Partition Analysis and Symmetrizing Operators

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Abstract.

Using a symmetrizing operator, we give a new expression for the Omega operator used by MacMahon in Partition Analysis, and given a new life by Andrews and his coworkers. Our result is stated in terms of Schur functions.

In his book "Combinatory Analysis", MacMahon introduced an Omega operator. Recently, Andrews et al [1–3] further developed the theory of Partition Analysis. We show in theorem 4 that the Omega operator can be expressed by a symmetrizing operator. As a consequence, we can formulate:

$$\underset{\geq}{\Omega}\, \lambda^k / \prod_{x\in\mathbb{X}} (1-x\lambda) \prod_{y\in\mathbb{Y}} (1-\frac{y}{\lambda})$$

in terms of Schur functions of $\mathbb{X}$ and $\mathbb{Y}$ (and therefore in terms of the elementary symmetric functions in $\mathbb{X}$ and $\mathbb{Y}$).

Recall the definitions of MacMahon's Omega operator $\underset{\geq}{\Omega}$ and of the symmetrizing operator π_ω.

Definition 1

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\cdots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\cdots,s_r}.$$

By iteration, it is sufficient to treat the case of one variable λ only .

Definition 2 *[6] Given $\mathbb{X} = \{x_1, x_2, \cdots, x_n\}$ of cardinality $|\mathbb{X}| = n$, the symmetrizing operator π_ω is defined by:*

$$\forall f(x_1,\ldots,x_n),\ \pi_\omega f(x_1,\cdots,x_n) = \sum_{\sigma\in\mathfrak{S}(\mathbb{X})} \sigma\left(\frac{f(x_1,\cdots,x_n)}{\triangle(\mathbb{X})} x_1^{n-1}\cdots x_n^0\right),$$

writing $\triangle(\mathbb{X})$ for the Vandermonde $\prod_{1\le i<j\le n}(x_i - x_j)$, the sum being over all permutations σ in the symmetric group $\mathfrak{S}(\mathbb{X})$.

Recall that complete symmetric functions $S^j(\mathbb{X})$ are defined by the generating function:

$$\sum_{j=0}^{\infty} S^j(\mathbb{X})\lambda^j = \frac{1}{\prod_{i=1}^{n}(1 - x_i\lambda)}.$$

Complete symmetric functions are compatible with union of alphabets (denoted '+'). Given $\mathbb{Y} = \{y_1, y_2, \cdots, y_m\}$, we have:

$$S^n(\mathbb{X}+\mathbb{Y}) = \sum_{k=0}^{n} S^k(\mathbb{X})S^{n-k}(\mathbb{Y}).$$

Schur functions have two classical expressions:

$$S_\mu(\mathbb{X}) \quad = \quad \left|x_i^{\mu_j+j-1}\right|_{1\le i,j\le n} / \triangle(\mathbb{X}) = \left|S^{\mu_i-i+j}(\mathbb{X})\right|_{1\le i,j\le n},$$

where $\mu = [\mu_1, \cdots, \mu_n]$ with $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n \ge 0$.

From the definition of π_ω, we get [6] :

$$\pi_\omega\, x_1^{\mu_1}\cdots x_n^{\mu_n} = \left|x_i^{\mu_j+j-1}\right|_{1\le i,j\le n} / \triangle(\mathbb{X}) = S_\mu(\mathbb{X}). \tag{1}$$

This formula is still valid if $\mu \in \mathbb{Z}^n$, $\mu_1 > -n, \ldots, \mu_n > -1$:

$$\pi_\omega\, x_1^{\mu_1}\cdots x_n^{\mu_n} = S_\mu(\mathbb{X}), \tag{2}$$

the Schur function S_μ, still defined as the determinant $|S^{\mu_i-i+j}|_{1\le i,j\le n}$, being either null or equal to $\pm$ a Schur function indexed by a partition.

Symmetrizing first in $x_2, \ldots, x_n$, one also has, with the same hypotheses on μ :

$$\pi_\omega\, x_1^{\mu_1}\, S_{\mu_2,\ldots,\mu_n}(x_2, \ldots, x_n) = S_\mu(\mathbb{X})\,. \tag{3}$$

Lemma 3 *Given $\mathbb{X}$, $\mathbb{Y}$ and k such that $0 \le k < |\mathbb{X}|$, then one has:*

$$\pi_\omega \left(\sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{Y})\right) = \sum_{j=0}^{\infty} S^{j-k}(\mathbb{X})S^j(\mathbb{Y})\,. \tag{4}$$

Proof. Since powers of x_1 range from $-k$ to ∞, we can apply (3):

$$\pi_\omega \left(\sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{Y})\right) = \sum_{j=0}^{\infty} S_{j-k,\,0^{n-1}}(\mathbb{X})\, S^j(\mathbb{Y})\,.$$

The terms such that $j < k$ are all null, being determinants with two identical rows, and the sum reduces to the expression stated in the lemma. ∎

Let us remark that the operator $\underset{\geq}{\Omega}$ relative to $x_1, \ldots, x_n$ can be obtained from the operator $x_1, \ldots, x_{n+r}$, $r \geq 0$ by specializing $x_{n+1}, \ldots, x_{n+r}$ to 0. Therefore we can suppose that n be bigger than any given integer. This allows us in the following theorem to suppose that $n > k$.

Theorem 4 *Given two alphabets $\mathbb{X} = \{x_1, x_2, \cdots, x_n\}$ and $\mathbb{Y} = \{y_1, y_2, \cdots, y_m\}$ of cardinality n and m, let $\mathbb{B} = 1 + \mathbb{Y} = \{1\} \cup \mathbb{Y}$. If $k < n$, then we have:*

$$\begin{aligned} \pi_\omega \sum x_1^{j-k} S^j(\mathbb{B}) &= \underset{\geq}{\Omega} \frac{\lambda^k}{(1 - x_1\lambda) \cdots (1 - x_n\lambda)(1 - \frac{y_1}{\lambda}) \cdots (1 - \frac{y_m}{\lambda})} \\ &= \frac{\sum_\mu (-1)^{|\mu|} S_{\mu'}(\mathbb{B}) S_{-k,\,\mu}(\mathbb{X})}{R(1, \mathbb{X}\mathbb{B})}, \end{aligned} \tag{5}$$

where $R(1, \mathbb{X}\mathbb{Y})$ is equal to $\prod_{x \in \mathbb{X}, y \in \mathbb{Y}} (1 - xy)$, and where the sum is over all partitions μ (the sum is in fact finite). The vector $[-k, \mu_1, \ldots, \mu_{n-1}]$ is denoted $-k, \mu$.

Proof. We first recall Cauchy's formula [7, p. 65]:

$$R(1, \mathbb{X}\mathbb{Y}) = \sum_\mu (-1)^{|\mu|} S_\mu(\mathbb{X}) S_{\mu'}(\mathbb{Y}),$$

where $\mu \to \mu'$ is the conjugation of partitions.

$$\begin{aligned} \underset{\geq}{\Omega} \sum_{i,j=0}^{\infty} S^i(\mathbb{X}) S^j(\mathbb{Y}) \lambda^{i-j+k} &= \underset{\geq}{\Omega} \frac{\lambda^k}{(1 - x_1\lambda) \cdots (1 - x_n\lambda)(1 - \frac{y_1}{\lambda}) \cdots (1 - \frac{y_m}{\lambda})} \\ &= \sum_{i=0}^{\infty} S^i(\mathbb{X}) \sum_{j=0}^{i+k} S^j(\mathbb{Y}) = \sum_{i=0}^{\infty} S^i(\mathbb{X}) S^{i+k}(\mathbb{B}) \\ &= \sum_{j=0}^{\infty} S^{j-k}(\mathbb{X}) S^j(\mathbb{B}) \,. \end{aligned}$$

On the other hand, lemma 3 allows us to write this last sum as $\pi_\omega \left(\sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{B}) \right)$. We shall now directly compute the action of π_ω on $\sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{B})$, denoting $\mathbb{X} \setminus x_1 = \{x_2, \ldots, x_n\}$.

$$\begin{aligned} \pi_\omega \sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{B}) &= \pi_\omega \, x_1^{-k} \sum_{j=0}^{\infty} x_1^j S^j(\mathbb{B}) \\ &= \pi_\omega \, \frac{x_1^{-k}}{R(1, x_1\mathbb{B})} = \pi_\omega \, \frac{x_1^{-k} R(1, (\mathbb{X} \setminus x_1)\mathbb{B})}{R(1, \mathbb{X}\mathbb{B})} \\ &= \frac{\pi_\omega \, \left(x_1^{-k} \sum_\mu (-1)^{|\mu|} S_{\mu'}(\mathbb{B}) \, S_\mu(\mathbb{X} \setminus x_1) \right)}{R(1, \mathbb{X}\mathbb{B})} \\ &= \frac{\sum_\mu (-1)^{|\mu|} S_{\mu'}(\mathbb{B}) \, S_{-k,\,\mu}(\mathbb{X})}{R(1, \mathbb{X}\mathbb{B})} \end{aligned}$$

and the theorem is proved. ∎

The result can be expressed in terms of elementary symmetric functions because $e_i(\mathbb{B}) = e_i(\mathbb{Y}) + e_{i-1}(\mathbb{Y})$ and Schur functions are determinants in elementary symmetric functions.

In [4, Theorem 2.1], the authors give a "Fundamental Recurrence" for the numerator of (5).

In [5, Theorem 1.4], Guo-Niu Han expresses the Omega operator in terms of Lagrange interpolation:

$$\underset{\geq}{\Omega} \frac{\lambda^k}{A(\lambda)B(\lambda)} = \sum_{i=1}^{n} \frac{x_i^{n-1-k}}{(1-x_i)B(x_i)\prod_{j\neq i}(x_i - x_j)}, \tag{6}$$

where:

$$A(\lambda) = \prod_{i=1}^{n}(1 - x_i\lambda), B(\lambda) = \prod_{j=1}^{m}(1 - y_j\lambda).$$

Let us recall the definition [6] of the Lagrange operator $L_{\mathbb{X}}$:

Definition 5

$$\forall f \in \mathfrak{Sym}(1|n-1), \quad L_{\mathbb{X}} f(x_1, \ldots, x_n) = \sum_{x\in\mathbb{X}} \frac{f(x, \mathbb{X}\setminus x)}{R(x, \mathbb{X}\setminus x)},$$

where $\mathfrak{Sym}(1|n-1)$ *is the space of polynomials in* $x_1, x_2, \ldots, x_n$*, symmetrical in* $x_2, \ldots, x_n$*, and* $R(x, \mathbb{X}\setminus x) = \prod_{x'\in\mathbb{X}\setminus x}(x - x')$*.*

We can express the Lagrange operator in terms of π_ω.

Lemma 6 $\forall f \in \mathfrak{Sym}(1|n-1)$*, we have:*

$$\pi_\omega f(x_1, \ldots, x_n) = L_{\mathbb{X}}\left(f(x_1, \ldots, x_n)\, x_1^{n-1}\right). \tag{7}$$

Proof. Elements of $f(x_1, x_2, \ldots, x_n)$ can be written as sums of powers of x_1, with coefficients symmetrical in $x_1, \ldots, x_n$. Checking now that

$$L_{\mathbb{X}}(x_1^k\, x_1^{n-1}) = \pi_\omega(x_1^k) = S^k(\mathbb{X}),$$

is immediate. ∎

Formula (7) shows that the Lagrange operator in formula (6) can be replaced by π_ω, and therefore [5, Theorem 1.4] is a consequence of theorem 4.

One does not need to suppose that all the x_i's be distinct. Indeed, in a Schur function, one may specialize $x_1, \ldots, x_k$ to the same value a. This is more of a problem in the Lagrange interpolation formula, where one has in that case to use derivatives of different orders.

Let us finish with a small explicit example of the action of π_ω, for $n = 2$, $m = 1$, $k = 1$.

$$\begin{aligned}
\pi_\omega\left(\sum_{j=0}^{\infty} x_1^{j-1} S^j(\mathbb{B})\right) &= \frac{\sum_\mu (-1)^{|\mu|} S_{\mu'}(\mathbb{B}) S_{-1,\,\mu}(\mathbb{X})}{R(1, \mathbb{X}\mathbb{B})} \\
&= \frac{-S_1(\mathbb{B})S_{-1,\,1}(\mathbb{X}) + S_{1,1}(\mathbb{B})S_{-1,\,2}(\mathbb{X})}{R(1, \mathbb{X}\mathbb{B})} \\
&= \frac{(1+y) - y(x_1 + x_2)}{(1-x_1)(1-x_2)(1-x_1 y)(1-x_2 y)} \\
&= \Omega_{\geq} \frac{\lambda}{(1-\lambda x_1)(1-\lambda x_2)(1-y/\lambda)}.
\end{aligned}$$

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References

[1] G. E. Andrews, MacMahon's Partition Analysis II: Fundamental theorems, Ann. Combi. **4**, 2000, 327-338.

[2] G. E. Andrews, P. Paule and A. Riese, MacMahon's Partition Analysis VI: A new reduction algorithm, Ann. Combi. **5**, 2001, 251-270.

[3] G. E. Andrews, P. Paule, A. Riese and V. Strehl, MacMahon's Analysis V: bijections, recursions, and magic squares, In: Algebraic Combinatorics and Applications, Lecture Notes in Math. A. Betten et al., Eds., Springer, 2001, 1-39.

[4] G. E. Andrews, P. Paule, A. Riese, MacMahon's Analysis: The Omega Package, Europ. J. Combin. **22**, 2001, 887-904.

[5] Guo-Niu Han, A general algorithm for the MacMahon Omega operator, Ann. Combi. **7**, 2003, 1-14.

[6] A. Lascoux, Symmetric funtions & Combinatorial Operators on Polynomials, CBMS/AMS Lecture Notes, 2003.

[7] I. G. Macdonald, Symmetric functions and Hall polynomials, second edition, Oxford Mathematical Monographs, 1995.