Approximating minimum cocolourings

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Abstract

A cocolouring of a graph G is a partition of the vertex set of G such that each set of the partition is either a clique or an independent set in G. Some special cases of the minimum cocolouring problem are of particular interest.

We provide polynomial-time algorithms to approximate a mimimum cocolouring on graphs, partially ordered sets and sequences. In particular, we obtain an efficient algorithm to approximate within a factor of 1.71 a minimum partition of a partially ordered set into chains and antichains, and a minimum partition of a sequence into increasing and decreasing subsequences.

1 Introduction

A cocolouring of a graph G is a partition of the vertices such that each set of the partition is either a clique or an independent set in G. The cochromatic number of G is the smallest cardinality of a cocolouring of G. The cochromatic number was originally studied in [18]. Subsequent papers addressed various topics including the structure of critical graphs, bounds on the cochromatic numbers of graphs with certain properties (e.g., fixed number of vertices, bounded clique size, fixed genus) and algorithms for special graph classes (e.g., chordal graphs and cographs) [1, 4, 6, 7, 8, 11, 12, 20].

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In this paper, besides cocolouring of graphs in general we study cocolourings of permutation graphs, comparability graphs and cocomparability graphs. The cocolouring problem on permutation graphs is equivalent to the cocolouring problem on repetition-free sequences of integers (one may assume a permutation of the first n integers) which has the following motivation: if one has to sort such a sequence, it is desirable to have a partition into a small number of sets of already sorted elements, i.e., subsequences which are either increasing or decreasing. Now the minimum number of monotone subsequences partitioning the original sequence is exactly the cochromatic number of the permutation graph corresponding to the sequence. This problem was studied in [2, 21].

Wagner showed that the problem "Given a sequence and an integer k, decide whether the sequence can be partitioned into at most k monotone (increasing or decreasing) subsequences" is NP-complete [21]. In our paper we provide a first constant-factor approximation algorithm for cocolouring sequences. More precisely our algorithm approximates a minimum cocolouring of a sequence within a factor of 1.71.

In fact we derive our 1.71-approximation algorithm for the minimum cocolouring problem on comparability (or cocomparability) graphs. This problem is equivalent to the cocolouring problem on partially ordered sets, i.e. the problem to partition a partially ordered set P into a minimum number of subsets each being a chain or an antichain of P. Since every permutation graph is a comparability graph, our algorithm can also be used to approximate within a factor of 1.71 a minimum cocolouring of a permutation graph, and a minimum partition of a sequence of integers into increasing and decreasing subsequences.

We also present a greedy algorithm to approximate a minimum cocolouring of perfect graphs within a factor of $\log n$. Finally we show the following hardness result for the minimum cocolouring problem on graphs in general: for every $\epsilon > 0$, no polynomial-time algorithm approximates a minimum cocolouring within a factor of $n^{1/2-\epsilon}$, unless NP $\not\subseteq$ ZPP.

2 Definitions

We denote by $G = (V, E)$ a finite undirected and simple graph with n vertices and m edges. For every $W \subseteq V$, the subgraph of $G = (V, E)$ induced by W is denoted by $G[W]$. For simplicity we denote the graph $G[V \setminus A]$ by $G - A$.

A *clique* C of a graph $G = (V, E)$ is a subset of V such that all the vertices of C are pairwise adjacent. A subset of vertices $I \subseteq V$ is *independent* if no two of its elements are adjacent. We denote by $\omega(G)$ the maximum number of vertices in a clique of G and by $\alpha(G)$ the maximum number of vertices in an independent set of G.

An *r-colouring* of a graph $G = (V, E)$ is a partition $\{I_1, I_2, \dots, I_r\}$ of V such that for each $1 \leq j \leq r$, I_j is an independent set. The *chromatic number* $\chi(G)$ is the minimum size of such a partition and $\kappa(G) = \chi(\overline{G})$ is the minimum size of a partition $\{C_1, C_2, \dots, C_s\}$ of the vertices of G into cliques. Analogously, a *cocolouring* of G is a partition $\{I_1, I_2, \dots, I_r, C_1, C_2, \dots, C_s\}$ of V such that each I_j, $1 \leq j \leq r$, is an independent set and each C_j, $1 \leq j \leq s$, is a clique. The smallest cardinality of a cocolouring of G is the *cochromatic number* $z(G)$. Therefore, $z(G) \leq \min\{\chi(G), \kappa(G)\}$.

A graph $G = (V, E)$ is *perfect* if $\chi(G[W]) = \omega(G[W])$ for every $W \subseteq V$. Perfect graphs and classes of perfect graphs play an important role in graph theory and algorithmic graph theory. The following well-known classes of perfect graphs will be studied in the sequel: comparability, cocomparability and permutation graphs. For all information on these graph classes and their properties not given in our paper we refer to [3, 13].

Let $\mathcal{F} = \{F_1, F_2, \dots, F_s\}$ and $\mathcal{F}' = \{F'_1, F'_2, \dots, F'_t\}$ be two set families of subsets of a ground set U. We denote by $\bigcup \mathcal{F}$ the set $\bigcup_{i=1}^{s} F_i$. We denote by $\mathcal{F}_1 \circ \mathcal{F}_2$ the set family $\{F_1, F_2, \dots, F_s, F'_1, F'_2, \dots, F'_t\}$.

3 Partially ordered sets and comparability graphs

Let $P = (V(P), \prec)$ be a finite partially ordered set, i.e. $\prec$ is a reflexive, antisymmetric and transitive relation on the finite ground set $V(P)$. Two elements a, b of P are comparable if $a \prec b$ or $b \prec a$. Now a subset $C \subseteq V(P)$ is a *chain* of P if every two elements of P are comparable, and a subset $A \subseteq V(P)$ is an *antichain* of P if no two elements of A are comparable.

An orientation $H = (V, D)$ of an undirected graph $G = (V, E)$ assigns one of the two possible directions to each edge $e \in E$. The orientation is *transitive* if $(a, b) \in D$ and $(b, c) \in D$ implies $(a, c) \in D$. A graph $G = (V, E)$ is a *comparability graph* if there is a transitive orientation $H = (V, D)$ of G. A graph is a *cocomparability graph* if its complement is a comparability graph.

Consider the following well-known relation between partially ordered sets and comparability graphs. Let $P = (V(P), \prec)$ be a partially ordered set. We define an undirected graph with vertex set $V(P)$ and an edge between a and b iff a and b are comparable. Then this graph is a comparability graph, its cliques correspond to chains in P and its independent sets correspond to antichains in P. On the other hand, suppose $G = (V, E)$ is a comparability graph, and let H be a transitive orientation of graph G. Since H is an acyclic and transitive directed graph it induces a partially ordered set with ground set V where $u \prec w$ $(u \neq w)$ iff there is a directed path from u to w in H. Now every chain in the partially ordered set corresponds to a directed path in H which corresponds to a clique in G due to transitivity. Furthermore every antichain in the partially ordered set corresponds to an independent set in G. Thus the well known Dilworth theorem saying that the maximum cardinality of an antichain in P is equal to the minimum number of chains in a chain partition of $V(P)$ implies that comparability (and cocomparability) graphs are perfect.

More important for our paper, a cocolouring of a comparability (or cocomparability) graph G corresponds to a partition of a partially ordered set into chains and antichains. Now we study cocolourings of comparability graphs.

A *maximum k-colouring* $\mathcal{I}_k$ is a family of k independent subsets of a graph G covering a maximum number of vertices. Let $\alpha_k(G)$ denote the size of the maximum k-colouring, i.e. the number of vertices in a maximum k-chromatic subgraph of G. A *maximum h-covering* $\mathcal{C}_h$ is a family of h cliques of G covering a maximum number of vertices. We denote by $\kappa_h(G)$ the maximum size of an h-covering of G, i.e. the number of vertices in a maximum subgraph of G partitionable into h cliques.

Our approximation algorithm is based on the following results by Frank [10] which can be seen as algorithmic proofs of Greene and Kleitman's [14, 15] generalizations of Dilworth's theorem.

Theorem 1 ([10]). *There is an $O(nm)$ time algorithm which computes for any given comparability graph $G = (V, E)$ simultaneously*

(a) *for all integers k with $1 \leq k \leq \chi(G)$ a maximum k-colouring $\mathcal{I}_k$, and*

(b) *for all integers h with $1 \leq h \leq \kappa(G)$ a maximum h-covering $\mathcal{C}_h$.*

The essential part of the algorithm is a minimum-cost flow algorithm on a network associated to G (via a partially ordered set P having comparability graph G).

We shall also need the procedure SQRTPartition which is based on a result by Erdős et al. [7] (see also Brandstädt et al. [2]).

SQRTPartition

Input: perfect graph $G = (V, E)$ with $n < \frac{k(k+1)}{2}$ vertices, $k \geq 2$.

Output: cocolouring $\mathcal{Z}$ of G

- $\mathcal{Z} := \emptyset$; $U := V$;
- **while** $U \neq \emptyset$ **do**

  begin

  - **if** $\chi(G[U]) < k + 1$ **then** compute a k-colouring $\mathcal{I} = \{I_1, I_2, \ldots, I_k\}$ of $G[U]$; $\mathcal{Z} := \mathcal{Z} \circ \mathcal{I}$

    else choose a clique C of size at least $k + 1$ and add C to $\mathcal{Z}$;
  - $U := U - \bigcup \mathcal{Z}$; $k := k - 1$.

  end

Lemma 2 ([7]). *For every perfect graph $G = (V, E)$ with $n < \frac{k(k+1)}{2}$, $k \geq 2$, procedure* SQRTPartition *outputs a cocolouring of size at most k. Thus $z(G) \leq \lfloor \sqrt{2n + 1/4} - 1/2 \rfloor$ for every perfect graph.*

Proof. For the sake of completeness we provide the simple proof by induction on k. For $k = 2$ the theorem is true. Suppose that theorem is true for $k \geq 2$.

Let G be a perfect graph with $n < \frac{(k+1)(k+2)}{2}$ vertices. If $\chi(G) < k + 2$ then the procedure outputs a cocolouring of size at most $k + 1$.

If $\chi(G) \geq k + 2$ then the procedure chooses a clique C of G such that $|C| \geq k + 2$ which exists by the perfectness of G. The procedure removes all vertices of C from G, thus the number of remaining vertices in $G - C$ is less than $\frac{k(k+1)}{2}$. By induction hypothesis $G - C$ has a cocolouring of size at most k and thus the theorem is true for $k + 1$. □

Lemma 3. *Procedure* SQRTPartition *has polynomial running time on perfect graphs and its running time on comparability graphs is $O(\sqrt{n}m)$.*

Proof. The running time of SQRTPartition depends on the best known running time of an algorithm to solve the problems minimum colouring and maximum clique on our special classes of graphs.

The best known running time is polynomial on perfect graphs [16] and linear on comparability graphs [13]. □

Now we are in the position to describe our algorithm to approximate a minimum cocolouring on comparability graphs.

> APPROX COCOLOURING
> INPUT: comparability graph $G = (V, E)$
> OUTPUT: cocolouring $\mathcal{Z}$ of G
>
> - Compute a k-colouring $\mathcal{I}_k = \{I_1, I_2, \dots, I_k\}$ and an h-covering $\mathcal{C}_h = \{C_1, C_2, \dots, C_h\}$ of G such that the sum $k+h$ is minimum subject to the condition $\alpha_k(G) + \kappa_h(G) \geq n$;
> -
>   - If $|\bigcup \mathcal{I}_k| > |\bigcup \mathcal{C}_h|$ then $\mathcal{Z}' := \{I_1, I_2, \dots, I_k\}$;
>   - Else $\mathcal{Z}' := \{C_1, C_2, \dots, C_h\}$;
>
> $G' := G - \bigcup \mathcal{Z}'$;
> - Let n' be the number of vertices in G'. Compute a k'-colouring $\mathcal{I}_{k'} = \{I'_1, I'_2, \dots, I'_{k'}\}$ and an h'-covering $\mathcal{C}_{h'} = \{C'_1, C'_2, \dots, C'_{h'}\}$ of G' such that the sum $k' + h'$ is minimum subject to the condition $\alpha_{k'}(G') + \kappa_{h'}(G') \geq n'$;
> - $\mathcal{Z}'' := \{I'_1, I'_2, \dots, I'_{k'}, C'_1 \setminus \bigcup \mathcal{I}_{k'}, C'_2 \setminus \bigcup \mathcal{I}_{k'}, \dots, C'_{h'} \setminus \bigcup \mathcal{I}_{k'}\}$;
> - Compute a cocoluring $\mathcal{Z}'''$ of the graph $G' - \bigcup \mathcal{Z}''$ by calling SQRTPARTITION;
> - $\mathcal{Z} := \mathcal{Z}' \circ \mathcal{Z}'' \circ \mathcal{Z}'''$.

Theorem 4. *The $O(nm)$ time algorithm* APPROX COCOLOURING *approximates a minimum cocolouring of a comparability graph within a factor of 1.71.*

Proof. Let $\mathcal{I}_k = \{I_1, I_2, \dots, I_k\}$ and $\mathcal{C}_h = \{C_1, C_2, \dots, C_h\}$ be the sets produced at the first step of the algorithm. Then by the choice of k and h as well as $\mathcal{I}_k$ and $\mathcal{C}_h$ we have that $k + h \leq z(G)$. Let us think that $|\bigcup \mathcal{I}_k| > |\bigcup \mathcal{C}_h|$. (The case $|\bigcup \mathcal{I}_k| \leq |\bigcup \mathcal{C}_h|$ is similar.) Then the vertex set of G' can be covered by h cliques implying $k + k' + h' \leq k + h \leq z(G)$ and $|\bigcup \mathcal{I}_{k'}| + |\bigcup \mathcal{C}_{h'}| \geq |\bigcup \mathcal{C}_h|$. Hence the number of vertices in $\bigcup(\mathcal{Z}' \circ \mathcal{Z}'')$ is at least

$$
\begin{aligned}
&|\bigcup \mathcal{I}_k| + |\bigcup \mathcal{I}_{k'}| + |\bigcup \mathcal{C}_{h'}| - |(\bigcup \mathcal{I}_{k'}) \cap (\bigcup \mathcal{C}_{h'})| \geq \\
&|\bigcup \mathcal{I}_k| + |\bigcup \mathcal{C}_h| - |(\bigcup \mathcal{I}_{k'}) \cap (\bigcup \mathcal{C}_{h'})| \geq \\
&n - k'h'.
\end{aligned}
$$

(Since $\mathcal{I}_{k'}$ is a family of independent sets and $\mathcal{C}_{h'}$ is a family of cliques, implying $|(\bigcup \mathcal{I}_{k'}) \cap (\bigcup \mathcal{C}_{h'})| \leq k'h'$.)

Therefore, the graph $G' - \bigcup(\mathcal{Z}' \circ \mathcal{Z}'')$ has at most $k'h'$ vertices and by Lemma 2 procedure SQRTPartition computes a cocolouring of $G' - \bigcup(\mathcal{Z}' \circ \mathcal{Z}'')$ having size at most $\sqrt{2k'h'}$. Consequently, Approx Cocolouring computes a cocolouring $\mathcal{Z}$ of G of size at most

$$
k + k' + h' + \sqrt{2k'h'} \leq (k + k' + h')(1 + \frac{1}{\sqrt{2}}) \leq (1 + \frac{1}{\sqrt{2}})z(G) \leq 1.71 \cdot z(G).
$$

The time bound follows from Theorem 1 and Lemma 3. □

Corollary 5. *The algorithm* Approx Cocolouring *can also be used to approximate within a factor of* 1.71

(a) *a minimum cocolouring of a partially ordered set,*

(b) *a minimum partition of a (repetition-free) sequence (of integers) into increasing and decreasing subsequences,*

(c) *a minimum cocolouring of a permutation graph, and*

(d) *a minimum cocolouring of a cocomparability graph.*

4 Perfect graphs

We consider the following greedy algorithm for minimum cocolouring on graphs

GREEDY COCOLOURING

INPUT: graph $G = (V, E)$

OUTPUT: cocolouring $\mathcal{Z}$ of G

- $\mathcal{Z} := \emptyset$;
- $U := V$;
- **while** $U \neq \varnothing$ **do**

  begin

  - Compute a maximum independent set I_U and a maximum clique C_U of $G[U]$;
  - Choose X to be I_U or C_U such that $|X| = \max(|I_U|, |C_U|)$ and add X to $\mathcal{Z}$;
  - $U := U - X$.

  end

Theorem 6. *The* GREEDY COCOLOURING *algorithm approximates a minimum cocolouring of a perfect graph within a factor of* $\ln n$.

Proof. To obtain the approximation ratio of the algorithm let us consider a hypergraph $\mathcal{H} = (V, E_H)$, where the vertex set of $\mathcal{H}$ is the vertex set V of the input graph G and E_H is the set of all independent sets and cliques in G, i.e. every hyperedge of $\mathcal{H}$ is either an independent set or a clique in G.

Any minimum cocolouring on G is equivalent to a minimum set cover of $\mathcal{H}$ and vice versa. Moreover GREEDY COCOLOURING can be seen as the greedy algorithm for the minimum set cover problem on input $\mathcal{H}$ (the only difference is that GREEDY COCOLOURING won't inspect all hyperedges of $\mathcal{H}$). It is well known [5, 17, 19] that the greedy algorithm for the minimum set cover problem is an $\ln n$-approximation algorithm. □

By a well-known result of Grötschel et al. [16] a maximum independent set and a maximmum clique can be computed by a polynomial-time algorithm on perfect graphs.

Corollary 7. *The* GREEDY COCOLOURING *algorithm is a polynomial-time algorithm to approximate a minimum cocolouring within a factor of* $\ln n$ *on each graph class* $\mathcal{G}$ *for which there are polynomial-time algorithms to compute* *a maximum clique and a maximum independent set. In particular this is the case for perfect graphs.*

5 Hardness of approximation

The following lemma allows us to obtain a first hardness result concerning the approximation of minimum cocolouring.

Lemma 8. *[11] $\chi(G) = z(nG)$, where nG is the disjoint union of n copies of G.*

We shall rely on a well-known result on the hardness of approximating the chromatic number of a graph.

Theorem 9. *[9] It is hard to approximate the chromatic number to within $n^{1-\epsilon}$ for any $\epsilon > 0$, assuming $\mathrm{NP} \not\subseteq \mathrm{ZPP}$. (Here, ZPP denotes the class of languages decidable by a random expected polynomial-time algorithm that makes no errors.)*

Combining Lemma 8 with Theorem 9 we obtain the following

Theorem 10. *It is hard to approximate the cochromatic number to within $n^{1/2-\epsilon}$ for any $\epsilon > 0$, assuming $\mathrm{NP} \not\subseteq \mathrm{ZPP}$.*

Proof. Let G be any input to the problem minimum colouring. Then $G' = nG$ is our input to the problem minimum cocolouring. By Lemma 8, we have $\chi(G) = z(nG)$. Since the number of vertices of G' is the square of the number of vertices of G, by Theorem 9 we obtain the theorem. $\square$

6 Concluding remarks

We leave many questions unanswered, a few of them are:

1. The problem of finding a minimum partition of a sequence into monotone subsequences is NP-hard. We provide a 1.71-approximation algorithm for this problem. A natural question is if there exists a PTAS for this problem?

2. We have proved that GREEDY COCOLOURING is a $\ln n$-approximation algorithm for perfect graphs. Are there nontrivial classes of perfect graphs for which GREEDY COCOLOURING approximates the cochromatic number within a constant factor?

3. What is the computational complexity of computing a maximum k-colouring and a maximum h-covering for perfect graphs? A polynomial time algorithm computing these parameters for perfect graphs will imply that our 1.71-approximation algorithm for a minimum cocolouring on comparability graphs is also a polynomial time algorithm on perfect graphs.

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