Pseudo-Permutations I: First Combinatorial and Lattice Properties

Daniel Krob[*], Matthieu Latapy[†], Jean-Christophe Novelli[‡], Ha Duong Phan[§] and Sylviane Schwer[¶]

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Résumé

Cet article présente une étude de nouveaux objets combinatoires baptisés les pseudo-permutations, qui sont des généralisations des permutations. Les pseudo-permutations sont apparues naturellement dans divers cadres informatiques et mathématiques. Nous présentons ici une première étude de ces objets, en calculant les statistiques énumératives classiques, comme les inversions et les descentes et nous montrons que l'ensemble des pseudo-permutations est un treillis.

Abstract

In this paper, we introduce new combinatorial objects, the pseudo-permutations, which are a generalization of the permutations. Pseudo-permutations naturally appear in various fields of Computer Science and Mathematics. We provide the first combinatorial results on these objects: we study the classical statistics of enumeration, inversions, descents and we prove that the set of all pseudo-permutations has a lattice structure.

Keywords: Permutations, Enumerative Combinatorics, Lattices

1 Introduction

Representation and reasoning about time-dependent information is an active and important research area in Artificial Intelligence and in Computer Science. Vilain and Kautz's point algebra (see [VK86]) is one of the most influential models used for representing and reasoning with qualitative time information, because the reasoning tasks are solvable in polynomial time in that model, contrary to more powerful models, as Allen's Interval Algebra (see [All81]) in which the reasoning tasks are computationally intractable in general. The problem is the following: one considers a set of events which happen at certain dates, and wants to use this information to solve a problem, take a decision. However, it is often not meaningful *when* the events occur, while the relevant informations are the *temporal relations* between events: did event i happened before, during or after event j ? In this context, it is natural to represent the temporal relations between n events by an ordered sequence of nonempty parts of $[\![1,n]\!]$ such that each integer appears exactly once. If i is in a part of $[\![1,n]\!]$

*LIAFA – Université Paris VII – 2, Place Jussieu – 75005 Paris – France – *e-mail:* dk@liafa.jussieu.fr

†LIAFA – Université Paris VII – 2, Place Jussieu – 75005 Paris – France – *e-mail:* latapy@liafa.jussieu.fr

‡IGM – 5, Boulevard Descartes – Champs-sur-Marne, F-77454 Marne-La-Vallée Cedex 2, France – *e-mail:* novelli@weyl.univ-mlv.fr – Corresponding author

§LIAFA – Université Paris VII – 2, Place Jussieu – 75005 Paris – France – *e-mail:* latapy@liafa.jussieu.fr

¶LIPN, Institut Galilée – Université Paris XIII – 99 Bd J-B Clément – 93430 Villetaneuse – France – *e-mail:* schwer@lipn.univ-paris13.fr

which appears in the sequence before the part which contains j, then the event i happened before the event j. If they appear in the same part, they occured simultaneously. For example, the sequence $\{1\}\{3,4\}\{2\}$ means that event 1 occured first, event 2 occured last, and events 3 and 4 occured at the same time. In the following, we will call such a sequence a *pseudo-permutation of order* n (in the example, $n = 4$), we will use parentheses instead of braces, and we will remove unnecessary comas. Therefore, we will write the example as (1)(34)(2).

Having defined the pseudo-permutations, one can study their combinatorial properties. They appear to have a very rich structure and many interesting properties. Among these, one can see that their enumeration is related to Eulerian numbers, and that an inversion table associated with each pseudo-permutation can be defined. This induces a partial order on pseudo-permutations compatible with the inversion tables and we will see that this order is a lattice, as it is the case for the usual permutations. One can also define the descents of a pseudo-permutation and prove that this notion has the same properties as the usual descents on permutations. About the lattice structure, one can also see that the set can be divided in connected components that are hypercubes, the dimensions of which can naturally be interpretated in combinatorial terms. The aim of this paper is to show how rich can be this structure and to give a first insight in it, given that a new paper (see [BHKN01]) provides geometrical and representation theoretical properties about pseudo-permutations.

The paper is structured as follows. We first recall the definitions of a lattice and the usual properties of Eulerian numbers (Section 2). We then concentrate on the pseudo-permutations themselves and define the pseudo-permutohedron (Section 3). Next, we prove their first combinatorial properties (Section 4) and their first lattice properties (Section 5).

2 Preliminaries

We recall here a few definitions and basic results used in the rest of the paper. We will also use the following standard notations. We will denote by $[\![i,j]\!]$ the set $\{k \in \mathbb{N} \mid i \leq k \leq j\}$ and by $\mathfrak{S}_n$ the set of all the permutations of order n, *i.e.*, the set of all the sequences of n integers which contain each integer in $[\![1,n]\!]$ exactly once. The *permutohedron* of order n is then the directed graph defined over $\mathfrak{S}_n$ by asking that: there is an edge from σ to σ' if σ' is obtained from σ by switching two neighbour integers i and j in σ such that $i < j$ (see Figure 1 for the permutohedron of order 3).

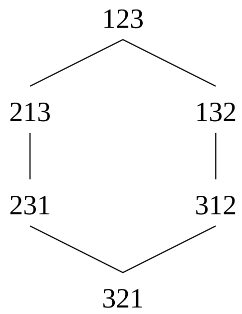

Figure 1: The permutohedron of order 3.

2.1 Rises and Eulerian numbers

The Eulerian numbers are very classical numbers and the reader can refer to [FS70, Com70] for a complete overview about these. In this paper, we will only recall their usual definitions and properties.

Let σ be a permutation. One says that σ has a *rise* in i, or equivalently that i is a *rise* of σ, if $\sigma(i) > \sigma(i+1)$. The *rise number* of σ is then the number of rises of σ. The *rise set* of σ is the set of the rises of σ. If one denotes by $a_{n,k}$ the number of elements of $\mathfrak{S}_n$ with k rises, one has the following formulas:

$$a_{n,k} = a_{n,n-k-1} = (n-k)a_{n-1,k-1} + (k+1)a_{n-1,k},$$

with $n > 1$ and $k > 1$. Moreover, $a_{n,0} = 1$ and $a_{0,k} = 0$ for all n and k. The *Eulerian numbers* are very close to these numbers since they can be defined by $A_{n,k} = a_{n,k-1}$. Moreover, the generating series of these numbers is known and one has:

$$1 + \sum_{1 \le k \le n} a_{n,k-1} \frac{t^n}{n} u^{k-1} = \frac{1-u}{e^{t(u-1)} - u}.$$

One can define in the same way the descents of a permutation and the descent number of a permutation (see, *e.g.*, [Com70] for more details).

2.2 Lattice theory

We recall that an *order relation* is a binary relation $\le$ over a set, such that for all x, y and z in this set, $x \le x$ (reflexivity), $x \le y$ and $y \le z$ implies $x \le z$ (transitivity), and $x \le y$ and $y \le x$ implies $x = y$ (antisymmetry). Such a relation is often called a *partial order*. The set is then a *partially ordered set* or, for short, a *poset*.

A *lattice* is a poset such that any two elements a and b have a least upper bound (called *supremum* of a and b and denoted by $a \vee b$) and a greatest lower bound (called *infimum* of a and b and denoted by $a \wedge b$). The element $a \vee b$ is the smallest element among the elements greater than both a and b. The element $a \wedge b$ is defined dually. Lattices are strongly structured sets, and many general results are known about them. For example, efficient coding and algorithms are known for lattices. For more details, see, *e.g.*, [DP90, Pha99].

3 Pseudo-permutations

3.1 Definitions

Let n be an integer. The set $\mathfrak{P}(n)$ of the pseudo-permutations of order n is the set of sequences of non-empty parentheses such that each integer in $[\![1,n]\!]$ appears exactly once (in [Gro62], one can find the same sort of combinatorial object; they are called *preferential arrangements*). In other words, the set $\mathfrak{P}(n)$ is the set of ordered partitions[1] of $[\![1,n]\!]$ with nonempty parts.
For example, here is the complete set $\mathfrak{P}(3)$:

$$\begin{aligned}\mathfrak{P}(3) = \{ \quad & (1)(2)(3),\ (1)(3)(2),\ (2)(1)(3),\ (2)(3)(1),\ (3)(1)(2),\ (3)(2)(1), \\ & (1)(23),\ (2)(13),\ (3)(12),\ (23)(1),\ (13)(2),\ (12)(3),\ (123) \quad \}\end{aligned}$$

Since the order inside the parentheses is irrelevant, we will generally write the integers in the parentheses in increasing order. Notice also that there exists an obvious embedding from the permutations into the pseudo-permutations that maps a permutation $\mathfrak{s} = (\sigma_1, \ldots, \sigma_n)$ to $(\sigma_1)(\sigma_2)\cdots(\sigma_n)$.

As for the usual permutations, one can generate a graph which vertices are the elements of $\mathfrak{P}(n)$ and which edges are defined according to the following operators:

[1] A *partition* of a set S is a set of subsets $S_1, S_2, \ldots, S_k$ of S such that $\cup_{i=1}^{k} S_i = S$ and for all $i \neq j$, $S_i \cap S_j = \emptyset$. The S_i are then the *parts* of the partition. A partition is *ordered* if we consider a sequence of subsets instead of a set of subsets.

- The operator M_i acts on the i-th and the $i+1$-th parentheses of a pseudo-permutation as follows: if each element of the i-th parenthese is smaller than all the elements of the $(i+1)$-th, then one can merge these two parentheses into one single parenthese which contains the union of the elements of these two parentheses.

- The operator $S_{i,j}$ acts on the i-th parenthese of a pseudo-permutation as follows: it splits this parenthese into two parentheses, the *second* one containing the j smallest elements of the initial parenthese and the first one containing the others.

For example, one can apply the operators M_2, M_4, $S_{3,1}$, $S_{3,2}$ and $S_{4,1}$ to the permutation $\sigma = (7)(3)(568)(12)(4)$:

$$\begin{aligned}
\sigma &\xrightarrow{M_2} (7)(3568)(12)(4)\\
&\xrightarrow{M_4} (7)(3)(568)(124)\\
&\xrightarrow{S_{3,1}} (7)(3)(68)(5)(12)(4)\\
&\xrightarrow{S_{3,2}} (7)(3)(8)(56)(12)(4)\\
&\xrightarrow{S_{4,1}} (7)(3)(568)(2)(1)(4)
\end{aligned}$$

We can now define the *pseudo-permutohedron*, denoted by $\mathfrak{G}(n)$, as the directed graph one obtains by iterating these two operators from the element $(1)(2)\cdots(n)$. For example, we give in Figure 2 the graphs $\mathfrak{G}(3)$ and $\mathfrak{G}(4)$.

This allows us to define a natural partial order on the pseudo-permutations as $\sigma < \tau$ if one can obtain σ from τ by iterating operators M and S. We will see in Section 4.2 that this order can also be defined with the notion of inversions and that it shows that $\mathfrak{G}(n)$ is a lattice.

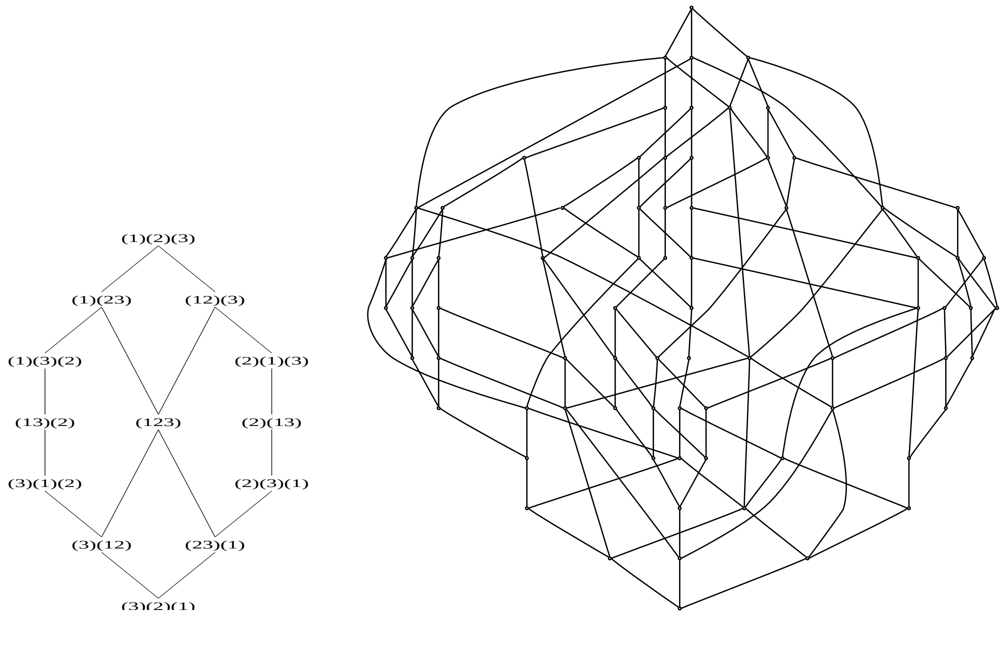

Figure 2: The graphs $\mathfrak{G}(3)$ (left) and $\mathfrak{G}(4)$ (right). The orientation of the edges is not displayed since it is always from the topmost vertex to the other.

We will now see some properties of the pseudo-permutations and their associated pseudo-permutohedron, first from the classical combinatorial point of view, then from the lattice point of view.

4 Combinatorial properties of $\mathfrak{P}(n)$

In this section, we give enumeration results, combinatorial properties and structural properties of the set $\mathfrak{P}(n)$. Let us begin with the most simple enumerations.

4.1 Enumeration of $\mathfrak{P}(n)$

As defined, the set $\mathfrak{P}(n)$ is the set of ordered partitions of n elements with nonempty parts. Therefore, it satisfies the induction relation:

$$\mathrm{Card}(\mathfrak{P}(\mathrm{n})) = \sum_{\mathrm{i}=0}^{\mathrm{n}-1} \binom{\mathrm{n}}{\mathrm{i}} \mathrm{Card}(\mathfrak{P}(\mathrm{i})).$$

One can also transform the set $\mathfrak{P}(n)$ in order to find a direct (well-known) formula, related to the Eulerian numbers.

Let n be an integer. As already noticed, the order of the elements in a given parenthesis is irrelevant, so we can assume that they are written in increasing order.

Definition 1. *Given an element $\mathfrak{s}$ of $\mathfrak{P}(n)$, one can then define its* horizontal reading*, denoted as $\mu(\mathfrak{s})$, which is the permutation obtained by removing all the parentheses, reading the integers in the parentheses in increasing order.*

For example, the horizontal reading of (1)(34)(2)(5) is 13425. One can then consider the classes of the pseudo-permutations which have the same horizontal reading: for example, the pseudo-permutations (123), (1)(23), (12)(3), and (1)(2)(3) form a class of $\mathfrak{P}(n)$. It is clear that, given a permutation σ, the number of pseudo-permutations in the corresponding class is equal to the number of correct sets of parentheses one can generate. These sets correspond to compositions of n and one can easily see that a composition $I = (i_1, \ldots, i_p)$ is "correct" iff the descent set of σ is included in the set $\{i_1, i_1 + i_2, \ldots, i_1 + \cdots, i_p\}$. It then comes that the number of correct parentheses for a given permutation σ of order n is equal to 2^{n-k-1} where k is the number of descents of σ.

Finally, since the number of pseudo-permutations generated from a permutation only depends on the order of the descent set of this permutation, one makes the connection with the Eulerian numbers and one derives the following formula for the cardinality of $\mathfrak{P}(n)$:

$$\mathrm{Card}(\mathfrak{P}(\mathrm{n})) = \sum_{\mathrm{k}=0}^{\mathrm{n}-1} \mathrm{A}_{\mathrm{n,k}}\, 2^{\mathrm{n}-\mathrm{k}-1} = \sum_{\mathrm{k}=0}^{\mathrm{n}-1} \mathrm{A}_{\mathrm{n,k}}\, 2^{\mathrm{k}}.$$

4.2 Inversions

In this subsection, we generalize to the pseudo-permutations the well-known statistics of inversions over usual permutations.

4.2.1 Table of inversions

Let n be an integer and let $\mathfrak{s}$ be an element of $\mathfrak{P}(n)$. For every pair (i, j) with $1 \leq i < j \leq n$, we define the value of the inversion (i, j) as follows:

- If i and j are in the same parenthese, this value is equal to $\frac{1}{2}$.
- If i and j are in distinct parentheses, this value is equal to 0 if the parenthese of i is before the parenthese of j in $\mathfrak{s}$ and it is equal to 1 in the other case.

Note 2. *In order to simplify the presentation of the next properties, we define the inversions of (j,i) with $j > i$ as the inversions of (i,j), even if these types of "formal" inversions will not be considered in the table of inversions of the pseudo-permutations.*

The *table of inversions* of $\mathfrak{s}$ is then the list of the non zero-valued pairs (i,j) with $1 \le i < j \le n$ of integers given with the value of the inversion (i,j). The *number of inversions* of $\mathfrak{s}$ is then the sum for all $1 \le i < j \le n$ of the values of the inversions (i,j). For example, the table of inversions of the pseudo-permutation (45)(13)(2) is equal to:

$$\left\{\frac{1}{2}(1,3),\ (1,4),\ (1,5),\ (2,3),\ (2,4),\ (2,5),\ (3,4),\ (3,5),\ \frac{1}{2}(4,5)\right\},$$

and its number of inversions is 8.

Definition 3. *Let $\mathfrak{s}$ and $\mathfrak{t}$ be two pseudo-permutations. One says that the table of inversions of $\mathfrak{s}$ is included in the table of $\mathfrak{t}$ if one has:*

$$\forall 1 \le i < j \le n,\ (i,j)_{\mathfrak{s}} \le (i,j)_{\mathfrak{t}}\,.$$

4.2.2 Characterization of the inversion sets

One can now ask about the reverse problem: given an inversion table, does it exist a pseudo-permutation that has this inversion table and when the answer is yes, how can one rebuild the corresponding pseudo-permutation.

As for the usual permutations, the answer to the first question is negative in general and one can get the following characterization which generalizes the characterization of the correct sets for the usual inversion tables:

Proposition 4. *Let n be an integer and L be a list of the pairs (i,j), with $1 \le i < j \le n$, each pair having a value in the set $\{0, \frac{1}{2}, 1\}$. Then L is the inversion table of a pseudo-permutation if and only if each of the following implications holds:*

- *If the value of an element (i,j) of L is $\frac{1}{2}$, then*
  - *If $k \in [\![i,j]\!]$, the sum of the value of (i,k) and the value of (k,j) is equal to 1 in L.*
  - *If k is not in $[\![i,j]\!]$, the value of (i,k) is equal to the value of (k,j) in L.*
- *If the value of two elements (i,j) and (j,k) of L are both equal to 1, and if $i < j < k$ then the value of (i,k) is also 1 in L.*
- *If the value of an element (i,k) of L is 1, then for every $j \in [\![i,k]\!]$, either the value of (i,j) or the value of (j,k) is also 1 in L.*

Proof — Let us first see why if L is an inversion table of a pseudo-permutation, L satisfies the characterization.

- Let i,j,k be such that the value of (i,j) is $\frac{1}{2}$.
  - Assume that $k \in [\![i,j]\!]$. If k is to the left of the parenthesis of i and j then (i,k) is equal to 1 and (k,j) is equal to 0, so that the sum of these values equals 1. If k belongs to the same parenthesis, both values are equal to $\frac{1}{2}$, so that the result holds. If k is to the right of the parenthesis of i and j then $(i,k) = 0$ and $(k,j) = 1$, so that the result holds.
  - Let now assume that k is not in $[\![i,j]\!]$. As in the previous case, one studies the three possible positions for the parenthesis of k and then sees that the result holds in each case.

- Let now $i < j < k$ be such that $(i,j) = 1$ and $(j,k) = 1$. By definition, that means that the parenthesis of i is to the right of the parenthesis of j that is itself to the right of the parenthesis of k. So the parenthesis of i is to the right of the parenthesis of k, so that $(i,k) = 1$.

- Let now $i < j < k$ be such that $(i,k) = 1$. By definition, that means that the parenthesis of i is to the right of the parenthesis of k. Since they are not in the same parenthesis, that means that either the parenthesis of k is to the left of the parenthesis of j, either the parenthesis of i is to the right of the parenthesis of i. So the result holds.

Conversely, let us now prove that if a list L of weighted pairs satisfies the three previous conditions then it is the inversion table of a permutation. Let L_i be the sublist of L that only contains the values of the pairs of elements x and y both smaller than or equal to i. We are going to prove that there exist a unique pseudo-permutation that satisfies all the conditions of list L by building by induction the unique pseudo-permutation satisfying the conditions of each L_i. First, let us notice that L_1 necessarily is empty, so that $\pi_1 = 1$. Next, let us assume that there exist a unique pseudo-permutation on $[\![1, i]\!]$ that satisfies the conditions of L_i.

Let us now consider L_{i+1}. Since the i first integers satisfy by definition the conditions of L_i, these element necessarily are in the same position as in L_i. So there only remains to prove that $i+1$ can be placed in a unique position in this pseudo-permutation.

- If there exist an integer j such that the coefficient of $(j, i+1)$ equals $\frac{1}{2}$, then put $i+1$ is the parenthesis of the smallest such j (notice that it implies uniqueness since $i+1$ *needs* to be in such a parenthesis). Let k be in a parenthesis to the left of j and assume that $k < j$ (the other case can be done in the same way). So the weight of (k,j) is 0. Thanks to the first condition, it then comes that $(k, i+1) = 0$, that is compatible with the respective positions of k and $i+1$. The proof is the same if k is in the parenthesis of j or if it is to the right of j.

- If there does not exist such an integer j, but that there exist and integer j' such that the coefficient of $(j', i+1)$ equals 1, put $i+1$ to the left of the left-most parenthesis containing such a j'. It then remains to check that the values of $(k, i+1)$ are the right ones depending on the respective positions of k and j'. The proof is as simple as in the previous case.

- If $i+1$ does not appear in L_{i+1}, put $i+1$ to the right end of the previous pseudo-permutation and conclude that this case is the only right one given the conditions on the weights.

■

Theorem 5. *Let $\mathfrak{s}$ and $\mathfrak{t}$ be two pseudo-permutations. Then $\mathfrak{s} < \mathfrak{t}$ if and only if the table of inversions of $\mathfrak{t}$ is included in the table of inversions of $\mathfrak{s}$.*

Proof — Let us first notice that if $\mathfrak{s} < \mathfrak{t}$ then the table of inversions of $\mathfrak{t}$ is included in the table of inversions of $\mathfrak{s}$ since it is true for a single interation of an operator M or S.

Conversely, let us assume that the table of inversions of $\mathfrak{t}$ is included in the table of inversions of $\mathfrak{s}$. We make the proof on the difference of the number of inversions of $\mathfrak{t}$ and $\mathfrak{s}$.

Among the set of pairs $i < j$ which coefficients of both inversion tables are not the same, choose a pair (i,j) such that $j - i$ is minimal. There are three cases depending on the possible values of (i,j) in the table of inversions of $\mathfrak{s}$ and $\mathfrak{t}$. Let us only prove the case when this coefficient is 0 in $\mathfrak{t}$ and $\frac{1}{2}$ in $\mathfrak{s}$, the other proofs being the same. Then one has:

$$\mathfrak{s} = \alpha_1(i, j, \alpha_2)\alpha_3 \text{ and } \mathfrak{t} = \beta_1(i, \beta_{1'})\beta_2(j, \beta_{2'})\beta_3.$$

Since $j - i$ is minimal among all the different values between the inversion tables, all letters of the words β_2, $\beta_{1'}$ and $\beta_{2'}$ belong to $[\![1, i-1]\!] \cup [\![j+1, n]\!]$. Indeed, if there exist such an integer k, it cannot belong to any word α_* in $\mathfrak{s}$: if k belongs to α_1 or α_2, the pair (i,k) does not have the same value in $\mathfrak{s}$ and in $\mathfrak{t}$. If k belongs to α_3, the pair (k, j) does not have the same value in $\mathfrak{s}$ and in $\mathfrak{t}$. So one can write β_2, $\beta_{1'}$ and $\beta_{2'}$ as a sequence of x_k all smaller than i shuffled with a sequence of z_k all greater than j.

- If β_2 is the empty word, let $\mathfrak{t}' = \beta_1(\beta_{1'}, i, j, \beta_{2'})\beta_3$.

  It is then clear that $\mathfrak{t}' < \mathfrak{t}$ since one obtains $\mathfrak{t}'$ from $\mathfrak{t}$ by one iteration of operator M. Moreover, by construction, the table of inversions of $\mathfrak{t}'$ is included in the table of inversions of $\mathfrak{s}$, so $\mathfrak{s} < \mathfrak{t}'$ by induction, and $\mathfrak{s} < \mathfrak{t}$, by transitivity.

- If β_2 is not the empty word, let $\mathfrak{t}' = \alpha_1 z_1 \dots z_k(\beta_{1'}, i)(j, \beta_{2'})x_1 \dots x_{k'}\alpha_3$.

  It is then clear that the table of inversions of $\mathfrak{t}$ is strictly included in the table of inversions of $\mathfrak{t}'$. Moreover, let us consider an inversion of $\mathfrak{t}'$. If it is not one of $\mathfrak{t}$, then it is an inversion between i and a z_k or between j and an x_k, or between an x_k and a z_k or between a z_k and $\beta_{1'}$ or betwee an x_k and $\beta_{2'}$. Let us prove that all these inversions belong to $\mathfrak{s}$ too. For example, let us consider an inversion between i and a z_k. In $\mathfrak{t}$, there is a full inversion (j, z_k). So, in $\mathfrak{s}$, there is this inversion too. Since z_k is not in $\llbracket i, j \rrbracket$, it comes from Proposition 4 that (i, z_k) is also an inversion. Doing the same in the other cases, one then gets that the table of inversions of $\mathfrak{t}'$ is included (strictly since it does not contain the inversion (i, j)) in the table of inversions of $\mathfrak{s}$. By induction, one can then see that $\mathfrak{s} < \mathfrak{t}'$ and $\mathfrak{t}' < \mathfrak{t}$ that proves that $\mathfrak{s} < \mathfrak{t}$.

■

4.3 Rises of elements of $\mathfrak{P}(n)$

Let $\mathfrak{s}$ be a pseudo-permutation. We define its *rise set* and its *rise number* respectively as the rise set and the rise number of its horizontal reading. One then has the following property.

Proposition 6. *Let $\mathfrak{s}$ be a pseudo-permutation of order n. Then the number of outgoing edges of $\mathfrak{s}$ in the pseudo-permutohedron $\mathfrak{G}(n)$ is equal to the rise number of $\mathfrak{s}$.*

Proof — First, notice that the operator M can only act between two parentheses if the horizontal reading of these two is increasing. The operator S makes transforms inside a parenthesis, that is, by definition, is increasing order in the horizontal reading. So this proves that the outgoing edges that are applications of the operators M and S can be split into the outgoing edges coming from the different maximal increasing subsequences of the horizontal reading of $\mathfrak{s}$. So the number of outgoing edges of $\mathfrak{s}$ is equal to the sum of the number of outgoing edges of each maximal increasing subsequence of $\mu(\mathfrak{s})$. So, to conclude, it only remains to prove the result for an increasing word.

Let us now consider the case when $\mu(\mathfrak{s})$ is an increasing word. Let us consider two consecutive letters. If they belong to the same parenthesis, one can apply the operator S to split the parenthesis so that these letters belong in the end to two different parentheses. If they belong to two parentheses, one can apply the operation M to merge these parentheses into one. So there is at least as many outgoing edges of $\mathfrak{s}$ as there is consecutive letters in $\mathfrak{s}$, that is the rise number of $\mathfrak{s}$. There cannot exist any other outoing edge since we studied all the cases of application of operators M and S. So we conclude to the proposition. ■

Remark 7. *One can also provide a dual property that gives the number of incoming edges of a pseudo-permutations in terms of its descent number. But one has to define the descent number in a sligthly different way (that is, write the numbers in the parentheses in* decreasing *order and take the descent number of the horizontal reading).*

With the help of one of the previous properties, one can easily establish the following enumeration result:

Corollary 8. *Let n be an integer. The number of edges of the pseudo-permutohedron of order n is given by:*

$$\sum_{k=0}^{n-1} A_{n,k}\, 2^k\, k.$$

5 Lattice properties of $\mathfrak{P}(n)$

5.1 $\mathfrak{G}(n)$ is a lattice

In this subsection, we prove that the pseudo-permutohedron is a lattice. To prove this property, we first define the infimum and the supremum of two given elements. We actually use the fact that the graph of the order $\mathfrak{G}(n)$ is nothing but the inversion graph of $\mathfrak{P}(n)$ (see Theorem 5) and we prove that this order is a lattice.

To achieve this, we introduce the notion of the trans-union of two given pseudo-permutations. First consider the table of inversions of these elements; make then the union of these tables, taking the maximum value for each pair (i, j); finally compute the closure of this table that is the intersection of all inversion tables that contain this union table. This last table is the *trans-union* of the two elements. For example, if we consider $(1)(23)$ and $(12)(3)$ with inversion tables respectively $\{\frac{1}{2}(2,3)\}$ and $\{\frac{1}{2}(1,2)\}$, their trans-union is $\{\frac{1}{2}(1,2), \frac{1}{2}(2,3), \frac{1}{2}(1,3)\}$. Then one can state the following result:

Proposition 9. *Let us consider two elements of a pseudo-permutohedron. Then their trans-union table is the inversion table of a pseudo-permutation that belong to the same pseudo-permutohedron. Moreover, this element is their infimum.*

Proof — We will not give the details of this proof since it is very technical and gives no more insight within the theory. Indeed, thanks to Theorem 5, it is sufficient to prove that the intersection of some inversion tables is an inversion table (this property is false in general but true when one considers the trans-union tables) since it will then clearly be the greatest pseudo-permutation smallest than both tables, so the infimum of these tables. To prove that the intersection of some inversion tables is an inversion table, we make the proof by induction on n. So let L_i and L'_i be the sublists of both lists that only contain the values of the pairs of elements x and y both smaller than or equal to i. Thanks to the induction hypothesis, one can see that the sub-pseudo-permutations of order $n-1$ that contain the union of L_{n-1} and L'_{n-1} have an infimum. It then only remains to put n inside this infimum in the best possible way (there are different cases that can appear that essentially are similar to the cases of Proposition 4). It is then easy to see why, thanks to the inversion table, this last element is the infimum (in the case of the infimum, n is put to the right-most possible position and cannot be any further, even if one tries to put n in other elements than the infimum of order $n-1$). ∎

As it is the case for other lattices, one cannot define the supremum of two elements in such a simple way. However, the pseudo-permutohedron has an interesting property that we can use for this purpose. If one says that the dual of an element is obtained by reading it from right to left, then the graph obtained from $\mathfrak{G}(n)$ by replacing each pseudo-permutation by its dual and by reversing each edge is $\mathfrak{G}(n)$ itself. In other words, $\mathfrak{G}(n)$ is *auto-dual*. Then one obviously has the following duality result:

Proposition 10. *Let us consider two elements of a pseudo-permutohedron. Their* supremum *is the dual of the infimum of their duals.*

We can therefore state the main result of this section that is a consequence of the last two propositions:

Theorem 11. *For all integer n, the pseudo-permutohedron of order n is a lattice.*

5.2 Cutting the pseudo-permutohedron

Let us consider a pseudo-permutohedron and let us consider the connected components obtained by removing the edges which correspond to the S operators. Then each connected component is an hypercube that contains all the elements of the pseudo-permutohedron which have the same horizontal reading. The bottom-most element is built from this permutation (the horizontal reading) by cutting it as the concatenation of the smallest possible number of increasing sequences and putting these sequences into the same parentheses. The top-most element is also built from this permutation by putting each integer in a different parenthesis. For example, if the horizontal reading is 1673425, then the bottom-most element is (167)(34)(25) and the top-most element is (1)(6)(7)(3)(4)(2)(5).

The dimension of the hypercube is equal to the number of descents of its horizontal reading, *i.e.*, the number of ingoing edges of its elements. As an example, we give the decomposition of $\mathfrak{P}(3)$ as an union of hypercubes in Figure 3.

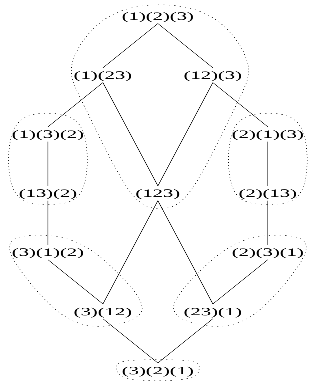

Figure 3: $\mathfrak{G}(3)$ is a disjoint union of hypercubes.

As we saw before, the pseudo-permutohedron is auto-dual, therefore all the properties one can establish for it can be dualized. Since the M edges and the S edges are exchanged in this process, one can derive that the number of M edges is the half of the total number of edges of $\mathfrak{G}(n)$. The previous results then allows to make in another way the enumeration of the edges of $\mathfrak{G}(n)$.

Conclusion

In this paper, we have shown a few properties of a new combinatorial object. These properties are fundamental since they are generalizations of the key properties of the symmetric group, which allows us to think that many other statistics defined on the symmetric group have a real meaning in this context. We believe that the pseudo-permutohedron is an interesting and meaningful extension of the permutohedron, and we hope that some work will be done to extend the classical results to this case. For example, in [BHKN01], we obtained a geometrical interpretation of the pseudo-permutohedron a representation theoretical interpretation too. These interpretations seem very deep and seem connected with many interesting properties of the hyperplane arrangements. It would certainly be very interesting to investigate these last properties too.

Another interesting direction of investigation with many applications is to consider pseudo-permutations with multiple occurences of the integers. This gives a very general and powerful model which makes it possible to represent the relations between multiple events which can have a temporal length. It seems that most of the results presented here are still valid in this context and some new questions arise.

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