

# $q$ -Identities related to overpartitions and divisor functions

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**Abstract.** We prove some  $q$ -Identities related to overpartitions and divisor functions

## 1. Introduction

In this note, using the classical notations  $(z; q)_i = (1 - z) \cdots (1 - zq^{i-1})$ , and  $\begin{bmatrix} n \\ i \end{bmatrix} = (q; q)_n / (q; q)_i (q; q)_{n-i}$ , we prove the following two identities :

**Theorem 1.1** *For any pair positive integers  $m, n$ , one has*

$$\begin{aligned} \sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} (x+1) \cdots (x+q^{i-1})}{(1-q^i)^m} q^{mi} \\ = \sum_{i=1}^n \frac{(-1)^{i-1} (x^i - (-1)^i)}{1-q^i} q^i \sum_{i \leq i_2 \leq \dots \leq i_m \leq n} \frac{q^{\sum_{j=2}^m i_j}}{\prod_{j=2}^m (1-q^{i_j})}, \end{aligned} \quad (1.1)$$

$$\frac{(z; q)_{n+1}}{(q; q)_n} \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} (x+1) \cdots (x+q^{i-1})}{1-zq^i} q^i = \sum_{i=0}^n (-1)^{i-1} \frac{(z; q)_i}{(q; q)_i} x^i q^i. \quad (1.2)$$

In the next section, we shall show that (1.1) and (1.2) can be obtained from the Newton interpolation in points  $\{-1, -q, -q^2, \dots\}$ , using complete functions in the variables  $\{q/(1-q), q^2/(1-q^2), \dots\}$ .

Using the same methods, we have already given in [5] an identity which generalizes the case  $x = 0$  of (1.1).

Given  $\mathbb{X} = \{x_1, x_2, \dots\}$ , Newton gave the following interpolation formula, for any function  $f(x)$ :

$$f(x) = f(x_1) + f\partial_1(x - x_1) + f\partial_1\partial_2(x - x_1)(x - x_2) + \dots,$$

where  $\partial_i$ , acting on its left, is defined by

$$f(x_1, \dots, x_i, x_{i+1}, \dots)\partial_i = \frac{f(\dots, x_i, x_{i+1}, \dots) - f(\dots, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}}.$$

Taking in particular  $f(x) = x^n$ , we have

$$x_1^n \partial_1 \dots \partial_i = h_{n-i}(x_1, x_2, \dots, x_{i+1}), \quad (1.3)$$

where  $h_k$  is the complete function of degree  $k$ , defined by

$$h_k(x_1, x_2, \dots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}.$$

Recall the following properties of  $h_k$ :

1. The generating function of  $h_k$  is:

$$\sum_{k=0}^{\infty} h_k(x_1, \dots, x_n) t^k = \frac{1}{(1 - tx_1) \dots (1 - tx_n)}. \quad (1.4)$$

2. More generally, given two alphabets  $\mathbb{X}$  and  $\mathbb{Y}$ , then the generating functions of  $h_k(\mathbb{X} + \mathbb{Y})$  and  $h_k(\mathbb{X} - \mathbb{Y})$  are:

$$\begin{aligned} \sum_{k=0}^{\infty} h_k(\mathbb{X} + \mathbb{Y}) t^k &= \frac{1}{\prod_{x \in \mathbb{X}} (1 - xt) \prod_{y \in \mathbb{Y}} (1 - yt)}, \\ \sum_{k=0}^{\infty} h_k(\mathbb{X} - \mathbb{Y}) t^k &= \frac{\prod_{y \in \mathbb{Y}} (1 - yt)}{\prod_{x \in \mathbb{X}} (1 - xt)}. \end{aligned} \quad (1.5)$$

As a consequence, one has:

$$h_n(\mathbb{X} + \mathbb{Y}) = \sum_{k=0}^n h_k(\mathbb{X}) h_{n-k}(\mathbb{Y}). \quad (1.6)$$

3. Given  $\{x_1, x_2, \dots, x_n\}$ , and a positive integer  $m$ , we have:

$$\sum_{i=1}^n x_i h_{m-1}(x_i, x_{i+1}, \dots, x_n) = h_m(x_1, x_2, \dots, x_n). \quad (1.7)$$

Taking  $\mathbb{X} = \{-1, -q, -q^2, \dots\}$ , it is easy to check from (1.3) and (1.4):

$$x_1^n \partial_1 \dots \partial_i = h_{n-i}(-1, -q, \dots, -q^i) = (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix}.$$

## 2. Proofs of (1.1) and (1.2)

The Gauss polynomials  $\begin{bmatrix} n \\ k \end{bmatrix}$  satisfy the following recursion (cf. [1]):

$$\sum_{j=0}^n \begin{bmatrix} m+j \\ m \end{bmatrix} q^j = \begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix}. \quad (2.1)$$

In this paper, we need the following more general relations.

**Lemma 2.1** *Let  $k, m$  and  $n$  be nonnegative integers. Then we have the following formulas:*

$$\sum_{i=k}^n \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} \sum_{i \leq i_2 \leq \dots \leq i_m \leq n} \frac{q^{\sum_{j=2}^m i_j}}{\prod_{j=2}^m (1-q^{i_j})} = \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{km}}{(1-q^k)^m}, \quad (2.2)$$

and

$$\sum_{i=0}^n \frac{(z; q)_i}{(q; q)_i} q^i = \frac{(zq; q)_n}{(q; q)_n}. \quad (2.3)$$

*Proof.* Taking  $\mathbb{X} = \{1, q, \dots, q^l\}$  and  $\mathbb{Y} = \{q^{l+1}, q^{l+2}, \dots\}$ , we obtain from (1.5) and (1.6):

$$\frac{1}{(q; q)_n} = h_n(\mathbb{X} + \mathbb{Y}) = \sum_{i=0}^n h_i(\mathbb{Y}) h_{n-i}(\mathbb{X}) = \sum_{i=0}^n \frac{1}{(q; q)_i} \begin{bmatrix} n-i+l \\ l \end{bmatrix} q^{(l+1)i}. \quad (2.4)$$

Letting  $f(m)$  be the left side of (2.2), then we have:

$$\begin{aligned} \sum_{m=1}^{\infty} f(m) z^m &= \sum_{m=1}^{\infty} \sum_{i=k}^n \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} h_{m-1} \left( \frac{q^i}{1-q^i}, \frac{q^{i+1}}{1-q^{i+1}}, \dots, \frac{q^n}{1-q^n} \right) z^m \\ &\stackrel{(1.4)}{=} z \sum_{i=k}^n \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} \frac{1}{(1-q^i z / (1-q^i)) \dots (1-q^n z / (1-q^n))} \\ &= z \sum_{i=k}^n \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} \sum_{l=0}^{\infty} (q^i; q)_{n-i+1} \begin{bmatrix} n-i+l \\ l \end{bmatrix} (q^i(1+z))^l \\ &= z \begin{bmatrix} n \\ k \end{bmatrix} \sum_{i=k}^n \frac{(q; q)_{n-k}}{(q; q)_{i-k}} \sum_{l=0}^{\infty} \begin{bmatrix} n-i+l \\ l \end{bmatrix} q^{(l+1)i} \sum_{m=0}^l \binom{l}{m} z^m \\ &= z \begin{bmatrix} n \\ k \end{bmatrix} \sum_{l=0}^{\infty} \sum_{m=0}^l \binom{l}{m} z^m \sum_{i=k}^n \frac{(q; q)_{n-k}}{(q; q)_{i-k}} \begin{bmatrix} n-i+l \\ l \end{bmatrix} q^{(l+1)i} \\ &\stackrel{(2.4)}{=} \begin{bmatrix} n \\ k \end{bmatrix} \sum_{l=0}^{\infty} \sum_{m=0}^l \binom{l}{m} z^{m+1} q^{(l+1)k} \end{aligned}$$

$$= \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^k z}{1 - q^k(1 + z)} = \sum_{m=1}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{km}}{(1 - q^k)^m} z^m.$$

Taking  $\mathbb{X} = \{1\}$ ,  $\mathbb{Y} = \{q, q^2, \dots\}$  and  $\mathbb{Z} = \{zq, zq^2, \dots\}$ , we get from (1.5) and (1.6), the proof of (2.3):

$$\begin{aligned} \frac{(zq; q)_n}{(q; q)_n} &= h_n((\mathbb{X} + \mathbb{Y}) - \mathbb{Z}) \\ &= h_n(\mathbb{X} + (\mathbb{Y} - \mathbb{Z})) = \sum_{i=0}^n h_i(\mathbb{Y} - \mathbb{Z}) h_{n-i}(\mathbb{X}) = \sum_{i=0}^n \frac{(z; q)_i}{(q; q)_i} q^i. \end{aligned}$$

■

Taking

$$f(x) = \sum_{i=1}^n \frac{(-1)^{i-1} (x^i - (-1)^i)}{1 - q^i} q^i \sum_{i \leq i_2 \leq \dots \leq i_m \leq n} \frac{q^{\sum_{j=2}^m i_j}}{\prod_{j=2}^m (1 - q^{i_j})},$$

and  $\mathbb{X} = \{-1, -q, -q^2, \dots\}$ , then we have,

$$\begin{aligned} f(x) &= f(x_1) + \sum_{k=1}^n f(x_1) \partial_1 \dots \partial_k (x+1) \dots (x+q^{k-1}) \\ &= \sum_{k=1}^n \sum_{i=k}^n (-1)^{k-1} \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1 - q^i} \sum_{i \leq i_2 \leq \dots \leq i_m \leq n} \frac{q^{\sum_{j=2}^m i_j}}{\prod_{j=2}^m (1 - q^{i_j})} (x+1) \dots (x+q^{k-1}) \\ &= \sum_{k=1}^n (-1)^{k-1} (x+1) \dots (x+q^{k-1}) \sum_{i=k}^n \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1 - q^i} \sum_{i \leq i_2 \leq \dots \leq i_m \leq n} \frac{q^{\sum_{j=2}^m i_j}}{\prod_{j=2}^m (1 - q^{i_j})} \\ &= \sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^{k-1} (x+1) \dots (x+q^{k-1})}{(1 - q^k)^m} q^{mk}, \end{aligned}$$

as stated in (1.1).

■

Taking

$$f(x) = \sum_{i=0}^n (-1)^{i-1} \frac{(z; q)_i}{(q; q)_i} x^i q^i, \quad \text{and} \quad \mathbb{X} = \{-1, -q, -q^2, \dots\},$$

we have,

$$\begin{aligned} f(x) &= \sum_{k=0}^n f(x_1) \partial_1 \dots \partial_k (x+1) \dots (x+q^{k-1}) \\ &= \sum_{k=0}^n (x+1) \dots (x+q^{k-1}) \sum_{i=k}^n (-1)^{k-1} \begin{bmatrix} i \\ k \end{bmatrix} \frac{(z; q)_i}{(q; q)_i} q^i \\ &= \sum_{k=0}^n (-1)^{k-1} (x+1) \dots (x+q^{k-1}) q^k \frac{(z; q)_k}{(q; q)_k} \sum_{i=k}^n \frac{(zq^k; q)_{i-k}}{(q; q)_{i-k}} q^{i-k} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^n (-1)^{k-1} (x+1) \dots (x+q^{k-1}) q^k \frac{(z; q)_k (zq^{k+1}; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} \\
&= \frac{(z; q)_{n+1}}{(q; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^{k-1} (x+1) \dots (x+q^{k-1})}{1-zq^k} q^k,
\end{aligned}$$

and this proves (1.2). ■

### 3. Special Cases

In their study of overpartitions [3, Theorem 4.4], Corteel and Lovejoy obtained a combinatorial interpretation of the identity:

$$\sum_{i=1}^{\infty} (-1)^{i-1} \frac{(-1; q)_i}{(q; q)_i} \frac{q^i}{1-q^i} = \sum_{i=1}^{\infty} \frac{2q^{2i-1}}{1-q^{2i-1}} = \sum_{i=1}^{\infty} \frac{2q^i}{1-q^{2i}}. \quad (3.1)$$

Corteel asked us a proof of the following two related identities :

$$\begin{aligned}
\sum_{i=1}^{2n-1} \begin{bmatrix} 2n-1 \\ i \end{bmatrix} \frac{(-1)^{i-1} (-1; q)_i}{(1-q^i)^m} q^{mi} \\
= \sum_{i=1}^n \frac{2q^{2i-1}}{1-q^{2i-1}} \sum_{2i-1 \leq i_2 \leq \dots \leq i_m \leq 2n-1} \frac{q^{\sum_{j=2}^m i_j}}{\prod_{j=2}^m (1-q^{i_j})}, \quad (3.2)
\end{aligned}$$

and

$$\sum_{i=1}^{2n} \begin{bmatrix} 2n \\ i \end{bmatrix} \frac{(-1)^{i-1} (-1; q)_i}{1-q^{i+2}} q^i = \sum_{i=1}^n \frac{2q^{2i-1} (1-q)}{(1+q^2)(1-q^{2i-1})(1-q^{2i+1})}, \quad (3.3)$$

the first one becoming (3.1) for  $m = 1$  and  $n = \infty$ , because

$$\sum_{i=1}^{\infty} \frac{2q^i}{1-q^{2i}} = \sum_{i=1}^{\infty} \frac{2q^{2i-1}}{1-q^{2i-1}}.$$

In fact, (3.2) is the special case of (1.1) when  $x = 1$ .

Changing  $n$  into  $2n$ , specializing  $x = 1$  and  $z = q^2$ , one gets from (1.2)

$$\frac{(1-q^{2n+1})(1-q^{2n+2})}{1-q} \sum_{i=0}^{2n} \begin{bmatrix} 2n \\ i \end{bmatrix} \frac{(-1)^{i-1} (-1; q)_i}{1-q^{i+2}} q^i = \sum_{i=0}^{2n} (-1)^{i-1} q^i \frac{1-q^{i+1}}{1-q}.$$

Suppressing in the left member the term corresponding to  $i = 0$ , and taking into account that the right member of (3.3) sums to  $2q(1-q^{2n})(1-q^4)^{-1}(1-q^{2n+1})^{-1}$ , one obtains (3.3).

The case  $x = 0$ ,  $m = 1$  of (1.1) is due to Van Hamme [6] (see also [2], [5], [7]):

$$\sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1-q^i} = \sum_{i=1}^n \frac{q^i}{1-q^i}.$$

Taking  $x = 0$ , and (1.7), we get the formula of Dilcher [4]:

$$\sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} (-1)^{i-1} \frac{q^{\binom{i}{2}+mi}}{(1-q^i)^m} = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq n} \frac{q^{i_1}}{1-q^{i_1}} \cdots \frac{q^{i_m}}{1-q^{i_m}}.$$

When  $x = 0$  and  $z = q^m$  in (1.2), we get Uchimura's identity [8]:

$$\sum_{i=1}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+2}{2}}}{1-q^{i+m}} = \sum_{i=1}^n \frac{q^i}{1-q^i} \Big/ \begin{bmatrix} i+m \\ i \end{bmatrix}.$$

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