Baker–Campbell–Hausdorff type computations using noncommutative symmetric functions

D. Krob, * B. Leclerc, * J.-C. Novelli, * and J.-Y. Thibon †

Abstract

This paper presents some relations between noncommutative symmetric functions and free Lie algebras. We show in particular how to decompose a Lie projector of Solomon's descent algebra on Dynkin's and Klyachko's bases. We illustrate these methods on the example of the Hausdorff series.

1 Introduction

The problem of expanding the Hausdorff series

$$H(a_1, \ldots, a_N) = \ln(e^{a_1} \cdots e^{a_N})$$

(where the a_i are non commuting variables) on a basis of the free Lie algebra generated by the a_i's involves, as demonstrated in [1], certain idempotents of the group algebra of the symmetric group. A more general problem is to find an expression for $f(E-1)$, where $f(z) = \sum c_n z^n$ is an arbitrary analytic function, and E is either $e^{a_1} \cdots e^{a_N}$, or the evolution operator of a first order linear evolution equation, *i.e.* satisfies

$$\frac{\partial}{\partial t} E(t; t_0) = H(t)\, E(t; t_0)$$

with $E(t_0; t_0) = 1$. This also can be accomplished by the use of appropriate idempotents. It turns out that all the idempotents encountered in this situation belong to a certain remarkable sub-algebra of the algebra of the symmetric group, called the *descent algebra*,introduced by L. Solomon in 1976 (see [8]). The recent theory of *noncommutative*

* Address : Université Paris 7 – LITP – 2, place Jussieu – 75251 Paris Cedex 05 – France; e-mails : dk@litp.ibp.fr (D. Krob), bl@litp.ibp.fr (B. Leclerc), jcn@litp.ibp.fr (J.-C. Novelli)

† Address : Université de Marne–la–Vallée – IFI – 1, rue de Vaugirard – 93166 Noisy-le-Grand Cedex – France; e-mail : jyt@litp.ibp.fr

symmetric functions (cf [3]) provides a simple interpretation of all these calculations, as well as a variety of new results (cf [2]). In this article, we illustrate the formalism on the case of *Lie idempotents*, corresponding to the classical case $f(z) = \ln(1+z)$ of the above mentioned problem. In particular, we derive a formula for expanding any Lie idempotent of the descent algebra on a basis of multi-linear alternants.

2 Lie projectors

This section provides the necessary background on free Lie algebras and Lie idempotents. More details can be found in [7]. We denote by K a field of characteristic zero.

2.1 Definition

Let A be an alphabet. A *Lie projector* is a projection from the free associative algebra $K\langle A\rangle$ onto the free Lie algebra $L(A)$. In other words, a Lie projector is an endomorphism π of $K\langle A\rangle$ with the following properties :

- $\pi^2 = \pi$ (π is a *projector*);
- $\pi(w) \in L(A)$ for every word $w \in A^*$ (π is a *Lie* projector).

The basic property of a Lie projector is that it maps any Lie element on itself. This simple fact can be used for obtaining Lie decompositions in different contexts. Let us explicit this classical method on the example of the Baker-Campbell-Hausdorff formula which states that the Hausdorff series

$$H(a,b) = \log(\exp(a)\,\exp(b)) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} (\exp(a)\,\exp(b) - 1)^n$$

is a Lie series. One can easily expand the right-hand side in $\mathbb{Q}\langle\langle A\rangle\rangle$:

$$H(a,b) = \sum \frac{(-1)^{n-1}}{n} \frac{1}{p_1!\,q_1!\,\ldots\,p_n!\,q_n!}\, a^{p_1}\, b^{q_1}\,\ldots\, a^{p_n}\, b^{q_n} \ ,$$

the summation being taken over all $n \geq 1$ and $p_i, q_i \geq 0$ with $p_i + q_i \geq 1$ for $i = 1, \ldots, n$. If we apply a Lie projector π on both sides of this equation, we get the identity

$$H(a,b) = \sum \frac{(-1)^{n-1}}{n} \frac{1}{p_1!\,q_1!\,\ldots\,p_n!\,q_n!}\, \pi(a^{p_1}\, b^{q_1}\,\ldots\, a^{p_n}\, b^{q_n}) \ , \qquad (1)$$

which is a Lie decomposition of the Hausdorff series $H(a,b)$. The problem is therefore to choose a good Lie projector π in order to exploit in the best possible way the above relation. The same kind of idea can clearly be used in other situations. It should be noted that there exists an explicit formula (due to Goldberg, see [7]) for the coefficient of a word in the Hausdorff series (in any number of variables).

2.2 Relations with the symmetric group

Formula (1) shows that it is important to be able to compute the most efficiently possible the image of a word by a Lie projector. This is possible in the special case that we are now going to present (which is essentially the only one used in practical applications).

Recall that the symmetric group S_n acts on the homogeneous component of order n of $K_n\!<\!A\!>$ (and hence on $L_n(A)$) by

$$a_1 \ldots a_n\,\sigma = a_{\sigma(1)} \ldots a_{\sigma(n)}$$

for $a_i \in A$ and $\sigma \in S_n$. Recall also that an endomorphism f of $K\!<\!A\!>$ commutes with letter substitutions iff one has

$$f(s(w)) = s(f(w))$$

for every endomorphism s of $K < A >$ which maps every letter $a \in A$ onto another letter $s(a) \in A$ (such an endomorphism is called a letter substitution). Note that an endomorphism commuting with letter substitutions also commutes with the right action of S_n.

Suppose now that we work with an infinite alphabet $A = \{\,1, 2, \ldots\}$. We shall only consider in the sequel Lie projectors with the following properties

1. π is *finely homogeneous* ($\pi(K_\lambda\!<\!A\!>) \subset L_\lambda(A)$ for every multi-degree λ).
2. π commutes with letter substitutions.

If π is a Lie projector which satisfies these properties, it is easy to see that one can recover π from the sequence

$$\pi_n = \pi(1\,2\,\ldots\,n) \in L_{1^n}(1, 2, \ldots, n)\ , \quad (n \geq 1)$$

of elements of the multi-linear components $L_{1^n}(1, \ldots, n)$ of the free Lie algebras on the alphabets $[n] = \{1, \ldots, n\}$. Indeed,

$$\pi(a_1 \ldots a_n) = \pi(s(1\,2\,\ldots\,n)) = s(\pi_n)$$

where s denotes the letter substitution of $K < 1, 2, \ldots, n >$ mapping i onto a_i. Since π is finely homogeneous, one can consider π_n as an element of the group algebra $K[S_n]$ (permutations being identified with standard words), which is clearly an idempotent ($\pi_n^2 = \pi_n$) of this algebra. The study of Lie projectors that satisfy to the two conditions above can therefore be reduced to the study of *Lie idempotents* in $K[S_n]$, i.e. of those idempotents of $K[S_n]$ that can be expressed as Lie polynomials over the alphabet $\{1, 2, \ldots, n\}$.

2.3 Two classical Lie projectors

This subsection is devoted to the description of two well known Lie idempotents that will be of fundamental use in our computations. It appears that these idempotents belong to the descent algebra Σ_n which is a sub-algebra of $K[S_n]$ that we shall now present.

Recall that an integer $i \in [1, n-1]$ is said to be a *descent* of a permutation $\sigma \in S_n$ iff $\sigma(i) > \sigma(i+1)$. The *descent set* of a permutation $\sigma \in S_n$ is the subset of $[1, n-1]$ that consists of all descents of σ. If $I = (i_1, \ldots, i_r)$ is a composition of n (that is, a r-tuple of positive integers which sums to n), we associate with it the subset $A(I) = \{ d_1, \ldots, d_{r-1} \}$ of $[1,n]$ defined by $d_k = i_1 + \ldots + i_k$. We define then D_I as the sum in $\mathbb{Z}[S_n]$ of all permutations with descent set $A(I)$. Solomon showed that the D_I form a basis of a sub-algebra of $\mathbb{Z}[S_n]$ which is called the *descent algebra* of S_n and denoted by Σ_n (cf [8]).

The so-called *Dynkin idempotent*, denoted ϑ_n, is defined by

$$\vartheta_n = \frac{1}{n} [[\ldots[[1,2],3],\ldots],n] = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k D_{1^k,n-k} \ .$$

This idempotent is in general the only one used in the literature for obtaining Lie decompositions of Hausdorff type series (see for instance [7] p. 81), even though it does it does not lead in general to very efficient algorithms.

A second important idempotent has been introduced by Klyachko [4]. Let us first define the *major index* $maj(I)$ of a composition $I = (i_1, i_2, \ldots, i_r)$ by setting $maj(I) = (r-1)\, i_1 + (r-2)\, i_2 + \ldots + i_{r-1}$. Let now ζ denote a primitive n-th root of unity (in $\mathbb{C}$). It can be shown that

$$\kappa_n(\zeta) = \frac{1}{n} \sum_{|I|=n} \zeta^{maj(I)}\, D_I$$

is a Lie idempotent, called *Klyachko's idempotent*. This idempotent has a very simple action on words. To describe it explicitly, we need some definitions.

A word w is said to be *primitive* iff it is not a non trivial power of some word, i.e. iff one does not have $w = u^n$ with $n \geq 2$. Two words w and w' are said to be *conjugate* iff one has $w = uv$ and $w' = vu$. Suppose now that the alphabet A is totally ordered. One can then choose in every conjugacy class of non primitive words the unique minimal element for the lexicographic order : such a word is said to be a *Lyndon word*. In other terms, a word l is a Lyndon word iff one has

$$l = uv \implies l < vu$$

for any two non empty words u, v. We can now state the two following combinatorial properties of Klyachko's idempotent (cf [7]).

1. If w is not a primitive word,
$$w\, \kappa_n(\zeta) = 0 \ . \qquad (2)$$

2. If w is a primitive word, there is a unique Lyndon word l_w in the conjugacy class of w. In this case,
$$w\, \kappa_n(\zeta) = \zeta^{|v|}\, l_w\, \kappa_n(\zeta) \ , \qquad (3)$$
where u denotes the unique prefix of w such that $w = uv$ and $l_w = vu$.

Using these two properties, it is not difficult to prove that the family $(l\, \kappa_n(\zeta))_{l \in Ly_n}$ is a basis of $L_n(A)$ (where l describes the family Ly_n of Lyndon words of length n).

3 Noncommutative symmetric functions

3.1 The algebra of noncommutative symmetric functions

The algebra of *formal noncommutative symmetric functions* is the free associative algebra $\mathbf{Sym} = \mathbb{Q}\langle \mathbf{S_1}, \mathbf{S_2}, \ldots \rangle$ generated by an infinite sequence of noncommutative indeterminates S_k, called the *complete* symmetric functions. We set for convenience $S_0 = 1$. Let t be another variable commuting with all the S_k. Introducing the generating series

$$\sigma(t) := \sum_{k=0}^{\infty} S_k\, t^k \ ,$$

one can define other noncommutative symmetric functions by the following relations

$$\lambda(t) = \sigma(-t)^{-1} \ , \quad \frac{d}{dt}\,\sigma(t) = \sigma(t)\,\psi(t) \ ,$$

where $\lambda(t)$ and $\psi(t)$ are the generating series

$$\lambda(t) := \sum_{k=0}^{\infty} \Lambda_k\, t^k \ , \quad \psi(t) := \sum_{k=1}^{\infty} \Psi_k\, t^{k-1} \ .$$

The noncommutative symmetric functions Λ_k are called *elementary functions*. The elements Ψ_k are called *power sums* of *first kind*.

The algebra $\mathbf{Sym}$ is graded by the weight function w defined by $w(S_k) = k$. Its homogeneous component of weight n will be denoted by $\mathbf{Sym_n}$. If (F_n) is a sequence of noncommutative symmetric functions with $F_n \in \mathbf{Sym_n}$ for every $n \geq 1$, we set

$$F^I = F_{i_1}\, F_{i_2}\, \ldots\, F_{i_r}$$

for every composition $I = (i_1, i_2, \ldots, i_r)$. The families (S^I), (Λ^I) and (Ψ^I) are then homogeneous bases of $\mathbf{Sym}$.

The set of all compositions of a given integer n is equipped with the *reverse refinement order*, denoted $\preceq$. For instance, the compositions J of 4 such that $J \preceq (1,2,1)$ are $(1,2,1)$, $(3,1)$, $(1,3)$ and (4). The *ribbon Schur functions* (R_I) can then be defined by one of the two equivalent relations

$$S^I = \sum_{J \preceq I} R_I \ , \quad R_I = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)}\, S^J \ ,$$

where $\ell(I)$ denotes the *length* of the composition I. One can easily show that the family (R_I) is a homogeneous basis of $\mathbf{Sym}$.

The *commutative image* of a noncommutative symmetric function F is the commutative symmetric function f obtained by applying to F the algebra morphism which maps S_n onto h_n, using here the notations of [6]. The commutative image of Λ_n is then e_n. On the other hand, Ψ_n is mapped to p_n. Finally R_I is sent to an ordinary ribbon Schur function, which is denoted by r_I.

One can endow $\mathbf{Sym}$ with a structure of Hopf algebra, its co-multiplication Δ being defined by one of the following equivalent formulas

$$\Delta(S_n) = \sum_{i=1}^{n} S_i \otimes S_{n-i} \ , \quad \Delta(\Lambda_n) = \sum_{i=1}^{n} \Lambda_i \otimes \Lambda_{n-i} \ ,$$

$$\Delta(\Psi_n) = 1 \otimes \Psi_n + \Psi_n \otimes 1 \ .$$

It is in fact this Hopf structure which explains in a unified way the properties of Lie idempotents, and leads to generalizations (see [3, 2]).

3.2 Relations with Solomon's descent algebra

There is also a strong connection between noncommutative symmetric functions and the descent algebra of the symmetric group. One can indeed define an isomorphism of graded vector spaces

$$\alpha : \ \Sigma = \bigoplus_{n=0}^{\infty} \Sigma_n \ \longrightarrow \ \mathbf{Sym} = \bigoplus_{\mathbf{n=0}}^{\infty} \mathbf{Sym_n}$$

by setting

$$\alpha(D_I) = R_I$$

for any composition I. The existence of this isomorphism shows that an element of **Sym** is just a certain encoding of an element of the descent algebra Σ_n. Note that the interpretation of S_n and Λ_n in this encoding is simple since

$$\alpha^{-1}(S_n) = D_n = Id_n \ , \ \alpha^{-1}(\Lambda_n) = D_{1^n} = \omega_n \ ,$$

where Id_n denotes the identity of order n and where ω_n denotes the maximal permutation $n\, n{-}1 \ldots 1$ of S_n.

3.3 Relations with free Lie algebras

An element π of the group algebra $K[S_n]$ is a *Lie element* if it can be expressed as a Lie polynomial over the alphabet $\{1, 2, \ldots, n\}$. A *Lie quasi-idempotent* is a Lie element of $K[S_n]$ which is a quasi-idempotent. (an element x of a K-algebra is said to be quasi-idempotent iff $x^2 = k\, x$ with $k \in K$). It appears that the inverse image under α of the noncommutative symmetric function Ψ_n is (up to a scalar) equal to Dynkin's idempotent. Indeed, one can check that

$$\alpha^{-1}(\Psi_n) = [[\ldots[[1,2],3],\ldots],n] = n\, \vartheta_n \ .$$

This shows that there is a certain relation between the free Lie algebra and noncommutative symmetric functions.

Denote by $L(\Psi)$ the free Lie algebra generated by the family $(\Psi_n)_{n\geq 1}$. We can then state the following result that gives a characterization of all Lie quasi-idempotents of the descent algebra.

Theorem 3.1 [3] *Let F_n be an element of* $\mathbf{Sym_n}$ *and let $f_n = \alpha^{-1}(F_n)$ be the associated element of Σ_n. The following assertions are equivalent:*

1. *f_n is a Lie quasi-idempotent.*
2. *F_n belongs to the free Lie algebra $L(\Psi)$.*
3. *F_n is a primitive element for the coproduct Δ.*

4 Decompositions on Lie bases of Lie idempotents

4.1 Left derivative

The *left derivative* Λ_1^{-1} is the linear morphism of **Sym** defined on the basis Λ^I by the relations

$$\Lambda_1^{-1} \cdot \Lambda^{i_1,\ldots,i_r} = \begin{cases} \Lambda^{i_2,\ldots,i_r} & \text{if } i_1 = 1 \ , \\ 0 & \text{if } i_1 \neq 1 \ . \end{cases}$$

Equivalently, one can characterize Λ_1^{-1} by the following properties:

1. Λ_1^{-1} is an additive morphism of **Sym**,
2. Λ_1^{-1} satisfies
$$\Lambda_1^{-1}(FG) = \Lambda_1^{-1}(F)\, G + c(F)\, \Lambda_1^{-1}(G)$$
for $F, G \in$ **Sym**, where $c(F)$ denotes the constant term of F.
3. the action of Λ_1^{-1} on elementary functions is given by
$$\Lambda_1^{-1}(\Lambda_n) = \begin{cases} 1 & \text{if } n = 1 \ , \\ 0 & \text{if } n \neq 1 \ . \end{cases}$$

One can also describe the action of Λ_1^{-1} on the usual bases of **Sym**. Note that these formulas allow to effectively compute the left derivative of any noncommutative symmetric function given on the (S^I), the (Ψ^I) or the (R_I) basis.

Proposition 4.1 *For every $n \geq 1$ and for every composition $I = (i_1, \ldots, i_r)$, one has*

$$\Lambda_1^{-1}(S_n) = S_{n-1} \ , \ \ \Lambda_1^{-1}(R_I) = \begin{cases} R_{i_1-1,i_2,\ldots,i_r} & \text{if } i_1 \neq 1 \ , \\ 0 & \text{if } i_1 = 1 \ , \end{cases}$$

$$\Lambda_1^{-1}(\Psi_n) = S_{n-1} = \sum_{j_1+\ldots+j_s=n-1} \frac{1}{j_1(j_1+j_2)\ldots(j_1+\ldots+j_s)} \, \Psi^{j_1,\ldots,j_s} \ .$$

4.2 Decomposition on Dynkin's basis

We give here a first method for decomposing a Lie quasi-idempotent π_n of the descent algebra Σ_n on a Lie basis. Recall that one obtains a Lie basis of the standard component $L_{1^n}(1, 2, \ldots, n)$ of the free Lie algebra $L(1, 2, \ldots, n)$ by taking the family

$$\sigma\,(n\,\vartheta_n) = [[\ldots[[1, \sigma(2)], \sigma(3)], \ldots], \sigma(n)]$$

where σ runs through the set of permutations of S_n that fix 1. This family, called Dynkin's basis of $L_{1^n}(1, 2, \ldots, n)$, will be used in the next proposition for decomposing an arbitrary Lie quasi-idempotent of Σ_n.

Let us first define the K-linear mapping i_{n-1}^n of $K[S_{n-1}]$ into $K[S_n]$ that associates with every permutation $\sigma \in S_{n-1}$ the permutation

$$i_{n-1}^n(\sigma) = (1, \sigma(1)+1, \ldots, \sigma(n-1)+1) \ .$$

The following proposition gives an effective way of expressing a Lie quasi-idempotent of Σ_n on Dynkin's basis.

Proposition 4.2 *Let Π_n be a Lie quasi-idempotent of* $\mathbf{Sym_n}$. *The decomposition of the associated element $\pi_n = \alpha^{-1}(\Pi_n)$ of Σ_n on Dynkin's basis is given by the following formula:*

$$\pi_n = i_{n-1}^n(\alpha^{-1}(\Lambda_1^{-1}(\Pi_n)))\,(n\,\vartheta_n)\ . \tag{4}$$

Example 4.3 Let us consider the Lie quasi-idempotent Φ_3 defined by

$$\Phi_3 = \Psi_3 + \frac{1}{4}\,[\Psi_1, \Psi_2]\ .$$

A simple computation gives

$$\Lambda_1^{-1}(\Phi_3) = S_2 + \frac{1}{4}\,(\Psi_2 - S_1\,\Psi_1)$$

$$= R_2 + \frac{1}{4}\,((R_2 - R_{1,1}) - (R_2 + R_{1,1})) = R_2 - \frac{1}{2}\,R_{1,1}\ .$$

The proposition shows that

$$\alpha^{-1}(\Phi_3) = [[1,2],3] - \frac{1}{2}\,[[1,3],2]\ .$$

The following formula provides an explicit decomposition of Klyachko's idempotent on Dynkin's basis.

Corollary 4.4 *Let ζ be a primitive n-th root of unity. The decomposition of Klyachko's idempotent $\kappa_n(\zeta)$ on Dynkin's basis is given by*

$$\kappa_n(\zeta) = \Big(\sum_{|I|=n-1} \zeta^{maj(I)+l(I)-1}\,D_I\Big)\,(n\,\vartheta_n)\ . \tag{5}$$

Example 4.5 For $n = 4$, one has for instance

$$\kappa_4(\zeta) = [[[1,2],3],4] + \zeta^3\,([[[1,2],4],3] + [[[1,3],4],2])$$

$$+\zeta^2\,([[[1,3],2],4] + [[[1,4],2],3]) + \zeta\,[[[1,4],3],2]\ .$$

4.3 Application to Baker-Campbell-Hausdorff type computations

According to the Baker-Campbell-Hausdorff formula, the series

$$H = \log(\exp(a_1)\,\exp(a_2)\,\ldots\,\exp(a_n)\,\ldots)$$

is a Lie series. Let H_λ denote the component of multi-degree $\lambda = (l_1, \ldots, l_m)$ in H (i.e. the part of H involving l_1 times the letter a_1, l_2 times the letter a_2, etc). The methods presented in this paper allow to compute the standard component $H_{(1^n)}$. To describe the method, we first define the q-analogue of Klyachko's element $\kappa_n(q)$ by

$$\kappa_n(q) = \frac{1}{n}\sum_{|I|=n} q^{maj(I)}\,D_I\ .$$

To compute H_{1^n}, we start from the expansion of H on the word basis (which can be obtained by means of Goldberg's formula, see [7] p. 63). Next, we apply Klyachko's idempotent to the result. The properties of this idempotent (i.e. relations (2) and (3)) give then immediately a decomposition of H on Klyachko's basis. To avoid the use of complex numbers, all the computations should be carried out with $\kappa_n(q)$ instead of $\kappa_n(\zeta)$. This leads to a decomposition of every H_λ of the form

$$H_\lambda = \sum_{w \in Ly_\lambda} P_w(q)\, w\, \kappa_n(q) \ ,$$

where Ly_λ denotes the set of Lyndon words of multi-degree λ and where $P_w(q)$ is some polynomial in $\mathbb{Q}[q]$. The use of formula (5) gives then an explicit Lie decomposition of H_λ with coefficients in $\mathbb{Q}[q]$. If the family of alternants involved in the result is a basis of the corresponding multi-homogeneous component of the free Lie algebra, it suffices to reduce all the polynomial coefficients modulo the n-th cyclotomic polynomial $\Phi_n(q)$ in order to obtain the actual coefficients (since they belong to $\mathbb{Q}$). This situation does not happen in general, but this is clearly the case when $\lambda = 1^n$. Hence, formula (5) allows to compute explicitly the standard component of the Hausdorff series in a basis of multi-linear alternants.

This technique can also be used for computing any multi-homogeneous component H_λ of H. It suffices indeed to choose a basis $\mathcal{B}_\lambda$ of the component of multi-degree λ of the associated free Lie algebra and to develop on this basis every Lie element obtained by our method (i.e. developing H_λ on the word basis, applying Klyachko's idempotent for obtaining a decomposition on Klyachko's basis and using then formula (5) for an explicit Lie expression). The reduction modulo the corresponding cyclotomic polynomial of the polynomial coefficients of $\mathbb{Q}[q]$ that we obtain, gives them a decomposition of H_λ on $\mathcal{B}_\lambda$.

5 Conclusion

When the technique presented in the last subsection is used for computing an arbitrary component of the Hausdorff series or for solving the more general problem mentioned in the introduction, we are lead to the problem of developing the elements of the general Klyachko basis, i.e. the Lie elements $l\,\kappa_n(\zeta)$ where l is a Lyndon word of length n, on other bases. Two types of questions could be of interest for obtaining results in this direction:

1) If one tries to use the Lyndon basis, one must study the transition matrices which express the basis $(l\,\kappa_n)_{l \in Ly_n}$ into the classical Lyndon basis.

2) If one prefers to use Dynkin Lie elements, one must be able to extract a basis from the family $(w\,\vartheta_n)$. A solution to this problem appears for example in [5].

References

[1] I. Bialynicki-Birula, B. Mielnik and J. Plebański, *Explicit solution of the continuous Baker-Campbell-Hausdorff problem*, Ann. Phys. **51** (1969), 187-200.

[2] G. Duchamp, D. Krob, B. Leclerc and J.-Y. Thibon, *Déformations de projecteurs de Lie*, C.R. Acad. Sci. Paris (ser. I) **319** (1994), 909-914.

[3] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh and J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. in Math. (1995), at press.

[4] A. Klyachko, *Lie elements in the tensor algebra*, Siberian Math. J. **15** (1975), 914-920.

[5] G.P. Kukin, *Bases of free Lie algebras*, Mat. Zametki. **24** (1978), 375-382.

[6] I.G. Macdonald , *Symmetric functions and Hall polynomials*, Oxford, 1979.

[7] C. Reutenauer , *Free Lie algebras*, Oxford, 1994.

[8] L. Solomon , *A Mackey formula in the group ring of a Coxeter group*, J. Algebra **41** (1976), 255-268.

[9] R.M. Wilcox, *Exponential operators and parameter differentiation in Quantum Physics*, J. Math. Phys. **8** (1967), 962-982.