}, a_1]_q, a_1] = 0.$$

Lemma 6. *For all $1 < i < n$, one has:*

$$[a_i, \Lambda_n] = 0,$$

except in the case $i = n - 1 = 2$ and $q = -1$.

Proof of the proposition — Taking $j = n-1$ and $i = 1$, Lemma 4 proves that a_n commutes with Λ_n. Taking $j = n$ and $i = 2$, Lemma 5 proves that a_1 commutes with Λ_n. Finally, Lemma 6 proves that a_i commutes with Λ_n for all $1 < i < n$, except in the case $q = -1$, $i = n - 1 = 2$. ∎

Proof of the lemmas *Proof of lemma 2* — We make the proof using the recursive definition of B_q:

$$B \in B_q \quad \text{and} \quad \alpha, \beta \in B_q \Longrightarrow [\alpha, \beta]_q \in B_q.$$

- $u = b \in B$. The relation is $[b,[a,c]] = 0$, which is true since it is the third defining relation of $PPl_q(A)$ (see relations 3).
- $u = [\alpha, \beta]_q$ with $\alpha, \beta \in B_q$. The proof comes from the following identity

  $$\forall x, y, z, \quad [[x,y]_q, z] = [x,[y,z]]_q + [[x,z],y]_q \tag{7}$$

  with $x = \alpha$, $y = \beta$, $z = [a,c]$. This identity itself is proved by direct expansion.

∎

Proof of lemma 3 — We make the proof by induction on $j - i$.

– If $j-i=0$ or $j-i=1$, there is nothing to prove.
– If $j-i=2$, we use the following identity

$$\forall x, y, z, \quad [[x,[y,z]_q]_q = [[x,y]_q, z]_q + q[y,[x,z]] \tag{8}$$

with $x=a_j$, $y=a_{j-1}$ and $z=a_{j-2}=a_i$ and then remind that $[y,[x,z]]=0$ thanks to Lemma 2. We prove this identity by direct expansion.
– Let us now make the proof in the general case:

$$\begin{aligned}
\Lambda_{j,i} &= [a_j, \Lambda_{j-1,i}]_q \\
&= [a_j, [\Lambda_{j-1,i+1}, a_i]_q]_q && \text{(by induction hypothesis)} \\
&= [[a_j, \Lambda_{j-1,i+1}]_q, a_i]_q + q[\Lambda_{j-1,i+1}, [a_j, a_i]] && \text{(by identity (8))} \\
&= [[a_j, \Lambda_{j-1,i+1}]_q, a_i]_q && \text{(by Lemma 2)} \\
&= [\Lambda_{j,i+1}, a_i]_q && \text{(by definition of } \Lambda_{j,i+1}) \\
&= [\cdots[a_j, a_{j-1}]_q, \ldots, a_{i+1}]_q, a_i]_q && \text{(by induction hypothesis)}
\end{aligned}$$

■

Proof of lemma 4 — We make the proof by induction on $j-i$.

– If $j-i=0$, we have to prove

$$[a_n, [a_n, a_i]_q] = 0$$

that is true since it is the second defining relation of $PPl_q(A)$ with $a=a_i$ and $b=a_n$ (see relations 3).
– Let us make the proof in the general case. Let us denote by λ the expression $\Lambda_{j,i+1}$. We will use the two following relations in the proof:
  - This relation comes from the induction hypothesis:

$$a_n^2 \lambda = (1+q) a_n \lambda a_n - q \lambda a_n^2. \tag{9}$$

  - This relation comes from the second defining relation of $PPl_q(A)$ as we saw before:

$$a_n^2 a_i = (1+q) a_n a_i a_n - q a_i a_n^2. \tag{10}$$

The proof follows then from the following calculations:

$$\begin{aligned}
[a_n, [a_n, \Lambda_{j,i}]_q] &= [a_n, [a_n, [\lambda, a_i]_q]_q] \\
&= a_n^2 [\lambda, a_i]_q - (q+1) a_n [\lambda, a_i]_q a_n + q[\lambda, a_i]_q a_n^2 \\
&= a_n^2 \lambda a_i - q a_n^2 a_i \lambda \\
&\quad -(1+q) a_n \lambda a_i a_n + q \lambda a_i a_n^2 \\
&\quad -q^2 a_i \lambda a_n^2 + q(1+q) a_n a_i \lambda a_n \\
&= (1+q) a_n \lambda a_n a_i - q \lambda a_n^2 a_i \quad \text{(by identity (9))}
\end{aligned}$$

$$
\begin{aligned}
& -q(1+q)a_n a_i a_n \lambda + q^2 a_i a_n^2 \lambda \quad \text{(by identity (10))} \\
& -(1+q)a_n \lambda a_i a_n + q\lambda a_i a_n^2 \\
& -q^2 a_i \lambda a_n^2 + q(1+q)a_n a_i \lambda a_n \\
= {} & (1+q)(a_n\lambda[a_n, a_i] + q a_n a_i[\lambda, a_n]) \\
& +q\lambda(a_i a_n^2 - a_n^2 a_i) + q^2 a_i(a_n^2\lambda - \lambda a_n^2) \\
= {} & (1+q)(a_n\lambda[a_n, a_i] + q a_n a_i[\lambda, a_n]) \\
& -q(1+q)\lambda[a_n, a_i]a_n \qquad \text{(by identity (9))} \\
& +q^2(1+q)a_i[a_n, \lambda]a_n \qquad \text{(by identity (10))} \\
= {} & (1+q)(a_n\lambda[a_n, a_i] - q\lambda[a_n, a_i]a_n) \\
& +(1+q)(q a_n a_i[\lambda, a_n] + q^2 a_i[a_n, \lambda]a_n) \\
= {} & (1+q)(a_n\lambda[a_n, a_i] - \lambda a_n[a_n, a_i]) \\
& \qquad \text{(by identity (10))} \\
& +(1+q)(-q a_n a_i[a_n, \lambda] + q a_i a_n[a_n, \lambda]) \\
& \qquad \text{(by Lemma 2 and identity (9))} \\
= {} & (1+q)(a_n[a_n, a_i]\lambda - q\lambda[a_n, a_i]_n) \\
& \qquad \text{(by Lemma 2 and identity (10))} \\
& +(1+q)(-q[a_n, a_i][a_n, \lambda]) \\
= {} & (1+q)(q[a_n, a_i]a_n\lambda - q[a_n, a_i]\lambda a_n) \\
& \qquad \text{(by identity (10) and Lemma 2)} \\
& -(1+q)(q[a_n, a_i]a_n\lambda - q[a_n, a_i]\lambda a_n) \\
= {} & 0.
\end{aligned}
$$

■

Proof of lemma 5 — One can easily make the proof in a very similar way to the proof of Lemma 4. We will avoid this proof using Schützenberger's involution. Thanks to Lemma 4, we know that

$$[a_n, [a_n, \Lambda_{j,i}]_q] = 0,$$

for all $i < j < n$. So, we have

$$
\begin{aligned}
& [a_n, [a_n, \Lambda_{j,i}]_q] = 0 \\
\Longleftrightarrow {} & \#([a_n, [a_n, \Lambda_{j,i}]_q]) = 0 && \text{(by involution } \#) \\
\Longleftrightarrow {} & [\#([a_n, \Lambda_{j,i}]_q), \#a_n] = 0 && \text{(by identity (6) with } q = 1) \\
\Longleftrightarrow {} & [[\#(\Lambda_{j,i}), \#a_n]_q, a_1] = 0 && \text{(by identity (6))} \\
\Longleftrightarrow {} & [[\Lambda_{n-i+1,n-j+1}, a_1]_q, a_1] = 0 && \text{(by induction on identity (6))}
\end{aligned}
$$

Since the application that sends all pairs (i, j) such that $i < j < n$ to the pairs (i', j') such that $1 < i' < j'$ defined by $i' = n - j + 1$ and $j' = n - i + 1$ is a bijection, we conclude our proof. ■

Proof of lemma 6 — We make the proof by induction on n. First, we have the following equalities:

$$\begin{aligned}
[a_i, A_n] &= [a_i, [a_n, A_{n-1,1}]_q] \\
&= -[[a_n, A_{n-1,1}]_q, a_i] \\
&= -[[a_n, a_i], A_{n-1,1}]_q - [a_n, [A_{n-1,1}, a_i]]_q \quad \text{(by identity (7))} \\
&= [[a_i, a_n], A_{n-1,1}]_q \qquad\qquad\qquad \text{(by induction hypothesis)}
\end{aligned}$$

Using the induction hypothesis we derive the following identity:

$$\text{if } i \neq n-1, \qquad [[a_i, a_n], A_{n-2,1}]_q = 0. \tag{11}$$

Starting from the last equation, we derive

$$\begin{aligned}
[a_i, A_n] &= [[a_i, a_n], [a_{n-1}, A_{n-2,1}]_q]_q \\
&= -q(a_{n-1}A_{n-2,1} - qA_{n-2,1}a_{n-1})[a_i, a_n] \\
&\quad +[a_i, a_n](a_{n-1}A_{n-2,1} - qA_{n-2,1}a_{n-1}) \\
&= -q(a_{n-1}A_{n-2,1} - qA_{n-2,1}a_{n-1})[a_i, a_n] \\
&\quad +q(a_{n-1}A_{n-2,1} - qA_{n-2,1}a_{n-1})[a_i, a_n] \\
&\qquad \text{(by identity (11) and relations of } Ppl_q(A) \text{ with } i \neq n-1) \\
&= 0.
\end{aligned}$$

If $i = n-1$, we can apply Schützenberger's involution and prove the identity with $i = 2$. So the only case that remains to study is when $n-1 = 2 = i$, that is $n = 3$. The equation is:

$$\begin{aligned}
& [a_2, [a_3, [a_2, a_1]_q]_q] = 0 \\
\Longleftrightarrow\ & [[a_3, a_2], [a_2, a_1]_q]_q = 0 \quad \text{(thanks to the previous computations)} \\
\Longleftrightarrow\ & [a_3, a_2][a_2, a_1]_q = q[a_2, a_1]_q[a_3, a_2]
\end{aligned}$$

Let us consider the left member of the last equation.

$$\begin{aligned}
[a_3, a_2][a_2, a_1]_q &= (a_3a_2 - a_2a_3)(a_2a_1 - qa_1a_2) \\
&= (a_3a_2^2a_1 - a_2a_3a_2a_1 - q(a_3a_2a_1a_2 - a_2a_3a_1a_2) \\
&= q(a_2a_3a_2a_1 - a_2^2a_3a_1 - a_3a_2a_1a_2 + a_2a_3a_1a_2) \\
&\qquad\qquad \text{(by relations of } Ppl_q(A))
\end{aligned}$$

Let us define $X = a_2a_3a_2a_1 - a_2^2a_3a_1 - a_3a_2a_1a_2 + a_2a_3a_1a_2$. We are going to prove that $(1+q)X = (1+q)[a_2, a_1]_q[a_3, a_2]$.

$$\begin{aligned}
(1+q)X &= (1+q)(a_2a_3a_2a_1 - a_2^2a_3a_1 - a_3a_2a_1a_2 + a_2a_3a_1a_2) \\
&= (a_3a_2^2a_1 + qa_2^2a_3a_1) - (1+q)a_2^2a_3a_1 \\
&\quad -(a_3a_2^2a_1 + qa_3a_1a_2^2) + (1+q)a_2a_3a_1a_2 \\
&\qquad\qquad \text{(by the first two relations of } Ppl_q(A)) \\
&= -a_2^2a_3a_1 + a_2a_3a_1a_2 + qa_2a_3a_1a_2 - qa_3a_1a_2^2 \\
&\qquad\qquad \text{(after simplifications)}
\end{aligned}$$

$$
\begin{aligned}
&= a_2(-a_2a_3a_1 + a_3a_1a_2) + q(a_2a_3a_1 - a_3a_1a_2)a_2 \\
&= a_2(a_1a_3a_2 - a_2a_3a_1) + q(a_2a_1a_3 - a_1a_3a_2)a_2 \\
&\qquad\qquad\qquad\qquad \text{(by the third relation of } Ppl_q(A)) \\
&= (1+q)(a_2a_1a_3a_2) - a_2^2a_3a_1 - qa_1a_3a_2^2 \\
&= (1+q)(a_2a_1a_3a_2) - (qa_1a_2^2a_3 + a_2^2a_1a_3) \\
&\quad +(qa_1a_2^2a_3 + a_1a_3a_2^2) - (1+q)a_1a_3a_2^2 \\
&= (1+q)(a_2a_1a_3a_2 - a_2a_1a_2a_3 + a_1a_2a_3a_2 - a_1a_3a_2^2) \\
&= (1+q)(a_2a_1a_3a_2 - a_2a_1a_2a_3 - qa_1a_2a_3a_2 + qa_1a_2^2a_3) \\
&= (1+q)[a_2, a_1]_q[a_3, a_2]
\end{aligned}
$$

Dividing by $1+q$, one obtains that $[a_2, \Lambda_3] = 0$.

Notice that when $q = -1$, the equality does not hold since

$$
\begin{aligned}
& [a_3, a_2][a_2, a_1]_q = q[a_2, a_1]_q[a_3, a_2] \\
\iff & a_3a_2^2a_1 + a_3a_2a_1a_2 - a_2a_3a_2a_1 - a_2a_3a_1a_2 \\
& = -a_2a_1a_3a_2 + a_2a_1a_2a_3 - a_1a_2a_3a_2 + a_1a_2a_2a_3 \\
\iff & (a_3a_1a_2 - a_2a_3a_1)a_2 + a_3a_2a_1a_2 - a_2a_3a_2a_1 \\
& = (-a_2a_1a_3 + a_1a_3a_2)a_2 + a_2a_1a_2a_3 - a_1a_2a_3a_2 \\
\iff & a_3a_2a_1a_2 - a_2a_3a_2a_1 = a_2a_1a_2a_3 - a_1a_2a_3a_2 \\
& \qquad\qquad\qquad \text{(by the third relation of } PPl_q(A))
\end{aligned}
$$

This last equation is not true since we cannot rewrite any of the corresponding words using the relations of $PPl_q(A)$. ■

3.4 The center of the pseudoplactic algebra is equal to $\mathbb{C} < \Lambda_n >$

In this subsection, we complete the proof of Proposition 1.

Proposition 2. *Let P be in the center of the pseudoplactic algebra. Then P is a polynomial (over $\mathbb{C}$) in Λ_n.*

Proof of the proposition — We only give a sketch of this proof which is too long to be put here.

- First, we prove this proposition with $q = 1$, in which case the pseudoplactic algebra is the enveloping algebra of a Lie algebra.
- Next, thanks to the first part, we know that P is equal to Λ_n plus $q-1$ times a new polynomial. We can then divide by $q-1$ and return to the first step. Since this process can only repeat a finite number of times, we conclude to the theorem.

■

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