

q -Identities from Lagrange and Newton Interpolation

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ABSTRACT: Combining Newton and Lagrange interpolation, we give q -identities which generalize results of Van Hamme, Uchimura, Dilcher and Prodinger.

1. introduction

Van Hamme [8] gave the following identity involving Gauss polynomials, see also Andrews [1],

$$\sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1 - q^i} = \sum_{i=1}^n \frac{q^i}{1 - q^i}, \quad (1.1)$$

where $\begin{bmatrix} n \\ i \end{bmatrix}$ is the Gauss polynomials defined by $\begin{bmatrix} n \\ i \end{bmatrix} = (q; q)_n ((q; q)_i (q; q)_{n-i})^{-1}$ with $(z; q)_n = (1 - z)(1 - zq) \cdots (1 - zq^{n-1})$.

Uchimura [7] generalize (1.1) as following:

$$\sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1 - q^{i+m}} = \sum_{i=1}^n \frac{q^i}{1 - q^i} / \begin{bmatrix} i+m \\ i \end{bmatrix}, \quad m \geq 0 \quad (1.2)$$

and by Dilcher [2]:

$$\sum_{1 \leq i \leq n} \binom{n}{i} (-1)^{i-1} \frac{q^{\binom{i}{2} + mi}}{(1 - q^i)^m} = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq n} \frac{q^{i_1}}{1 - q^{i_1}} \cdots \frac{q^{i_m}}{1 - q^{i_m}}. \quad (1.3)$$

Prodinger [6] mentioned the following identity as a q -analogue of Kirchner's [4] formula,

$$\sum_{i=0, i \neq M} \binom{n}{i} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1 - q^{i-M}} = (-1)^M q^{\binom{M+1}{2}} \binom{n}{M} \sum_{i=0, i \neq M} \frac{q^{i-M}}{1 - q^{i-M}} \quad (1.4)$$

and explained how to obtain all these formulas by using Cauchy residues.

We shall show that in fact, all the above formulas are a direct consequence of Newton and Lagrange interpolation.

Given two finite sets of variables \mathbb{A} and \mathbb{B} , we denote by $R(\mathbb{A}, \mathbb{B})$ the product $\prod_{a \in \mathbb{A}, b \in \mathbb{B}} (a - b)$, and by $\mathbb{A} \setminus \mathbb{B}$ the set difference of \mathbb{A} and \mathbb{B} .

Let $\mathbb{A} = \{x_1, x_2, \dots\}$, $\mathbb{A}_n = \{x_1, x_2, \dots, x_n\}$, for any $n \geq 0$. Lagrange wrote the following summation:

$$f(x) = \sum_{i=1}^n f(x_i) \frac{R(x, \mathbb{A} \setminus x_i)}{R(x_i, \mathbb{A} \setminus x_i)}, \text{ mod } R(x, \mathbb{A}_n). \quad (1.5)$$

On the other hand, Newton's development is:

$$f(x) = f(x_1) + f \partial_1 R(x, \mathbb{A}_1) + f \partial_1 \partial_2 R(x, \mathbb{A}_2) + \dots, \quad (1.6)$$

where ∂_i , $i \geq 1$, operating on its left, is defined by

$$f(x_1, x_2, \dots, x_i, x_{i+1}, \dots) \partial_i = \frac{f(\dots, x_i, x_{i+1}, \dots) - f(\dots, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}}.$$

Taking $x_{n+1} = x$, Newton's summation terminates:

$$f(x) = f(x_1) + f \partial_1 R(x, \mathbb{A}_1) + f \partial_1 \partial_2 R(x, \mathbb{A}_2) + \dots + f \partial_1 \cdots \partial_n R(x, \mathbb{A}_n).$$

Newton's and Lagrange's expressions imply the same remainder $f \partial_1 \cdots \partial_n R(x, \mathbb{A}_n)$, and the polynomial $g_n(x) = f(x) - f \partial_1 \cdots \partial_n R(x, \mathbb{A}_n)$ is the only polynomial of degree $\leq n - 1$ such that $f(x_i) = g(x_i)$, $1 \leq i \leq n$.

Taking now $f(x) = (y - x)^{-1}$, since

$$\frac{1}{y - x_1} \partial_1 \cdots \partial_{n-1} = \frac{1}{(y - x_1) \cdots (y - x_n)},$$

one gets, by comparing Newton's and Lagrange's expressions, the identity:

$$\sum_{i=0}^{n-1} \frac{R(x, \mathbb{A}_i)}{R(y, \mathbb{A}_{i+1})} = \sum_{i=1}^n \frac{f(x_i) R(x, \mathbb{A} \setminus x_i)}{R(x_i, \mathbb{A} \setminus x_i)} = \frac{1}{y - x} - \frac{R(x, \mathbb{A}_n)}{R(y, \mathbb{A}_n)(y - x)}. \quad (1.7)$$

Letting $x = 1$, we derive:

$$\sum_{i=0}^{n-1} \frac{R(1, \mathbb{A}_i)}{R(y, \mathbb{A}_{i+1})} = \sum_{i=1}^n \frac{f(x_i) R(1, \mathbb{A} \setminus x_i)}{R(x_i, \mathbb{A} \setminus x_i)}. \quad (1.8)$$

Expanding $(y - x)^{-1} = \sum_i y^{-i-1} x^i$ and taking the coefficient of y^{-m-1} , we have:

$$x_1^m \partial_1 \cdots \partial_{n-1} = \sum_{i=1}^n \frac{x_i^m}{\prod_{j \neq i} (x_i - x_j)} = h_{m-n+1}(x_1, x_2, \dots, x_n). \quad (1.9)$$

Recall that complete functions [5] h_k are defined by

$$h_k(x_1, x_2, \dots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

2. General identities

The identities that we present just corresponds to taking $\mathbb{A} = \left\{ \frac{a-bq}{c-zq}, \frac{a-bq^2}{c-zq^2}, \dots \right\}$ in Lagrange or Newton interpolation. In that case, the products $R(x_i, \mathbb{A} \setminus x_i)$ are immediate to write.

Proposition 2.1 *Let $m, n \in \mathbb{N}$, $\tau = m - n + 1$, a, b, z, q be variables, and $\mathbb{A} = \left\{ \frac{a-bq}{c-zq}, \frac{a-bq^2}{c-zq^2}, \dots, \frac{a-bq^n}{c-zq^n} \right\}$. We have:*

$$\sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_\tau \leq n} \frac{a - bq^{i_1}}{c - zq^{i_1}} \frac{a - bq^{i_2}}{c - zq^{i_2}} \cdots \frac{a - bq^{i_\tau}}{c - zq^{i_\tau}} = \frac{c^n (zq/c; q)_n}{(q; q)_n (az - bc)^{n-1}} \sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2} - ni} (1 - q^i) (a - bq^i)^m}{(c - zq^i)^{\tau+1}}. \quad (2.10)$$

In particular, for $m = n$:

$$\begin{aligned} \sum_{i=1}^n \frac{a - bq^i}{c - zq^i} &= \\ &= \frac{c^n(zq/c; q)_n}{(q; q)_n(az - bc)^{n-1}} \sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2} - ni} (1 - q^i) (a - bq^i)^n}{(c - zq^i)^2} \end{aligned} \quad (2.11)$$

Proof. For our choice of \mathbb{A} , then $R(x_i, \mathbb{A} \setminus x_i) = \prod_{j \neq i} \frac{(az - bc)(q^i - q^j)}{(c - zq^i)(c - zq^j)}$, and

$$\begin{aligned} &\sum_{i=1}^n \frac{((a - bq^i)/(c - zq^i))^m}{\prod_{j \neq i} \left(\frac{a - bq^i}{c - zq^i} - \frac{a - bq^j}{c - zq^j} \right)} \\ &= \frac{c^n(zq/c; q)_n}{(q; q)_n(az - bc)^{n-1}} \sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2} - ni} (1 - q^i) (a - bq^i)^m}{(c - zq^i)^{m-n+2}}. \end{aligned}$$

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Consider the case $a = 0, b = -1, c = 1$ and $z = 1$, i.e. $\mathbb{A} = \left\{ \frac{q}{1-q}, \frac{q^2}{1-q^2}, \dots, \frac{q^n}{1-q^n} \right\}$.

Eq (2.11) implies Van Hamme identity :

$$\begin{aligned} \sum_{i=1}^n \frac{q^i}{1 - q^i} &= \frac{(q; q)_n}{(q; q)_n} \sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2} - ni} (1 - q^i) q^{ni}}{(1 - q^i)^2} \\ &= \sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1 - q^i}. \end{aligned}$$

From (2.10), we have,

$$\sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_{m-n+1} \leq n} \frac{q^{i_1}}{1 - q^{i_1}} \frac{q^{i_2}}{1 - q^{i_2}} \dots \frac{q^{i_{m-n+1}}}{1 - q^{i_{m-n+1}}} = \sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i+1} q^{\binom{i+1}{2} - ni} q^{mi}}{(1 - q^i)^{m-n+1}},$$

replacing $m - n + 1$ by m , we derive (1.3).

Take now $a = 1, b = 0, c = 0, z = -1$, then $\mathbb{A} = \{q^{-1}, q^{-2}, \dots, q^{-n}\}$, and Eq.(1.8) implies:

$$R.H.S = \sum_{i=0}^{n-1} \frac{R(1, \mathbb{A}_i)}{R(y, \mathbb{A}_{i+1})} = \sum_{i=0}^{n-1} \frac{\prod_{j=1}^i (1 - q^{-j})}{\prod_{j=1}^{i+1} (y - q^{-j})} = - \sum_{i=0}^{n-1} \frac{q^{i+1} (q; q)_i}{(yq; q)_{i+1}},$$

$$\begin{aligned}
L.H.S &= \sum_{i=1}^n \frac{f(x_i)R(1, \mathbb{A} \setminus x_i)}{R(x_i, \mathbb{A} \setminus x_i)} \\
&= \sum_{i=1}^n \frac{(y - q^{-i})^{-1} \prod_{j \neq i} (1 - q^{-j})}{\prod_{j \neq i} (q^{-i} - q^{-j})} = \sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^i q^{\binom{i+1}{2}}}{1 - yq^i}.
\end{aligned}$$

Clearly:

$$\sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1 - yq^i} = \sum_{i=0}^{n-1} \frac{q^{i+1}(q; q)_i}{(yq; q)_{i+1}} = \sum_{i=1}^n \frac{q^i(q; q)_{i-1}}{(yq; q)_i}.$$

In the special case $y = q^m$, we obtain (1.2).

We take now $\mathbb{A} = \{q^M, q^{M-1}, q^{M-2}, \dots, q, q^{-1}, \dots, q^{M-n}\}$, with $M \in \mathbb{N}$, and $y = 1$. Eq(1.8) becomes:

$$R.H.S = \sum_{i=0}^{n-1} \frac{R(1, \mathbb{A}_i)}{R(1, \mathbb{A}_{i+1})} = - \sum_{i=0, i \neq M}^n \frac{q^{i-M}}{1 - q^{i-M}},$$

and

$$\begin{aligned}
L.H.S &= \sum_{i=0, i \neq M}^n \frac{1}{1 - q^{M-i}} \frac{\prod_{j \neq i, M} (1 - q^{M-j})}{\prod_{j \neq i, M} (q^{M-i} - q^{M-j})} \\
&= \sum_{i=0, i \neq M}^n \frac{(-1)^{M+i} q^{\binom{i+1}{2} - \binom{M+1}{2}} (q; q)_M (q; q)_{n-M}}{(q; q)_i (q; q)_{n-i}},
\end{aligned}$$

which proves (1.4).

As a final comment, we would like to stress that we have just used simple alphabets in Newton and Lagrange interpolation; it is easy to generalize the above formulas by taking more sophisticated alphabets.

References

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