On the hypoplactic monoid

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Résumé

Cet article présente une étude combinatoire du monoïde hypoplaxique qui est l'analogue du monoïde plaxique dans le cadre des fonctions symétriques non-commutatives. Après en avoir rappelé la construction par réécritures, nous en donnons une nouvelle définition et utilisons celle-ci pour démontrer combinatoirement que chaque classe hypoplaxique contient un et un seul mot quasi-ruban. Nous démontrons ensuite des analogues hypoplaxiques des résultats classiques sur le monoïde plaxique et en particulier, nous étudions les analogues des fonctions de Schur.

Abstract

This paper presents a combinatorial study of the hypoplactic monoid that is the analog of the plactic monoid in the theory of noncommutative symmetric functions. After having recalled its definition using rewritings, we provide a new definition and use this one to combinatorially prove that each hypoplactic class contains exactly one quasi-ribbon word. We then prove hypoplactic analogues of classical results of the plactic monoid and, in particular, we make the study of the analogues of Schur functions.

1 Introduction

A new theory of noncommutative symmetric functions was developed in [5, 8, 3, 9, 10] (they were already implicitly studied in [13]). It appears that the dual of the algebra of noncommutative symmetric functions can be identified with the algebra of quasi-symmetric functions that was already defined by Gessel in [6]. This last algebra has a remarkable basis (the so-called quasi-ribbon functions) that plays the role of the Schur functions in the context of quasi-symmetric functions. There was therefore a need of a representation theoretical interpretation of quasi-symmetric functions. This interpretation was obtained in [9, 10] where Krob and Thibon showed that the quasi-ribbon functions are:

- the characteristics of the irreducible modules over the 0-Hecke algebra,
- the characters of the irreducible comodules considered at $q = 0$ over a deformation of the ring of polynomial functions on the variety of $n \times n$ matrices introduced by Dipper and Donkin in [2],

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– the characters of the polynomial irreducible modules over a crystalizable version of the quantum deformation $U_q(Gl_n)$ considered at $q = 0$ of the enveloping algebra of the Lie algebra gl_n, introduced in [10] (this deformation was also essentially obtained by Takeuchi in [16]).

Similar interpretations of noncommutative symmetric functions can also be obtained by considering indecomposable modules or comodules over the previous quantum structures. The characters of the different modules or comodules that occur in these different quantum groups considered at $q = 0$ belong in fact to a remarkable algebra: the hypoplactic algebra which is a quotient of the plactic algebra by new quartic relations. It is interesting, for instance, that the hypoplactic algebra has two remarkable subalgebras, one of which is isomorphic to the algebra of quasi-symmetric functions and the other of which is isomorphic to the algebra of symmetric functions.

The purpose of this paper is therefore to study combinatorially this character algebra. It is structured as follows. We first give the notations, recall some classical definitions (Section 2) and define the main objects of our study: the quasi-ribbon tableaux (Section 3). We then introduce the hypoplactic monoid recalling its classical definition and present a new definition (Section 4). We can then establish the formula for the order of each class, show that its structure is compatible with the involution of Schützenberger and with the restriction of the alphabet, that the quasi-ribbon words are the smallest elements of their class (Section 5). Next, we begin the study of the F_I functions and then compute how the product of two of them decomposes in their basis (Section 6).

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2 Basic definitions and notations

2.1 Compositions

A *composition* I of n is a sequence of integers $(i_1, i_2, \ldots, i_r)$ called the parts such that the sum of all the parts is equal to n. The *weight* of the composition is n, its *length* is r.

We associate with the composition $I = (i_1, \ldots, i_r)$ the set $D(I) = \{d_1, \ldots, d_{r-1}\}$ defined as $d_l = i_1 + \ldots + i_l$ *for* $l \in [1, r-1]$. A composition J is said to be *finer* than a composition I iff $D(J)$ contains $D(I)$. In this case, we write $J \geq I$ or equivalently, $I \leq J$. For example, $(1,1,3,2)$ is finer than $(1,1,3,2)$, $(2,3,2)$, $(1,4,2)$, $(1,1,5)$, $(5,2)$, $(2,5)$, $(1,6)$ and (7).

Let σ be a permutation. The *descent set* of σ is the set of all l such that $\sigma(l) > \sigma(l+1)$. This set is denoted by $D(\sigma)$. The *descent composition* associated with σ is by definition the unique composition $C(\sigma)$ of weight n such that $D(C(\sigma)) = D(\sigma)$.

If I denotes a composition, D_I is defined as the formal sum of all permutations that have their descent set equal to $D(I)$. This sum can be seen as an element of the algebra $\mathbb{Z}[\mathfrak{S}_n]$. These elements were first studied by Solomon [15] who proved that they form a basis of a subalgebra of $\mathbb{Z}[\mathfrak{S}_n]$ called the *descent algebra of* $\mathfrak{S}_n$ and denoted by Σ_n.

A *ribbon diagram* is a graphical representation of a composition by means of a skew Young diagram such that the l^{th} row contains i_l cells (see Figure 1). I is called the shape of the ribbon diagram.

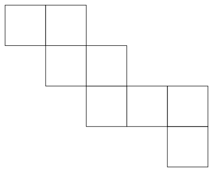

Figure 1: The ribbon diagram of $(2, 2, 3, 1)$

There is another way to encode a composition. We associate with a composition a word built on the alphabet $\{E, S\}$ in the following way. Take the associated ribbon diagram. Number its cells from top to bottom and from left to right in the rows. Then, the l^{th} letter of our word is E if one needs an east move to go from the cell numbered l to the cell numbered $l+1$ and S if one needs a south move. For example, the coding associated with the composition $(2, 2, 3, 1)$ is $ESESEES$. It is then easy to see that this process defines a bijection between the words of length n and the compositions of weight $n+1$.

The *conjugate* of a composition I can be defined geometrically: it is the composition corresponding to the sequence composed of the number of cells in each column (from right to left) of the ribbon diagram of I. We will denote it by $I^{\sim}$. For example, the conjugate of $(2, 2, 3, 1)$ is $(2, 1, 2, 2, 1)$.

2.2 The alphabet

In the sequel, A will always denote a finite ordered alphabet. We will use the natural integers for the elements of A. The letters will stand for variables in the statements. Most of the time, we will also omit the alphabet in our statements. But the reader should keep in mind that virtually all our results depend on the alphabet (i.e., on its length). Sometimes, it will be easier to work with an infinite alphabet to avoid some problems. This will be specified in the text. If w is a word, we will denote its length by $|w|$.

Definition 2.1 *We will denote by $\mathcal{E}$ the mapping that maps each word to its evaluation vector (i.e., the vector whose k^{th} component is equal to the number of occurrences of the k^{th} letter of A). We will denote by $\mathcal{E}_k$ the mapping that sends each word w to the k^{th} entry of $\mathcal{E}(w)$.*

2.3 Standardization and shuffle product

A word is said to be *standard* iff all its letters are distinct and the set of all its letters is a beginning interval of A. For example, if $A = \{1 < 2 < 3 < 4 < 5\}$, the word 312 is a standard word whereas 2354 is not.

Let w be a word. The *standardized* word of w is the word $St(w)$ built by the following process. Reading w from left to right, label with $1, 2, \ldots$ the successive occurrences of each letter. One obtains a word in distinct labelled letters. Regarding them as elements of the alphabet $A \times \mathbb{N}$, one can replace each labelled letter by the integers $1, 2, \ldots$ according to its rank in this new alphabet, endowed with the lexicographic order.

For example,

$$w = 1211343 \longrightarrow 1_1 2_1 1_2 1_3 3_1 4_1 3_2 \longrightarrow 1423576.$$

Let us take an equivalent definition of the standardization process which is more helpful in the proofs. Let $w = w_1 \ldots w_p$ be a word of length p. The standardized word of w is the permutation (in $\mathfrak{S}_p$) defined by $St(w)(k) < St(w)(l)$ iff $(w_k < w_l)$ or $(w_k = w_l$ and $k < l)$.

Lemma 2.2 *Let w and w' be two words having the same standardized word. Then each prefix of w has the same standardized word as the prefix of same length of w'. The property is the same for the suffixes of w and w'.*

Proof — The proof is immediate: w and w' have the same standardized word. So, $w_k \leq w_l$ iff $w'_k \leq w'_l$ for all pairs $k < l$. So it is true for all pairs $k < l$ smaller than the length of the prefix or greater than this length. □

Note 2.3 Notice that this process allows us to define the descent set (resp. the descent composition) associated with a word as the descent set (resp. the descent composition) associated with its standardized word.

Definition 2.4 *The* shuffle product *can be recursively defined by the formula*

$$au \sqcup\!\sqcup\, bv = a(u \sqcup\!\sqcup\, bv) + b(au \sqcup\!\sqcup\, v)$$

where $a, b \in A$ and $u, v \in A^*$.

For example, the shuffle product of 12 and 34 is $1234+1324+1342+3124+3142+3412$.

3 Quasi-ribbon tableaux

In this section, we define some objects that were first introduced by Gessel in [6] and used as a main tool in [9]. They are the central elements of our study and we will see later how they naturally appear in the study of the hypoplactic monoid. We then derive some of their properties. In the second part, we concentrate on the standard case to establish some specific properties and simplify some of the previous ones that were already proved in a more general case (see [9]).

3.1 Definition and Basic properties

Definition 3.1 *Let I be a composition. A* quasi-ribbon tableau *of shape I is a ribbon diagram r of shape I filled with letters of A in such a way that*

- *each row of r is nondecreasing from left to right,*
- *each column of r is strictly increasing from top to bottom.*

Notice that this notion is different from the classical notion of ribbon tableau: in a ribbon tableau, the columns are strictly increasing from bottom to top.

We now define two different readings of a filling of a ribbon diagram. The *canonical reading* consists in reading from *bottom to top* and from left to right the columns of the filling. The *column reading* consists in reading from *top to bottom* and from left to right the columns of the filling.

Let us consider the filling of a ribbon shown in Figure 2.

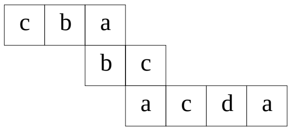

Figure 2: A filling of a ribbon diagram

The canonical reading of the ribbon diagram is the word $cbbaaccda$. Its column reading is $cbabcacda$.

Notice that, if I is a composition of weight n, then each of the previous readings gives a bijection between the fillings of the ribbon diagram of shape I and the words of length n. Thanks to the definition of a quasi-ribbon tableau, it is obvious to see that its column reading is an increasing word.

Definition 3.2 *We say that a word w is a* quasi-ribbon word *of shape I if it is the canonical reading of a quasi-ribbon tableau of shape I.*

Proposition 3.3 *A word is a quasi-ribbon word iff it is the concatenation of maximal strictly decreasing words such that the smallest letter of a given word is greater than or equal to the greatest one of the previous word (when reading from left to right).*

Proof — It is clear that the canonical reading of a quasi-ribbon tableau satisfies this characterization. Conversely, a word that satisfies these conditions is a quasi-ribbon word, *i.e.*, it is the canonical reading of a quasi-ribbon tableau: the strictly decreasing words are the columns of it read from bottom to top and the condition about the comparison gives the nondecreasing property in the rows. □

The word baa is not a quasi-ribbon word since it decomposes as $ba.a$ and that $b \not\leq a$. The word $aabaedcfe$ is a quasi-ribbon word since it decomposes as $a.a.ba.edc.fe$ and that $a \leq a$, $a \leq a$, $b \leq c$, $e \leq e$. Its corresponding quasi-ribbon tableau is given Figure 3.

The next proposition will be useful later in many proofs: we will be able to restrict to the standard case to prove some properties of the quasi-ribbon words.

Proposition 3.4 *Let w be a word. Then the standardized word of w is a quasi-ribbon word iff w is a quasi-ribbon word.*

Proof — There exists a unique decomposition of w written $w = w_1 \ldots w_p$ where the words w_i are the longest strictly decreasing factors of w. We then decompose $St(w)$ as $St(w) = St(w)_1 \ldots St(w)_p$ in such a way that $St(w)_i$ has the same length as w_i.

a	a	a		
		b	c	
			d	
			e	e
				f

Figure 3: The quasi-ribbon tableau of $aabaedcfe$

Thanks to the second definition of the standardization process, two consecutive letters of w are in decreasing order iff two consecutive letters of $St(w)$ are in decreasing order. So, the words $St(w)_i$ are the longest strictly decreasing factors of $St(w)$.

Moreover, if there exists an integer i such that the first letter of w_i is smaller (resp. weakly greater) than the last letter of w_{i+1}, the first letter of $St(w)_i$ is smaller (resp. greater) than the last letter of $St(w)_{i+1}$. So, $St(w)$ is a quasi-ribbon word iff w is a quasi-ribbon word. $\square$

In particular, this proposition shows that the standardized word of a quasi-ribbon word is a quasi-ribbon word.

Proposition 3.5 *Let I and J be two compositions. There exists a quasi-ribbon word of shape I and evaluation J iff J is finer than I. In this case, such a word is unique.*

Proof — We prove uniqueness first. Assume that there exists two such words. If they are different, the associated quasi-ribbon tableaux are different (using the bijection between words and fillings), and so are the column readings of these tableaux. This is impossible, since the column reading of a quasi-ribbon tableau is an increasing word, and that there is a unique increasing word of a given evaluation (both increasing words have the evaluation J).
Let us write J as $(j_1, \ldots, j_p)$. Let us consider r, the filling of the shape I associated with the word $(1^{j_1}2^{j_2} \ldots p^{j_p})$ by the column reading. If r is not a quasi-ribbon tableau, we know that there is no quasi-ribbon tableau of this shape and evaluation: if there was one, we would have two different fillings of the same shape that would give the same word by the column reading. It is impossible. So, in this case, there is no quasi-ribbon word. Thus, to establish our theorem, it is sufficient to prove that r is a quasi-ribbon tableau iff J is finer than I. In fact, r is always nondecreasing along the rows from left to right. It is strictly increasing along the columns from top to bottom iff J is finer than I: $D(I) \subset D(J)$ implies that when we change rows (south move), we also change letters, so that there is no cell whose content is equal to the content of the cell to its south. Conversely, if $D(I) \not\subset D(J)$, one can easily find such a cell. $\square$

3.2 The standard case

In the standard case, many things are simpler and we would like to make them clear since it is this case we will be interested in in the sequel. We also give some new properties that arise from the fact that standardized words are permutations.

Proposition 3.6 *There exists a unique standard quasi-ribbon word of a given shape.*

Proof — This is a direct consequence of Proposition 3.5 in the special case $J = (1^n)$: the composition (1^n) is finer than all the compositions. $\square$

Proposition 3.7 *A standard quasi-ribbon word is an involution (considered as a permutation).*

Proof — This proposition is a consequence of a special case of Proposition 3.3.

The standard quasi-ribbon words are the concatenation of the decreasing permutation on distinct alphabets such that all the letters of one of them are greater than all the letters of the previous ones. Since the decreasing permutation is an involution, it is also the case when it is concatenated with itself. $\square$

4 The hypoplactic monoid

4.1 Definition

Definition 4.1 *The* plactic monoid *is the quotient monoid of the free monoid by the* Knuth relations *(see [7]):*

$$\begin{cases} aba \equiv baa, \; bba \equiv bab & \text{for } a < b \;, \\ acb \equiv cab, \; bca \equiv bac & \text{for } a < b < c \;. \end{cases}$$

Definition 4.2 *The* hypoplactic *monoid is the quotient monoid of the plactic monoid by the* quartic relations *(see [9]):*

$$\begin{cases} baba \equiv abab, \; baca \equiv abac & \text{for } a < b < c \;, \\ cacb \equiv acbc, \; bcab \equiv cbba & \text{for } a < b < c \;, \\ cadb \equiv acbd, \; bdac \equiv dbca & \text{for } a < b < c < d \;. \end{cases}$$

We will write $w \equiv w'$ to mean that w and w' are hypoplactically equivalent. Let us remark that quartic relations can be reduced from six to two

$$\begin{cases} cadb \equiv acbd & \text{for } a \le b < c \le d \;, \\ bdac \equiv dbca & \text{for } a \le b < c \le d \;, \end{cases}$$

Lemma 4.3 *The standardization process defined earlier is compatible with the hypoplactic rewritings:*

$$w \equiv w' \quad \Rightarrow \quad St(w) \equiv St(w').$$

Proof — One has just to check it for all the rewritings. For example, $baba \equiv abab$ becomes $cadb \equiv acbd$. $\square$

We now define the Schensted algorithm in the hypoplactic case. It is the analog of Schensted algorithm that works in the plactic case [14]. The basic step of this algorithm builds upon a quasi-ribbon tableau Q and a letter a of A a new quasi-ribbon tableau denoted by $Q.a$. Hence, starting with the empty quasi-ribbon tableau, we build step by step a quasi-ribbon tableau corresponding to the word $a_1 a_2 \dots a_n$.

Algorithm 4.4
`INPUT : A quasi-ribbon tableau` Q `and a letter` a`.`
`OUTPUT : A quasi-ribbon tableau` Q'`.`

Let x be the right-most and bottom-most cell of Q such that its content is smaller than or equal to a. If such a cell does not exist, put a cell with content a and glue the previous quasi-ribbon tableau to the bottom of a.

If x exists, put then a new cell of content a at the right of it and glue the remaining part of the quasi-ribbon tableau to the bottom of a. The resulting quasi-ribbon tableau is the output.

Note 4.5 Let w be a quasi-ribbon word. Let Q be the result of the insertion algorithm applied to w. Then the canonical reading of Q is w.

The next proposition and the next theorem were established by Krob and Thibon in [9], using a quantum interpretation of the hypoplactic monoid.

Proposition 4.6 *Let w, w' be two words. We obtain the same quasi-ribbon tableaux applying the Schensted algorithm to both words iff they are hypoplactically equivalent.*

Theorem 4.7 *The quasi-ribbon words form a section of the hypoplactic monoid,* i.e., *each hypoplactic class contains exactly one quasi-ribbon word.*

As in the plactic case, it is possible to give a purely combinatorial proof of this theorem, by showing that insertion via the Schensted algorithm can be simulated by the use of the Knuth and quartic relations. But this proof is very complicated, much more than in the plactic case since there are many special cases to check. To make this combinatorial proof, we introduce in the next subsection another definition of the hypoplactic monoid and then prove the section theorem with this simpler definition (see Theorem 4.17).

4.2 An equivalent definition of the hypoplactic monoid

We present in this subsection a new equivalence relation on words. Two words are equivalent if their result by an algorithm is the same. We begin with presenting this algorithm and then prove that this equivalence is a congruence on words. We then establish some properties of the corresponding monoid and then show that it is equal to the hypoplactic monoid.

The description of this algorithm is difficult to follow but its result is simple: we in fact compute the descent composition of the inverse of the standardized word of a word (see Theorem 4.12). We decided to present this algorithm like this to simplify all the proofs.

Algorithm 4.8
`INPUT: A word` w`.`
`OUTPUT: A pair` (I, J) `of compositions.`

- Let $J = (j_1, \dots, j_p)$ be the evaluation vector $\mathcal{E}(w)$ of w. Let V be the binary vector of length $p-1$ whose k^{th} entry v_k is 1 if there is an occurrence of the $(k+1)^{st}$ letter of the alphabet to the left of an occurrence of the k^{th} one in w and 0 otherwise.

– Set $l := 1$, $k := 1$, $i_\alpha := 0$ for all $\alpha > 1$ and $i_1 := j_1$. While $k < p$ do:

$$\begin{array}{lll} \text{if } v_k = 1 & \text{then} & \{l := l+1,\ k := k+1 \text{ and } i_l := i_l + j_k\} \\ & \text{else} & \{k := k+1 \text{ and } i_l := i_l + j_k\}. \end{array}$$

– We denote by I the sequence $(i_1, i_2, \ldots)$. The output is the pair (I, J).

As we said above, Algorithm 4.8 computes nothing but the descent composition associated to the inverse of the standardized word of w. Let us give an example of the algorithm, taking $w = 2142135$. The vector J is in this case $(2,2,1,1,1)$ and the vector V is $(1,0,1,0)$. We then compute I using Step 1 and we find $I = (2,3,2)$. On the other hand, $St(w) = 3164257$, so that $St(w)^{-1} = 2514637$ and its descent composition is $(2,3,2)$.

Note 4.9 The composition J is finer than the composition I.

Note 4.10 I is completely determined given V and J. Conversely, V is completely determined given I and a finer composition J.

For example, if $V = (1,1,0,1)$ and $J = (2,2,4,1,3)$ then $I = (2,2,5,3)$. It is also possible to rebuild V knowing that $J = (2,2,4,1,3)$ and $I = (2,2,5,3)$ since $(2,2,5,3) = ((2),(2),(4+1),(3))$.

Definition 4.11 *We will denote by $\mathcal{I}$ the mapping that sends each word to the first component of its result when applying Algorithm 4.8. We will denote by $\mathcal{I}_k$ the mapping that sends each word w to the k^{th} entry of $\mathcal{I}(w)$ if it exists and 0 otherwise.*
We will denote by $\mathcal{V}$ the mapping that sends each word to its corresponding binary vector as defined in Algorithm 4.8. Finally, we will denote by $\mathcal{V}_k$ the mapping that sends each word w to the k^{th} entry of $\mathcal{V}(w)$ if it exists and 0 otherwise.

Theorem 4.12 *Let w be a word. $\mathcal{I}(w)$ is the descent composition associated with the inverse of $St(w)$.*

Proof — Let us first establish the following lemma.

Lemma 4.13 *Let w be a word. Then $\mathcal{I}(w) = \mathcal{I}(St(w))$.*

Proof of the lemma — Using the second definition of the standardization process, we easily see that $\mathcal{I}_1(w) = \mathcal{I}_1(St(w))$ and then conclude by induction on the length of $\mathcal{E}(w)$ since the standardization process is compatible with the restriction of alphabet. □

We now return to the proof of the theorem. Thanks to the previous lemma, it remains to prove that $\mathcal{I}(w)$ is equal to the descent composition of w^{-1} if w is a standard word.

Let w be a standard word. Looking at Algorithm 4.8, we see that we increment a new part of I iff $v_k = 1$ that is equivalent to say that $k+1$ is to the left of k in w. This is also equivalent to say that w^{-1} has a descent at position k. Thanks to the definition of the descent composition of a word, we conclude that the composition I computed by Algorithm 4.8 is the descent composition of w^{-1}. □

In the next proposition, we prove that Algorithm 4.8 leads to the definition of a new monoid.

Proposition 4.14 *We take the following equivalence relation on words:*

$$w \equiv' w' \quad \Leftrightarrow \quad \mathcal{I}(w) = \mathcal{I}(w') \quad \text{and} \quad \mathcal{E}(w) = \mathcal{E}(w'). \tag{1}$$

This relation is compatible with the usual concatenation product (denoted by .). Thus, $(A^*/\equiv', .)$ is a monoid.

In other words, take $w_1 \equiv' w'_1$ and $w_2 \equiv' w'_2$. Then $w_1.w_2 \equiv' w'_1.w'_2$.

Proof — The relation clearly is an equivalence relation. To prove the compatibility with the concatenation product, it suffices to show that for every pair (w, w') such that $w \equiv' w'$ and for every word w_1, $w.w_1 \equiv' w'.w_1$ and $w_1.w \equiv' w_1.w'$.

We will prove that $w.w_1 \equiv' w'.w_1$ and leave the other part to the reader since the proof is very similar. First, it is clear that $\mathcal{E}(w.w_1) = \mathcal{E}(w'.w_1)$. Thanks to Note 4.10, it remains to prove that both binary vectors are identical. Let d and e be two consecutive letters in the alphabet. Let v be a word. There are three different cases, depending on some properties of v and w_1.

If $\mathcal{E}_e(v) = 0$ (there is no occurrence of e in v), $\mathcal{V}_d(v.w_1) = \mathcal{V}_d(w_1)$. If $\mathcal{E}_e(v) \neq 0$, two cases appear: if $\mathcal{E}_d(w_1) \neq 0$, then $\mathcal{V}_d(v.w_1) = 1$, else $\mathcal{V}_d(v.w_1) = \mathcal{V}_d(v)$.
Since $\mathcal{E}_e(w) = \mathcal{E}_e(w')$, the words w and w' belong to the same case. Since $\mathcal{V}_d(w) = \mathcal{V}_d(w')$, they give the same result. So, $\mathcal{V}_d(w.w_1) = \mathcal{V}_d(w'.w_1)$. Since it is true for every d, we finally deduce that $\mathcal{V}(w.w_1) = \mathcal{V}(w'.w_1)$. □

Example 4.15 The class of 1323 is composed of three elements: 1323, 1332 and 3123. The class of 13245 is composed of nine elements: 13245, 13425, 13452, 31245, 31425,31452, 34125, 34152 and 34512.

The next theorem establishes the first link between the hypoplactic monoid and the previous monoid.

Theorem 4.16 *Let I be the shape of the quasi-ribbon tableau obtained by applying the Schensted algorithm to a word w. Then $I = \mathcal{I}(w)$.*

Proof — We first prove that two consecutive letters d and e of the alphabet belong to the same row of the quasi-ribbon tableau r associated with w by Schensted algorithm iff $\mathcal{V}_d(w) = 0$.

Assume that $\mathcal{V}(d) = 1$. When inserting the last d using Schensted algorithm, we glue this letter to the right of its previous occurrences (if there are) and the sequence of letters e (which is non empty) glue to the bottom of it. Since we never glue two different rows one to the right of another, letters d and e are on different rows of r. Conversely, if $\mathcal{V}(d) = 0$, when applying Schensted algorithm, we first glue all the occurrences of d and then glue to their right all the occurrences of e. So, d and e are on the same row of r since d and e are consecutive letters in the alphabet.

Let $J = (j_1, \ldots, j_p)$ be the evaluation vector of w. Using the first part of this proof, we compute the length of the rows of r by the following process. All the lengths are equal to 0 except the first one that is equal to j_1. While $(k < p)$, add j_k to the considered length if $\mathcal{V}(k) = 0$ and add j_k to next one if $\mathcal{V}(k) = 1$. Since this process is the exact copy of the process described in Algorithm 4.8, we conclude that $I = \mathcal{I}(w)$. □

We are now going to prove combinatorially the section theorem. It was already proved as mentioned above using the quantum interpretation of the hypoplactic monoid.

Theorem 4.17 *The quasi-ribbon words form a section of the monoid corresponding to the relation* $\equiv'$.

Proof — Let I be a composition and J a finer one. Let w be the quasi-ribbon word of shape I and evaluation J. Then, using Note 4.5 and Theorem 4.16, one can deduce that this word belongs to the class indexed by I and J. Moreover, all its classes are indexed by a pair of compositions such that the second one is finer than the first one, so the theorem follows. $\square$

Theorem 4.18 *Both monoids are the same, that is*

$$\forall\ w, w' \in A,\ w \equiv w' \quad \Leftrightarrow \quad w \equiv' w'.$$

Proof — Each hypoplactic class is naturally embedded in a class of the other monoid since one can check that the rewritings preserve the descent composition of the inverse of the standardized word (thanks to the standardization process, we only have to check it for the standard relations) and do not change the evaluation of a word. So, we have two monoids that have the same section (Theorems 4.7 and 4.17) such that one is included in the other. This proves that they are equal. $\square$

We finally state the anonymous referee's theorem. This theorem comes from this beautiful remark: the Knuth relations are the equivalences for words of length three that preserve the standardization inverse descent set. The hypoplactic relations are the equivalences for words of length four that preserve the standardization inverse descent set. The referee's theorem says that such equivalences for words of longer length generate nothing new. Its proof comes from the previous theorems.

Theorem 4.19 *Let us take two words of the same evaluation such that the descent set of the inverse of their standardized word is the same. Then these words are hypoplactically equivalent.*

Proof — This theorem is a direct consequence of the previous one since we can now say that two words are hypoplactically equivalent iff they have the same descent composition of the inverse of their standardized word and the same evaluation since it is the other definition of the hypoplactic monoid. $\square$

5 Basic properties of the hypoplactic monoid

The hypoplactic classes are indexed by two compositions. In the sequel, we will always denote a hypoplactic class as the class associated with two compositions.

5.1 Enumeration of a class

In this subsection, we study the number of elements of a given hypoplactic class. The order of a class will be denoted by $\#(I, J)$.

Theorem 5.1 *Let us consider a hypoplactic class. Let us denote by $I = (i_1, \ldots, i_l)$ and J its associated compositions. Then*

$$\sum_{I' \leq I} \#(I', J) = \binom{i_1 + \ldots + i_l}{i_1, \ldots, i_l}.$$

Proof — To prove this identity, we need to give an interpretation of the right-hand side of it: it is the number of words that belong to a certain shuffle product. We will then prove that these words are exactly the words belonging to the classes indexed by (I', J) with $I' \leq I$. Let us consider all the words that belong to the shuffle of the rows in the quasi-ribbon tableau associated with (I, J). Their cardinality is equal to the right-hand side of the identity. Let w be such a word. $\mathcal{E}(w) = J$. One has just to look at the definition of the shuffle to see that $\mathcal{I}(w) \leq I$. Conversely, it is also clear that all the words of the class (I, w) belong to the shuffle. So all the words of the classes such that $I' \leq I$ belong to the shuffle of the rows of their corresponding quasi-ribbon tableau and thus to the previous shuffle $(w_1.w_2 \sqcup\!\sqcup w_3 \in w_1 \sqcup\!\sqcup w_2 \sqcup\!\sqcup w_3)$. □

Example 5.2 Let us compute the shuffle product of 1, 23 and 45. We obtain the following words 12345, 12435, 12453, 14235, 14253, 14523, 21345, 21435, 21453, 23145, 23415, 23451, 24135, 24153, 24315, 24351, 24513, 24531, 41235, 41253, 41523, 42135, 42153, 42315, 42351, 42513, 42531, 45123, 45213, and 45231. They exactly are all the words of the classes $(1, 2, 2)$ [21435, 21453, 24135, 24153, 24315, 24351, 24513, 24531, 42135, 42153, 42315, 42351, 42513, 45213, 45231] , $(3, 2)$ [12435, 12453, 14235, 14253, 14523, 41235, 41253, 41523, 45123] , $(1, 4)$ [21345, 23145, 23415, 23451] , (5) [12345] .

This allows us to compute iteratively the order of a class since the order of the class (n) is always 1.

Note 5.3 The order of the class indexed by $((2^n), (1^{2n}))$ (resp. $((2^n, 1), (1^{2n+1}))$) is related to the expansion of $\sec(x)$ (resp. $\tan(x)$), and therefore, related to the Euler and Bernouilli numbers (see [1] and [4]). A result of Niven [12] proves that these classes are the greatest ones over the standard classes of a given length.

5.2 Schützenberger's Involution

Let us denote by $\sim$ the involution from A^* to itself sending each word to its mirror image $(a_1 \ldots a_n \longrightarrow a_n \ldots a_1)$ and by $\sharp$ the involution from A to itself such that $b \leq c$ iff $c^\sharp \leq b^\sharp$ extended to A^* as $w^\sharp = a_n^\sharp \ldots a_1^\sharp$.

Theorem 5.4 *Let w and w' be two words. Then*

$$w \equiv w' \quad \Leftrightarrow \quad w^\sharp \equiv w'^\sharp.$$

Proof — One has just to check it for all the rewritings. We already know that it is true for the plactic relations (see [11]). The quartic relations are sent to themselves by the involution except the second and the third one that are sent one to the other. □

Note 5.5 In general, this is not true for the involution $\sim$. It is true in the standard case. The explanation is clear when one thinks about Algorithm 4.8: we are only interested in the relative positions of the last occurrence of a letter and the first occurrence of its successor. Since we know nothing about the relative positions of the first occurrence of it and the last occurrence of its successor, it is quite clear that it cannot be compatible with the hypoplactic rewritings, except in one case: when the first and last occurrences of the same letter are identical. This is exactly the standard case.

5.3 Restricting the alphabet

Let B be an interval of A. We denote by $\mathcal{R}$ the morphism that sends each word to its longest subword that belongs to B^*. In other words, $\mathcal{R}$ deletes all the letters of our word that belong to $A \backslash B$.

Theorem 5.6 *Let w and w' be two words. Then*

$$w \equiv' w' \Longrightarrow \mathcal{R}(w) \equiv' \mathcal{R}(w').$$

Proof — This property was established by Lascoux and Schützenberger in [11] for the plactic rewritings. One can check that it is also true for the quartic rewritings. For example, if we reduce the alphabet to $\{1,2,3\}$, the last two relations become respectively $312 \equiv 132$ and $213 \equiv 231$, that are the plactic relations with three letters. □

Another way to do this is, one more time, to refer to Algorithm 4.8. In this context, we see that the class associated with $\mathcal{R}(w)$ is totally determined by a part of the binary vector and by a part of the evaluation vector independently from the values of these vectors. So, it is clear that $\mathcal{R}(w')$ is congruent to $\mathcal{R}(w')$.

5.4 The syntactic monoid of $\mathcal{I}$

In this subsection, we study more precisely the link between the function $\mathcal{I}$ and the hypoplactic monoid. That will lead us to show that it is the syntactic monoid of $\mathcal{I}$. In the next theorem, we give a sort of converse of Proposition 4.14.

Theorem 5.7 *Let w and w' be two words. If for all words w_1, $\mathcal{I}(w.w_1) = \mathcal{I}(w'.w_1)$, then $w \equiv w'$.*

Proof — Taking w_1 as the empty word, we deduce that $\mathcal{I}(w) = \mathcal{I}(w')$. It remains to show that $\mathcal{E}(w) = \mathcal{E}(w')$. Let z (resp. a) be the greatest (resp. smallest) letter of the alphabet.

Set $w_1 = zy\dots ba$. We then have

$$\begin{cases} \mathcal{I}(w.w_1) = (\mathcal{E}(w)+1,\ \mathcal{E}(w)+1, \dots, \mathcal{E}(w)+1,\ \mathcal{E}(w)+1) \\ \text{and} \\ \mathcal{I}(w'.w_1) = (\mathcal{E}(w')+1, \mathcal{E}(w')+1, \dots, \mathcal{E}(w')+1, \mathcal{E}(w')+1) \end{cases}$$

By hypothesis, these compositions are equal so that w and w' have the same evaluation. Finally, w and w' have the same shape and the same evaluation so, are hypoplactically equivalent. $\square$

Notice that the previous theorem is also true if one replaces "$\mathcal{I}(w.w_1) = \mathcal{I}(w'.w_1)$" by "$\mathcal{I}(w_1.w) = \mathcal{I}(w_1.w')$".

Example 5.8 The words 1131 and 1121 are not hypoplactically equivalent since we have $\mathcal{I}(1131.321) = (4, 1, 2)$ whereas $\mathcal{I}(1121.321) = (4, 2, 1)$.

Theorem 5.9 *The hypoplactic monoid is the syntactic monoid of the function $\mathcal{I}$,* i.e.,

$$w \equiv' w' \quad \Leftrightarrow \quad \forall\, u, v \in A^*,\ \mathcal{I}(uwv) = \mathcal{I}(uw'v).$$

Proof — This theorem is a consequence of Proposition 4.14 and Theorem 5.7 and the remark following its proof. $\square$

5.5 The standard case

Theorem 5.10 *Let w be a standard word and J a composition.*

$$(\exists\, w' \mid \mathcal{E}(w') = J \text{ and } St(w') = w) \quad \Leftrightarrow \quad J \geq \mathcal{I}(w).$$

Proof — Let us assume that there exists such a word w'. Then, thanks to Lemma 4.12, we know that w' belongs to the class indexed by $(\mathcal{I}(w), J)$ which is necessarily not empty. Thanks to Proposition 3.5 (uniqueness of a quasi-ribbon) and Theorem 4.17 (section of the hypoplactic monoid), we derive that J is necessarily finer than $\mathcal{I}(w)$.

Conversely, let us assume that J is finer than $\mathcal{I}(w)$. First, let us recall that two words having the same standardized word and the same evaluation are equal. Let us consider both classes indexed respectively by $(\mathcal{I}(w), 1^n)$ and $(\mathcal{I}(w), J)$. All the standardized words of the second one belong to the first one. Moreover, thanks to Theorem 5.1, we know that these classes have the same cardinality (the cardinality depends only on the first composition).

All this allows us to claim that the standardization process is an injection between two sets of the same cardinality and so is a surjection. This concludes the proof. $\square$

Let us do as an example the case of the word 152643. The only words whose standard word is 152643 are: 131321, 131421, 141432, 141532, 142432, 142532, 152543 and 152643 itself. The corresponding evaluations are $(3, 1, 2)$, $(3, 1, 1, 1)$, $(2, 1, 1, 2)$, $(2, 1, 1, 1, 1)$, $(1, 2, 1, 2)$, $(1, 2, 1, 1, 1)$, $(1, 1, 1, 1, 2)$ and $(1, 1, 1, 1, 1, 1)$ which are all the compositions finer than $\mathcal{I}(aebfdc) = (3, 1, 2)$.

The previous theorem gives another way to define the hypoplactic monoid.

Definition 5.11 *Let w be a word. We associate with it the composition I defined as the greatest composition such that there exists a word w' that has the same standardized word as w and has evaluation I. We say that two words are equivalent iff their associated compositions are the same and they have the same evaluation.*

5.6 A characterization of the quasi-ribbon words

Theorem 5.12 *The smallest word with respect to the lexicographic order of a non empty hypoplactic class is its quasi-ribbon word.*

Proof — We can reduce to the standard case, using the compatibility of the standardization process with the monoid structure. We are now in the case where $J = (1^{|I|})$. We are going to prove our result by induction on the number of columns of the ribbon diagram associated with I. Let us write $I = (1^{l-1}, i, I')$ with $i \neq 1$ and I' a composition. The length of the first column is then l. So the letters $1, 2, \ldots, l-1$ are to the north of their successors. In particular, they cannot go to the first position, so the smallest letter that can go to the first position is l. Knowing that we put l in the first position, we can put $l-1$ in the second one and then $l-2$ in the third one and so on. We obtain a strictly decreasing word $l.l-1\ldots 2.1$. Concatenating this word to the quasi-ribbon word associated with $(i-1, I')$ (on the new alphabet $l+1, \ldots$), one obtains a word w that is smaller than all the words of the considered class: we put the smallest possible prefix of length l and then put the smallest possible word (by the induction hypothesis). We remind the reader that we did not show yet that w belongs to this class. So, w is a quasi-ribbon word since it satisfies the criterion (see Proposition 3.3). Its shape is obviously I. So it belongs to the right class. □

6 Study of F_I

In this section, we define important functions in the hypoplactic case that play the same role as the Schur functions in the plactic case. We begin with some enumeration problems. We then define in another way our functions and establish some of their properties.

6.1 Definitions

We take the following notations about the alphabet. The smallest letter of the alphabet is denoted by a and its greatest one by z. We denote by $A\backslash\{z\}$ the alphabet containing all the letters of A except z, endowed with the same lexicographic order.

Definition 6.1 *We denote by $QR_I(A)$ the set of all quasi-ribbon words of shape I over A. We denote by $F_I(A)$ the sum of all quasi-ribbon words of shape I over A. These functions are called* quasi-ribbon functions.

We will always regard these functions as belonging to the hypoplactic algebra, that is to say: when we compute the product of such two functions, we allow hypoplactic rewritings for all the words obtained by doing this product formally. We will see further that they form a basis of a maximal commutative subalgebra of the hypoplactic algebra.

Example 6.2 Let $A = \{a < b < c\}$. There are four quasi-ribbon tableaux of ribbon shape $I = (2, 1)$ that are listed in Figure 4.

It follows that $F_{(2,1)}(a, b, c) = aba + aca + acb + bcb$.

$$\begin{array}{|c|c|}\hline a & a \\ \hline \multicolumn{1}{c|}{} & b \\ \cline{2-2}\end{array} \quad \begin{array}{|c|c|}\hline a & a \\ \hline \multicolumn{1}{c|}{} & c \\ \cline{2-2}\end{array} \quad \begin{array}{|c|c|}\hline a & b \\ \hline \multicolumn{1}{c|}{} & c \\ \cline{2-2}\end{array} \quad \begin{array}{|c|c|}\hline b & b \\ \hline \multicolumn{1}{c|}{} & c \\ \cline{2-2}\end{array}$$

Figure 4: The four quasi-ribbon tableaux of shape $I = (2, 1)$

6.2 Another definition of F_I

In this subsection, we give two different definitions of the same object and establish some simple properties. We will use them when we will compute the product of two F_I's.

Definition 6.3 *Let w be a word. We denote by $QR'(w)$ the set of all words of A^* that have the same standardized word as w.*

Theorem 6.4 *Let w be a quasi-ribbon word and I its shape. Then $QR'(w) = QR_I$.*

Proof — We prove this property by double inclusion. We denote by w_1 the unique standard quasi-ribbon word of shape I. It is also the standardized word of w (see Lemma 4.13 and Proposition 3.6).

Let w' be a word of QR_I. Thanks to Lemma 4.13, we know that $\mathcal{I}(St(w')) = \mathcal{I}(w') = I$. So $St(w') = w_1$: w' belongs to $QR'(w)$. So, $QR_I \subset QR'(w)$.

Conversely, let w' be a word of $QR'(w)$. First, we notice that $\mathcal{I}(w') = \mathcal{I}(w_1) = I$. It remains to show that w' is a quasi-ribbon word. We know that there exists a quasi-ribbon word w'' which is hypoplactically equivalent to w'. This implies that $St(w'') = w_1$. So, we have two words of the same evaluation that have the same standardized word. They are necessarily equal. So w' is a quasi-ribbon word. Thus $QR'(w) \subset QR_I$. □

6.3 Product of two F_I's.

In this subsection, we prove combinatorially a hypoplactic analog of a well-known result of Gessel ([6]). Gessel proved this result for the product of two functions that live in the algebra of quasi-symmetric functions.

Theorem 6.5 *Let w and w_0 be two words such that $w \equiv' w_0$. There is a natural bijection between $QR'(w)$ and $QR'(w_0)$: we send each word of $QR'(w)$ to the word of $QR'(w_0)$ which is congruent to it (hypoplactic relations).*

Proof — One can notice that the standardized words of w and w_0 are hypoplactically equivalent. Thanks to Theorem 5.10, it is obvious that there exists a word belonging to $QR'(w)$ of a given evaluation iff there exists a word belonging to $QR'(w_0)$ of the same evaluation. Moreover, these words are hypoplactically equivalent using Lemma 4.13. □

Theorem 6.6 *Let I and J be two compositions. Then*

$$F_I F_J = \sum_K \gamma_K F_K.$$

where γ_K is an integer.

Proof — First, we do the formal product of the functions. We obtain a sum of words, with coefficient 1. We can write it like this: $F_I F_J = \sum \gamma_w w$.
Let us consider a word w. It splits as a prefix of length $|I|$ and a suffix of length $|J|$. If its prefix is a quasi-ribbon word of shape I and its suffix is a quasi-ribbon word of shape J, then its coefficient is 1, else it is 0. Let us take $w' \in QR'(w)$. It decomposes as well into a prefix and a suffix. Lemma 2.2 implies that the coefficient of w is equal to the coefficient of w', since F_I is composed of all words that have a given standardized word ($QR'(w) = QR_I$, see Theorem 6.4) and so for F_J. So we proved that $\gamma_{w'}$ is constant for $w' \in QR'(w)$.

We can then apply Theorem 6.5 to the special case where w_0 is a quasi-ribbon word, and conclude since the bijection preserves the evaluation. In other terms, each word of a class $QR'(w)$ is sent to its quasi-ribbon word by the bijection. Doing the same for all the words, we obtain a sum of quasi-ribbon words with some coefficients. What we proved before implies that the coefficients of all quasi-ribbon words that are in the same F_K are equal. So, we can factorize by F_K. □

Thanks to Theorem 6.6, we know that the product of two F_I's can be developed as a sum of other F_I's. We now compute the coefficient of a given F_K.

Theorem 6.7 *Let I and J be two compositions. Let y (resp. y') be the standard quasi-ribbon word corresponding to I (resp. J) on the alphabet A (resp. A'). Let Sh be the set of all words that belong to the shuffle of y and y', seen as words on the ordinal sum of A and A'. Let us write $\sum_{Sh} C(y'') = \sum g_K K$, where $C(w)$ is the descent composition associated with w. Then*

$$F_I F_J = \sum g_K F_K.$$

Proof — We first define g_K in a simple way. We then find out an algorithm to compute it and finally derive its value. Thanks to Proposition 3.6, we know that in each F_K, there is exactly one permutation (evaluation 1^k). So g_K is equal to the coefficient of the corresponding permutation. By definition of the product $F_I F_J$, the coefficient of a permutation w is equal to the number of words, obtained by concatenating an element of QR_J to an element of QR_I that are hypoplactically equivalent to w. We first define the set we are interested in and show how it can be easily generated.

Definition 6.8 *Let U be the set of all words of the form $u.v$ with $u \in QR_I$, $v \in QR_J$ and $\mathcal{E}(u.v) = 1^k$.*

Our aim is to characterize the set $\mathcal{I}(w)$ where w spans U. We first characterize the smallest element w_0 of U and then generate U from w_0.

Lemma 6.9 *Let w_0 be the smallest element of U. Then w_0 is the standard quasi-ribbon word corresponding to the composition $K = (i_1, i_2, \ldots, i_l + j_1, \ldots, j_s)$ obtained by gluing J at the right of I.*

Proof of the lemma — The smallest word of U is necessarily the word obtained by taking the smallest standard word of shape I, that is, taking the $|I|$ first letters of the alphabet for this word and the J last ones for the quasi-ribbon word of shape J, that is w_0. □

Lemma 6.10 *A word w belongs to U iff there exists a sequence (c_l) such that*

- $c_0 = w$, $c_p = w_0$,
- *one can go from c_l to c_{l+1} by exchanging two consecutive letters (in the alphabet order) such that the greatest one belongs to the prefix (of length $|I|$) of the word and the smallest one to the suffix (of length $|J|$) of it.*

Proof of the lemma — Since we exchange two consecutive letters in our word, there is no letter of the prefix that was smaller than the first one and greater than the second one. So, it is clear that the standardized word of the prefix is constant in our path ($\mathcal{I}(u_0) = \mathcal{I}(u)$) and hence, all the words built by our process belong to U. Conversely, let U' be the set of the elements of U that are not obtained by this process. First, w_0 does not belong to U'. Since on all the other words of U, one can do at least one exchange, it is clear that there is no smallest element (for the lexicographic order) in U'. Since it is finite, this is impossible. So, U' is empty. □

We now consider $U^{-1} = \{w^{-1},\ w \in U\}$. First, let us notice that w_0 is an involution (see Proposition 3.7). Thanks to the previous lemma, we deduce that a word w' belongs to U^{-1} iff there exists a sequence (c'_l) such that

- $c'_0 = w'$, $c'_p = w'_0$,
- one can go from c'_l to c'_{l+1} by exchanging two such adjacent letters such that the left one belongs to the last part of the alphabet (letters of y') and the right one belongs to the first part of it (letters of y). This is exactly the definition of the shuffle. □

Example 6.11 Let $I = (1,1)$, $J = (2)$.

$$\begin{array}{rcccccccccccc}
21 \sqcup\!\sqcup 34 & = & 2134 & + & 2314 & + & 2341 & + & 3214 & + & 3241 & + & 3421 \\
\sum C(w) & = & 13 & + & 22 & + & 31 & + & 112 & + & 121 & + & 211 \\
\sum w''^{-1} & = & 2134 & + & 3124 & + & 4123 & + & 3214 & + & 4213 & + & 4312 \\
\sum \mathcal{I}(w''^{-1}) & = & 13 & + & 22 & + & 31 & + & 112 & + & 121 & + & 211 \\
F_{1,1}F_2 & = & F_{1,3} & + & F_{2,2} & + & F_{3,1} & + & F_{1,1,2} & + & F_{1,2,1} & + & F_{2,1,1}
\end{array}$$

Corollary 6.12 *Let I and J be two compositions. Then*

$$F_I F_J = F_J F_I.$$

Proof — This is a direct consequence of Theorem 6.7 since $y \sqcup\!\sqcup y' = y' \sqcup\!\sqcup y$. □

One can see that we did not define the F_I using the hypoplactic rewritings. We only assumed that they lived in the hypoplactic algebra. The next theorem is the hypoplactic equivalent of a result of Lascoux and Schützenberger (see [11]) for the plactic case. This result shows in particular that the F_I functions play the very same role as the Schur function in the classical theory.

Theorem 6.13 *Let A be an alphabet such that $|A| \geq 4$. The hypoplactic congruence is the smallest congruence on A^* commuting with the evaluation, with injective morphisms of ordered alphabets and with restriction morphisms of alphabets, and such that the F_I generate a commutative subalgebra of $\mathbb{Z}(A^*/\equiv)$.*

7 Study of R_I

In this section, we define another family of important functions that naturally live in the hypoplactic algebra. They play the same role as the ribbon Schur functions in the plactic case. We take the same notations as in the previous section about the alphabet.

We recall that a *ribbon tableau* is a ribbon shape filled in a way that its rows are weakly increasing from left to right and its columns strictly increasing from top to bottom. A *ribbon word* corresponds to the canonical reading of a ribbon tableau.

Definition 7.1 *Left I be a composition. We denote by $R_I(A)$ the sum of all ribbon words over A. These functions are called* ribbon functions.

Example 7.2 Let $A = \{a < b < c\}$. There are three ribbon diagrams of ribbon shape $I = (2, 1, 1)$ that are listed below.

$$\begin{array}{cc} a & c \\ & b \\ & a \end{array} \qquad \begin{array}{cc} b & c \\ & b \\ & a \end{array} \qquad \begin{array}{cc} c & c \\ & b \\ & a \end{array}$$

It follows that $R_{(2,1,1)}(a, b, c) = acba + bcba + ccba$.

Theorem 7.3 *The algebra generated by the functions R_I seen as living in the plactic algebra is isomorphic to the algebra generated by the functions R_I seen as living in the hypoplactic algebra.*

Proof — First, notice that in the plactic algebra, the algebra generated by the ribbon functions belongs to the algebra generated by the Schur functions that is isomorphic to the algebra of commutative symmetric functions. So, the algebras generated by the functions R_I seen as living in the plactic algebra or as living in the commutative algebra are isomorphic. Since the hypoplactic monoid is intermediate between the commutative monoid and the plactic monoid, we can conclude. $\square$

Let us now assume that the functions R_I live in the hypoplactic algebra. The next two corollaries are only consequences of the previous theorem and of the classical theory.

Corollary 7.4 *Let I and J be two compositions. We have in the hypoplactic algebra*

$$R_I R_J = R_J R_I.$$

Corollary 7.5 *Let I and J be two compositions. Let $I \top J$ (resp. $I \perp J$) denote the gluing of J at the end of I such that the first cell of J is to the right of the last cell of I (resp. the first cell of J is to the bottom of the last cell of I). We have in the hypoplactic algebra*

$$R_I R_J = R_{I \top J} + R_{I \perp J}.$$

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