

**BRANCHING RULES FOR SYMMETRIC MACDONALD  
POLYNOMIALS AND  $\mathfrak{sl}_n$  BASIC HYPERGEOMETRIC SERIES**

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ABSTRACT. A one-parameter generalisation  $R_\lambda(X; b)$  of the symmetric Macdonald polynomials and interpolations Macdonald polynomials is studied from the point of view of branching rules. We establish a Pieri formula, evaluation symmetry, principal specialisation formula and  $q$ -difference equation for  $R_\lambda(X; b)$ . We also prove a new multiple  $q$ -Gauss summation formula and several further results for  $\mathfrak{sl}_n$  basic hypergeometric series based on  $R_\lambda(X; b)$ .

1. INTRODUCTION

Let  $\lambda$  be a partition, i.e.,  $\lambda = (\lambda_1, \lambda_2, \dots)$  is a weakly decreasing sequence of nonnegative integers such that  $|\lambda| := \lambda_1 + \lambda_2 + \dots$  is finite. Let the length  $l(\lambda)$  of  $\lambda$  be the number of nonzero  $\lambda_i$ . For  $x = (x_1, \dots, x_n)$  and  $l(\lambda) \leq n$  the Schur function  $s_\lambda(x)$  is defined as

$$(1.1) \quad s_\lambda(x) := \frac{\det_{1 \leq i, j \leq n} (x_i^{\lambda_j + n - j})}{\det_{1 \leq i, j \leq n} (x_i^{n - j})} = \frac{\det_{1 \leq i, j \leq n} (x_i^{\lambda_j + n - j})}{\Delta(x)},$$

where  $\Delta(x) := \prod_{i < j} (x_i - x_j)$  is the Vandermonde product. If  $l(\lambda) > n$  then  $s_\lambda(x) := 0$ . From its definition it is clear that  $s_\lambda(x)$  is a symmetric polynomial in  $x$  of homogeneous degree  $|\lambda|$ , and that  $\{s_\lambda \mid l(\lambda) \leq n\}$  forms a basis of the ring of symmetric functions  $\Lambda_n := \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ .

A classical result for Schur functions is the combinatorial formula

$$(1.2) \quad s_\lambda(x) = \sum_T x^T.$$

Here the sum is over all semi-standard Young tableau  $T$  of shape  $\lambda$ , and  $x^T$  is shorthand for the monomial  $x_1^{\mu_1} x_2^{\mu_2} \dots x_n^{\mu_n}$  with  $\mu_i$  the number of squares of the tableau filled with the number  $i$ . One of the remarkable facts of (1.2) is that it actually yields a symmetric function.

The conventional way to view a semi-standard Young tableau of shape  $\lambda$  (and length at most  $n$ ) as a filling of a Young diagram with the numbers  $1, 2, \dots, n$  such that squares are strictly increasing along columns and weakly increasing along rows. Given two partitions (or Young diagrams)  $\lambda, \mu$  write  $\mu \preceq \lambda$  if  $\mu \subseteq \lambda$  and  $\lambda - \mu$  is a horizontal strip, i.e., if the skew diagram  $\lambda - \mu$  contains at most one square in each column. Then an alternative viewpoint is to consider a Young tableau of shape  $\lambda$  as a sequence of partitions

$$(1.3) \quad 0 = \lambda^{(0)} \preceq \lambda^{(1)} \preceq \dots \preceq \lambda^{(n)} = \lambda,$$

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where 0 denotes the empty partition. For example, for  $n = 6$  the tableau

1	1	1	2	2	2	4	6
2	2	4	5	5	5		
4	5	6					
5	6						
6							

may be encoded as

$$0 \preceq (3) \preceq (6, 2) \preceq (6, 2) \preceq (7, 3, 1) \preceq (7, 6, 2, 1) \preceq (8, 6, 3, 2, 1).$$

The above description implies that a recursive formulation of the Schur functions, equivalent to the combinatorial formula (1.2), is given by the *branching rule*

$$(1.4) \quad s_\lambda(x_1, \dots, x_n) = \sum_{\mu \preceq \lambda} x_n^{|\lambda - \mu|} s_\mu(x_1, \dots, x_{n-1}),$$

subject to the initial condition  $s_\lambda(-) = \delta_{\lambda, 0}$ .

If we let  $\mu \subseteq \lambda$  be a pair of partitions and define the skew Schur function  $s_{\lambda/\mu}$  of a single variable  $z$  as

$$s_{\lambda/\mu}(z) := \begin{cases} z^{|\lambda - \mu|} & \text{if } \mu \preceq \lambda \\ 0 & \text{otherwise,} \end{cases}$$

then the branching rule for Schur functions takes the more familiar form

$$(1.5) \quad s_\lambda(x_1, \dots, x_n) = \sum_{\mu \subseteq \lambda} s_{\lambda/\mu}(x_n) s_\mu(x_1, \dots, x_{n-1}).$$

The Macdonald polynomials  $P_\lambda(x) = P_\lambda(x; q, t)$  [11, 12] are an important  $q, t$ -generalisation of the Schur functions, and the  $P_\lambda$  for  $l(\lambda) \leq n$  form a basis of the ring  $\Lambda_{n, \mathbb{F}} := \Lambda_n \otimes \mathbb{F}$ , where  $\mathbb{F} = \mathbb{Q}(q, t)$ . A classical result in the theory is that the Macdonald polynomials satisfy a combinatorial formula not unlike that of the Schur functions;

$$P_\lambda(x) = \sum_T \psi_T x^T,$$

where  $\psi_T = \psi_T(q, t) \in \mathbb{F}$  is a function that admits an explicit combinatorial description. Importantly, if  $T$  has no more than  $n$  rows it factorises as

$$\psi_T = \prod_{i=1}^n \psi_{\lambda^{(i)}/\lambda^{(i-1)}},$$

where, as before,  $0 = \lambda^{(0)} \preceq \dots \preceq \lambda^{(n)} = \lambda$  is the sequence of partitions representing  $T$ . Probably the simplest (albeit non-combinatorial) expression for  $\psi_{\lambda/\mu}$  is [12, Page 342]

$$(1.6) \quad \psi_{\lambda/\mu} = \prod_{1 \leq i \leq j \leq l(\mu)} \frac{f(q^{\mu_i - \mu_j} t^{j-i}) f(q^{\lambda_i - \lambda_{j+1}} t^{j-i})}{f(q^{\lambda_i - \mu_j} t^{j-i}) f(q^{\mu_i - \lambda_{j+1}} t^{j-i})},$$

where  $f(a) = (at)_\infty / (aq)_\infty$  with  $(a)_\infty = \prod_{i \geq 0} (1 - aq^i)$ . (Note that  $\psi_{\lambda/\mu} \in \mathbb{F}$  since  $\mu \preceq \lambda$ .) It follows from the above that the Macdonald polynomials, like the Schur functions, can be described by a simple branching rule. Namely,

$$(1.7) \quad P_\lambda(x_1, \dots, x_n) = \sum_{\mu \preceq \lambda} x_n^{|\lambda - \mu|} \psi_{\lambda/\mu} P_\mu(x_1, \dots, x_{n-1}),$$

subject to the initial condition  $P_\lambda(-) = \delta_{\lambda,0}$ . Again we may define a single-variable skew polynomial  $P_{\lambda/\mu}(z) = P_{\lambda/\mu}(z; q, t)$  for  $\mu \subseteq \lambda$

$$(1.8) \quad P_{\lambda/\mu}(z) := \begin{cases} z^{|\lambda-\mu|} \psi_{\lambda/\mu} & \text{if } \mu \preceq \lambda \\ 0 & \text{otherwise} \end{cases}$$

to turn the branching formula for the Macdonald polynomials into

$$(1.9) \quad P_\lambda(x_1, \dots, x_n) = \sum_{\mu \subseteq \lambda} P_{\lambda/\mu}(x_n) P_\mu(x_1, \dots, x_{n-1}).$$

In view of the above two examples of symmetric functions admitting a recursive description in the form of a branching formula, a natural question is

*Can one find more general branching-type formulas that lead to symmetric functions?*

To fully appreciate the question we should point out that it is not at all obvious that if one were to take (1.4) (or, equivalently, (1.5)) as the definition of the Schur functions or (1.7) (or (1.9)) as the definition of the Macdonald polynomials, that the polynomials in question are symmetric in  $x$ .

Assuming throughout that  $|q| < 1$  let the (generalised)  $q$ -shifted factorials be defined as follows:

$$(1.10a) \quad (b)_\infty = (b; q)_\infty := \prod_{i=0}^{\infty} (1 - bq^i),$$

$$(1.10b) \quad (b)_k = (b; q)_k := \frac{(b)_\infty}{(bq^k)_\infty},$$

$$(1.10c) \quad (b)_\lambda = (b; q, t)_\lambda := \prod_{i=1}^{l(\lambda)} (bt^{1-i})_{\lambda_i},$$

and let  $(b_1, \dots, b_i)_k = (b_1)_k \cdots (b_i)_k$  and  $(b_1, \dots, b_i)_\lambda = (b_1)_\lambda \cdots (b_i)_\lambda$ . Then probably the best-known example of a branching rule generalising (1.9) and resulting in symmetric polynomials is

$$(1.11) \quad M_\lambda(x_1, \dots, x_n) = \sum_{\mu \subseteq \lambda} \frac{(t^{n-1}/x_n)_\lambda}{(t^{n-1}/x_n)_\mu} P_{\lambda/\mu}(x_n) M_\mu(x_1, \dots, x_{n-1}).$$

The  $M_\lambda(x) = M_\lambda(x; q, t)$  are the interpolation Macdonald polynomials of Knop, Okounkov and Sahi [5, 14, 15, 23], and (1.11) is [15, Proposition 5.3]. For comparison with [15], we have

$$M_\lambda(x_1, \dots, x_n) = t^{(n-1)|\lambda|} P_\lambda^*(t^{1-n}x_1, \dots, t^{-1}x_{n-1}, x_n).$$

From (1.11) it is clear that the top-homogeneous component of  $M_\lambda(x)$  is the Macdonald polynomial  $P_\lambda(x)$  so that  $\{M_\lambda \mid l(\lambda) \leq n\}$  forms an inhomogeneous basis of  $\Lambda_{n, \mathbb{F}}$ .

For  $\lambda = (\lambda_1, \dots, \lambda_n)$  and  $\mu = (\mu_1, \dots, \mu_{n-1})$  such that  $\lambda - \mu$  is a horizontal strip and such that  $\lambda_1 \leq m$  denote by  $m^n - \lambda$  and  $m^{n-1} - \mu$  the partitions  $(m - \lambda_n, \dots, m - \lambda_1)$  and  $(m - \mu_{n-1}, \dots, m - \mu_1)$ . Note that  $(m^n - \lambda) - (m^{n-1} - \mu)$  is again a horizontal strip. It follows from (1.6) that

$$P_{(m^n - \lambda)/(m^{n-1} - \mu)}(1/z; 1/q, 1/t) = z^{-m} P_{\lambda/\mu}(z; 1/q, 1/t) = z^{-m} P_{\lambda/\mu}(z)$$

for  $\lambda_1 \leq m$ . It also follows from (1.6) and (1.11) that

$$\frac{M_{m^n}(1/x_1, \dots, 1/x_n; 1/q, 1/t)}{M_{m^{n-1}}(1/x_1, \dots, 1/x_{n-1}; 1/q, 1/t)} = x_n^{-m}(x_n; 1/q)_m.$$

If we replace  $(x, q, t) \mapsto (1/x, 1/q, 1/t)$  in (1.11) and then change  $\lambda \mapsto (m - \lambda_n, \dots, m - \lambda_1)$  and  $\mu \mapsto (m - \mu_{n-1}, \dots, m - \mu_1)$  the branching rule for the interpolation Macdonald polynomials may thus be recast as

$$(1.12) \quad \frac{M_{m^n - \lambda}(1/x_1, \dots, 1/x_n; 1/q, 1/t)}{M_{m^n}(1/x_1, \dots, 1/x_n; 1/q, 1/t)} = \sum_{\mu \subseteq \lambda} \frac{(q^{1-m} x_n/t)_\mu}{(q^{1-m} x_n)_\lambda} P_{\lambda/\mu}(x_n) \frac{M_{m^{n-1} - \mu}(1/x_1, \dots, 1/x_{n-1}; 1/q, 1/t)}{M_{m^{n-1}}(1/x_1, \dots, 1/x_{n-1}; 1/q, 1/t)}$$

In this paper we consider a generalisation  $R_\lambda(x; b) = R_\lambda(x; b; q, t)$  of the Macdonald polynomials and the Macdonald interpolation polynomials defined recursively by the branching rule

$$(1.13) \quad R_\lambda(x_1, \dots, x_n; b) := \sum_{\mu \subseteq \lambda} \frac{(bx_n/t)_\mu}{(bx_n)_\lambda} P_{\lambda/\mu}(x_n) R_\mu(x_1, \dots, x_{n-1}; b).$$

Our interest in the functions  $R_\lambda(x; b)$  is not merely that they provide another example of a class of symmetric functions defined by a simple branching formula. Indeed, it may be shown that the more general

$$(1.14) \quad R_\lambda(x_1, \dots, x_n; a, b) := \sum_{\mu \subseteq \lambda} \frac{(x_n/a)_\lambda (bx_n/t)_\mu}{(x_n/a)_\mu (bx_n)_\lambda} P_{\lambda/\mu}(a) R_\mu(x_1, \dots, x_{n-1}; at, b)$$

is also symmetric. Moreover  $R_\lambda(x; a, b)$  is a limiting case (reducing  $BC_n$  symmetry to  $\mathfrak{S}_n$  symmetry and breaking ellipticity) of Rains'  $BC_n$  symmetric abelian interpolation functions [19, 21] so that  $R_\lambda(x; b)$  is not actually new. However, it turns out that the function  $R_\lambda(x; b)$  has a number of interesting properties, not shared by  $R_\lambda(x; a, b)$  or the more general  $BC_n$  abelian functions. For example,  $R_\lambda(x; b)$  satisfies a Pieri formula that not just implies the standard Pieri formulas for Macdonald polynomials, but also gives an  $\mathfrak{sl}_n$  generalisation of the famous  $q$ -Gauss summation formula. Specifically, with  $R_\lambda(x; b)$  a suitable normalisation of the functions  $R_\lambda(x; b)$  defined in equation (5.1), and  $X$  an arbitrary finite alphabet, we prove that

$$(1.15) \quad \sum_{\lambda} \left(\frac{c}{ab}\right)^{|\lambda|} (a, b)_\lambda R_\lambda(X; c) = \prod_{x \in X} \frac{(cx/a, cx/b)_\infty}{(cx, cx/ab)_\infty}.$$

For  $X = \{1\}$  this simplifies to the well-known  $q$ -Gauss sum

$$(1.16) \quad \sum_{k=0}^{\infty} \frac{(a, b)_k}{(q, c)_k} \left(\frac{c}{ab}\right)^k = \frac{(c/a, c/b)_\infty}{(c, c/ab)_\infty}.$$

## 2. PRELIMINARIES ON MACDONALD POLYNOMIALS

We begin with a remark about notation. If  $f$  is a symmetric function we will often write  $f(X)$  with  $X = \{x_1, \dots, x_n\}$  (and refer to  $X$  as an alphabet) instead of  $f(x)$  with  $x = (x_1, \dots, x_n)$ , the latter notation being reserved for function that are not (a priori) symmetric. Following this notation we also use  $f(X + Y)$  where

$X + Y$  denotes the (disjoint) union of the alphabets  $X$  and  $Y$ , and  $f(X + z)$  where  $X + z$  denotes the alphabet  $X$  with the single letter  $z$  added.

In the following we review some of the basics of Macdonald polynomial theory, most of which can be found in [11, 12].

Let  $T_{q,x_i}$  be the  $q$ -shift operator acting on the variable  $x_i$ :

$$(T_{q,x_i}f)(x_1, \dots, x_n) = f(x_1, \dots, x_{i-1}, qx_i, x_{i+1}, \dots, x_n).$$

Then the Macdonald polynomials  $P_\lambda(X) = P_\lambda(X; q, t)$  for  $X = \{x_1, \dots, x_n\}$  are the unique polynomial eigenfunctions of the Macdonald operator

$$(2.1) \quad D_n(c) := \sum_{I \subseteq [n]} (-c)^{|I|} t^{\binom{|I|}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q,x_i},$$

where  $[n] := \{1, \dots, n\}$ . Explicitly,

$$(2.2) \quad D_n(c)P_\lambda(X) = P_\lambda(X) \prod_{i=1}^n (1 - cq^{\lambda_i} t^{n-i}).$$

For later reference we state the coefficient of  $c^1$  of this equation separately; if

$$(2.3) \quad D_n^1 := \sum_{i=1}^n \left( \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \right) T_{q,x_i},$$

then

$$(2.4) \quad D_n^1 P_\lambda(X) = P_\lambda(X) \sum_{i=1}^n q^{\lambda_i} t^{n-i}.$$

For each square  $s = (i, j) \in \mathbb{Z}^2$  in the (Young) diagram of a partition (i.e., for each  $i \in \{1, \dots, l(\lambda)\}$  and  $j \in \{1, \dots, \lambda_i\}$ ), the arm-length  $a(s)$ , arm-colength  $a'(s)$ , leg-length  $l(s)$  and leg-colength  $l'(s)$  are given by

$$a(s) = \lambda_i - j, \quad a'(s) = j - 1$$

and

$$l(s) = \lambda'_j - i, \quad l'(s) = i - 1,$$

where  $\lambda'$  is the conjugate of  $\lambda$ , obtained by reflecting the diagram of  $\lambda$  in the main diagonal. Note that the generalised  $q$ -shifted factorial (1.10c) can be expressed in terms of the colengths as

$$(b)_\lambda = \prod_{s \in \lambda} (1 - bq^{a'(s)} t^{-l'(s)}).$$

With the above notation we define the further  $q$ -shifted factorials  $c'_\lambda = c'_\lambda(q, t)$ ,  $c_\lambda = c_\lambda(q, t)$  and  $b_\lambda = b_\lambda(q, t)$  as

$$c'_\lambda := \prod_{s \in \lambda} (1 - q^{a(s)+1} t^{l(s)}) \quad \text{and} \quad c_\lambda := \prod_{s \in \lambda} (1 - q^{a(s)} t^{l(s)+1})$$

and

$$b_\lambda := \frac{c_\lambda}{c'_\lambda}.$$

Then the Macdonald polynomials  $Q_\lambda(X) = Q_\lambda(X; q, t)$  are defined as

$$Q_\lambda(X) := b_\lambda P_\lambda(X).$$

We also need the skew Macdonald polynomials  $P_{\lambda/\mu}$  and  $Q_{\lambda/\mu}$  defined for  $\mu \subseteq \lambda$  by

$$(2.5a) \quad P_\lambda(X + Y) = \sum_{\mu \subseteq \lambda} P_{\lambda/\mu}(Y) P_\mu(X)$$

$$(2.5b) \quad Q_\lambda(X + Y) = \sum_{\mu \subseteq \lambda} Q_{\lambda/\mu}(Y) Q_\mu(X).$$

Note that  $P_{\lambda/0} = P_\lambda$  and  $Q_{\lambda/0} = Q_\lambda$ , and that  $P_{\lambda/\lambda} = Q_{\lambda/\lambda} = 1$ . To simplify some later equations it will be useful to extend the definitions of  $P_{\lambda/\mu}$  and  $Q_{\lambda/\mu}$  to all partition pairs  $\lambda, \mu$  by setting  $P_{\lambda/\mu} = Q_{\lambda/\mu} = 0$  if  $\mu \not\subseteq \lambda$ . From (2.1), (2.2) and (2.5) it follows that for  $a$  a scalar,

$$(2.6a) \quad P_{\lambda/\mu}(aX) = a^{|\lambda-\mu|} P_{\lambda/\mu}(X)$$

$$(2.6b) \quad Q_{\lambda/\mu}(aX) = a^{|\lambda-\mu|} Q_{\lambda/\mu}(X),$$

where  $aX := \{ax \mid x \in X\}$ .

For subsequent purposes it will be convenient to also introduce normalised (skew) Macdonald polynomials  $P_{\lambda/\mu}$  and  $Q_{\lambda/\mu}$  as

$$(2.7a) \quad P_{\lambda/\mu}(X) = t^{n(\lambda)-n(\mu)} \frac{c'_\mu}{c'_\lambda} P_{\lambda/\mu}(X) = t^{n(\lambda)-n(\mu)} \frac{c_\mu}{c_\lambda} Q_{\lambda/\mu}(X)$$

$$(2.7b) \quad Q_{\lambda/\mu}(X) = t^{n(\mu)-n(\lambda)} \frac{c'_\lambda}{c'_\mu} Q_{\lambda/\mu}(X) = t^{n(\mu)-n(\lambda)} \frac{c_\lambda}{c_\mu} P_{\lambda/\mu}(X),$$

where

$$n(\lambda) := \sum_{s \in \lambda} l'(s) = \sum_{i=1}^{l(\lambda)} (i-1)\lambda_i.$$

Note that no additional factors arise in the normalised form of (2.5):

$$P_\lambda(X + Y) = \sum_{\mu} P_{\lambda/\mu}(Y) P_\mu(X)$$

$$Q_\lambda(X + Y) = \sum_{\mu} Q_{\lambda/\mu}(Y) Q_\mu(X),$$

and that  $P_{\lambda/0} = P_\lambda$  and  $Q_{\lambda/0} = Q_\lambda$ . If we define the structure constants  $f_{\mu\nu}^\lambda = f_{\mu\nu}^\lambda(q, t)$  by

$$(2.8) \quad P_\mu(X) P_\nu(X) = \sum_{\lambda} f_{\mu\nu}^\lambda P_\lambda(X),$$

then

$$f_{\mu\nu}^\lambda = t^{n(\mu)+n(\nu)-n(\lambda)} \frac{c'_\lambda}{c'_\mu c'_\nu} f_{\mu\nu}^\lambda,$$

with  $f_{\mu\nu}^\lambda$  the  $q, t$ -Littlewood–Richardson coefficients.

Below we make use of some limited  $\lambda$ -ring notation (see [6] for more details). Let  $p_r$  be the  $r$ th power-sum symmetric function

$$p_r(X) := \sum_{x \in X} x^r$$

and let  $p_\lambda := \prod_{i \geq 1} p_{\lambda_i}$ . Then the  $p_\lambda(X)$  form a basis of  $\Lambda_n$  where  $n = |X|$ . Given a symmetric function  $f(X)$  we define

$$f\left[\frac{a-b}{1-t}\right] := \phi_{a,b}(f),$$

where  $\phi_{a,b}$  is the evaluation homomorphism given by

$$(2.9) \quad \phi_{a,b}(p_r) = \frac{a^r - b^r}{1 - t^r}.$$

In particular,  $f[(1-t^n)/(1-t)] = f(1, t, \dots, t^{n-1})$  is known as the principal specialisation, which we will also denote as  $f(\langle 0 \rangle)$ , and  $f[1/(1-t)] = f(1, t, t^2, \dots)$ . From [7, Equation (6.24)]

$$\mathbf{Q}_{\lambda/\mu}\left[\frac{a-b}{1-t}\right] = a^{|\lambda-\mu|} \sum_{\nu} (b/a)_{\nu} f_{\mu\nu}^{\lambda},$$

which, by  $f_{0\nu}^{\lambda} = \delta_{\lambda\nu}$ , also implies that

$$(2.10) \quad \mathbf{Q}_{\lambda}\left[\frac{1-a}{1-t}\right] = (a)_{\lambda} \quad \text{and} \quad \mathbf{P}_{\lambda}\left[\frac{1-a}{1-t}\right] = t^{2n(\lambda)} \frac{(a)_{\lambda}}{c_{\lambda} c'_{\lambda}}.$$

This last equation yields the well-known principal specialisation formula

$$\mathbf{P}_{\lambda}(\langle 0 \rangle) = t^{2n(\lambda)} \frac{(t^n)_{\lambda}}{c_{\lambda} c'_{\lambda}}.$$

The Cauchy identity for (skew) Macdonald polynomials is given by

$$\sum_{\lambda} \mathbf{P}_{\lambda/\mu}(X) \mathbf{Q}_{\lambda/\nu}(Y) = \left( \prod_{x \in X} \prod_{y \in Y} \frac{(txy)_{\infty}}{(xy)_{\infty}} \right) \sum_{\lambda} \mathbf{P}_{\nu/\lambda}(X) \mathbf{Q}_{\mu/\lambda}(Y).$$

The product on the right-hand side may alternatively be expressed in terms of the power-sum symmetric functions as

$$\exp\left(\sum_{r=1}^{\infty} \frac{1}{r} \frac{1-t^r}{1-q^r} p_r(X) p_r(Y)\right).$$

It thus follows from (2.9) and (2.10), as well as some elementary manipulations, that application of  $\phi_{a,c}$  (acting on  $Y$ ) turns the Cauchy identity into

$$(2.11) \quad \sum_{\lambda} \mathbf{Q}_{\lambda/\nu}\left[\frac{a-c}{1-t}\right] \mathbf{P}_{\lambda/\mu}(X) = \left( \prod_{x \in X} \frac{(cx)_{\infty}}{(ax)_{\infty}} \right) \sum_{\lambda} \mathbf{Q}_{\mu/\lambda}\left[\frac{a-c}{1-t}\right] \mathbf{P}_{\nu/\lambda}(X).$$

For  $\mu = \nu = 0$  (followed by the substitution  $X \rightarrow X/a$  and then  $a \rightarrow c/a$ ) this is the  $q$ -binomial identity for Macdonald polynomials [4, 13]

$$(2.12) \quad \sum_{\lambda} (a)_{\lambda} \mathbf{P}_{\lambda}(X) = \prod_{x \in X} \frac{(ax)_{\infty}}{(x)_{\infty}}.$$

For later reference we also state the more general  $(\mu, \nu) = (0, \mu)$  instance of (2.11)

$$(2.13) \quad \mathbf{P}_{\mu}(X) \prod_{x \in X} \frac{(bx)_{\infty}}{(ax)_{\infty}} = \sum_{\lambda} \mathbf{Q}_{\lambda/\mu}\left[\frac{a-b}{1-t}\right] \mathbf{P}_{\lambda}(X).$$

For reasons outlined below we will refer to this as a Pieri formula.

Let  $\phi_{\lambda/\mu} = \phi_{\lambda,\mu}(q, t)$  and  $\psi'_{\lambda/\mu} = \psi'_{\lambda,\mu}(q, t)$  be defined by

$$\phi_{\lambda/\mu} := \frac{b_\lambda}{b_\mu} \psi_{\lambda/\mu} \quad \text{and} \quad \psi'_{\lambda/\mu}(q, t) := \psi_{\lambda'/\mu'}(t, q).$$

(For combinatorial expressions for all of  $\psi_{\lambda/\mu}$ ,  $\psi'_{\lambda/\mu}$  and  $\phi_{\lambda/\mu}$ , see [12]). Further let  $g_{(r)}(X) := P_{(r)}(X)(t)_r/(q)_r$  and  $e_r(X)$  the  $r$ th elementary symmetric function. Then the Macdonald polynomials  $P_\lambda(X)$  satisfy the Pieri formulas

$$(2.14a) \quad P_\mu(X)g_r(X) = \sum_{\substack{\lambda \succcurlyeq \mu \\ |\lambda - \mu| = r}} \phi_{\lambda/\mu} P_\lambda(X)$$

$$(2.14b) \quad P_\mu(X)e_r(X) = \sum_{\substack{\lambda' \succcurlyeq \mu' \\ |\lambda - \mu| = r}} \psi'_{\lambda/\mu} P_\lambda(X).$$

Now observe that (2.13) for  $b = at$  yields

$$P_\mu(X) \prod_{x \in X} \frac{(atx)_\infty}{(ax)_\infty} = \sum_{\lambda} a^{|\lambda - \mu|} Q_{\lambda/\mu}(1) P_\lambda(X),$$

whereas for  $a = bq$  it yields

$$P_\mu(X) \prod_{x \in X} (1 - bx) = \sum_{\lambda} b^{|\lambda - \mu|} Q_{\lambda/\mu} \left[ \frac{q-1}{1-t} \right] P_\lambda(X).$$

Since

$$\sum_{r \geq 0} a^r g_r(X) = \prod_{x \in X} \frac{(atx)_\infty}{(ax)_\infty} \quad \text{and} \quad \sum_{r \geq 0} (-b)^r e_r(X) = \prod_{x \in X} (1 - bx)$$

we therefore have

$$P_\mu(X)g_r(X) = \sum_{|\lambda - \mu| = r} Q_{\lambda/\mu}(1) P_\lambda(X)$$

and

$$P_\mu(X)e_r(X) = (-1)^r \sum_{|\lambda - \mu| = r} Q_{\lambda/\mu} \left[ \frac{q-1}{1-t} \right] P_\lambda(X),$$

where we have also used (2.7b). Identifying

$$Q_{\lambda/\mu}(1) = \begin{cases} \phi_{\lambda/\mu} & \text{if } \mu \preccurlyeq \lambda \\ 0 & \text{otherwise} \end{cases}$$

and

$$(2.15) \quad Q_{\lambda/\mu} \left[ \frac{q-1}{1-t} \right] = \begin{cases} (-1)^{|\lambda - \mu|} \psi'_{\lambda/\mu} & \text{if } \mu' \preccurlyeq \lambda' \\ 0 & \text{otherwise} \end{cases}$$

these two formulas are equivalent to the Pieri rules of (2.14).

The skew polynomials can be used to define generalised  $q$ -binomial coefficients [9, 10, 14] as

$$(2.16) \quad \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q,t} := Q_{\lambda/\mu} \left[ \frac{1}{1-t} \right].$$

In particular has  $[\lambda]_{\mu} = 0$  if  $\mu \not\subseteq \lambda$  and

$$\left[ \begin{matrix} (m) \\ (k) \end{matrix} \right] = \prod_{i=1}^k \frac{1 - q^{i+m-k}}{1 - q^i} = \left[ \begin{matrix} m \\ k \end{matrix} \right]$$

with on the right the classical  $q$ -binomial coefficients  $\left[ \begin{matrix} m \\ k \end{matrix} \right] = \left[ \begin{matrix} m \\ k \end{matrix} \right]_q$ .

If  $\lambda_{(i)} := (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots, \lambda_i)$  then [9]

$$(1 - q)t^{i-1} \left[ \begin{matrix} \lambda \\ \lambda_{(i)} \end{matrix} \right] = \frac{c'_{\lambda}}{c'_{\lambda_{(i)}}} \psi'_{\lambda/\lambda_{(i)}}.$$

This, together with [9, Théorème 9, Bis]

$$(2.17) \quad (\omega_{\lambda} - \omega_{\mu}) \left[ \begin{matrix} \lambda \\ \mu \end{matrix} \right] = (1 - q) \sum_{i=1}^n q^{-\lambda_i} t^{i-n} \left[ \begin{matrix} \lambda \\ \lambda_{(i)} \end{matrix} \right] \left[ \begin{matrix} \lambda_{(i)} \\ \mu \end{matrix} \right],$$

where

$$(2.18) \quad \omega_{\lambda} = \omega_{\lambda}(q, t) := \sum_{i=1}^n q^{-\lambda_i} t^{i-n},$$

provides a simple recursive method to compute the generalised  $q$ -binomial coefficients.

**Lemma 2.1.** *Assume that  $l(\lambda), l(\mu) \leq n$ . Then*

$$(2.19a) \quad \left[ \begin{matrix} \lambda \\ \mu \end{matrix} \right] = \frac{s_{\mu}(\langle 0 \rangle)}{s_{\lambda}(\langle 0 \rangle)} \det_{1 \leq i, j \leq n} \left( \left[ \begin{matrix} \lambda_i + n - i \\ \mu_j + n - j \end{matrix} \right] \right) \quad \text{if } t = q$$

and

$$(2.19b) \quad \left[ \begin{matrix} \lambda \\ \mu \end{matrix} \right] = \sum_{u^+ = \mu} \prod_{i=1}^n \left[ \begin{matrix} \lambda_i \\ u_i \end{matrix} \right] \quad \text{if } t = 1.$$

In the above  $\sum_{u^+ = \mu}$  denotes a sum over compositions  $u \in \mathbb{N}^n$  in the  $\mathfrak{S}_n$  orbit of  $\mu$ .

*Proof.* Assume that  $t = q$ . Then (2.16) simplifies to

$$(2.20) \quad \left[ \begin{matrix} \lambda \\ \mu \end{matrix} \right] = q^{n(\mu) - n(\lambda)} \frac{c_{\lambda}}{c_{\mu}} s_{\lambda/\mu} \left[ \begin{matrix} 1 \\ 1 - q \end{matrix} \right],$$

where we have also used (2.7b) and the fact that for  $t = q$  the (skew) Macdonald polynomials reduce to the (skew) Schur functions. Let  $h_r(X)$  be the  $r$ th complete symmetric function. By application of the Jacobi–Trudi identity [12, Equation (I.5.4)]

$$s_{\lambda/\mu} = \det_{1 \leq i, j \leq n} (h_{\lambda_i - \mu_j - i + j}) \quad \text{for } n \geq l(\lambda),$$

and the principal specialisation formula [12, page 44]

$$(2.21) \quad s_{\lambda}(\langle 0 \rangle) = \frac{q^{n(\lambda)}}{c'_{\lambda}} \prod_{i=1}^n \frac{(q)_{\lambda_i + n - i}}{(q)_{n - i}}$$

the generalised  $q$ -binomial coefficient (2.20) can be expressed as a determinant

$$\left[ \begin{matrix} \lambda \\ \mu \end{matrix} \right] = \frac{s_{\mu}(\langle 0 \rangle)}{s_{\lambda}(\langle 0 \rangle)} \det_{1 \leq i, j \leq n} \left( \frac{(q; q)_{\lambda_i + n - i}}{(q, q)_{\mu_j + n - j}} h_{\lambda_i - \mu_j - i + j} \left[ \begin{matrix} 1 \\ 1 - q \end{matrix} \right] \right).$$

Since

$$h_r \left[ \frac{1}{1-q} \right] = \frac{1}{(q)_r}$$

this establishes the first claim.

The second claim follows in analogous manner. Since making the substitution  $t = 1$  in the right-hand side of (2.16) is somewhat problematic it is best to first use the symmetry [14, Equation (2.12)]

$$(2.22) \quad \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q,t} = \begin{bmatrix} \lambda' \\ \mu' \end{bmatrix}_{t^{-1}, q^{-1}}.$$

Since

$$P_{\lambda/\mu}(X; 1, t) = e_{\lambda'/\mu'}(X)$$

we therefore get

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q^{-1}, 1} = \begin{bmatrix} \lambda' \\ \mu' \end{bmatrix}_{1, q} = q^{n(\mu') - n(\lambda')} \frac{c_{\lambda'}(1, q)}{c_{\mu'}(1, q)} e_{\lambda'/\mu'} \left[ \frac{1}{1-q} \right].$$

Using

$$e_{\lambda/\mu} = \sum_{u^+ = \mu} \prod_{i=1}^n e_{\lambda_i - u_i} \quad \text{for } n \geq l(\lambda),$$

and

$$e_r \left[ \frac{1}{1-q} \right] = \frac{q^{\binom{r}{2}}}{(q; q)_r}$$

as well as  $c_{\lambda'}(q, 1) = \prod_i (q)_{\lambda_i}$ , it follows that

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q^{-1}, 1} = \sum_{u^+ = \mu} \prod_{i=1}^n q^{-\lambda_i u_i} \begin{bmatrix} \lambda_i \\ u_i \end{bmatrix}.$$

Finally replacing  $q \mapsto 1/q$  yields the second claim.  $\square$

### 3. SYMMETRIC FUNCTIONS AND BRANCHING RULES

In this section we consider the question posed in the introduction:

*Can one find new(?) branching-type formulas, similar to (1.4), (1.7) and (1.11), that lead to symmetric functions?*

Assume that  $k$  is a fixed nonnegative integer, and let  $\mathbf{a} = (a_1, a_2, \dots, a_k)$  denote a finite sequence of parameters. Then we are looking for branching coefficients  $f_{\lambda/\mu}(z; \mathbf{a})$  such that

$$(3.1a) \quad f_{\lambda}(x_1, \dots, x_n; \mathbf{a}) = \sum_{\mu \subseteq \lambda} f_{\lambda/\mu}(x_n; \mathbf{a}) f_{\mu}(x_1, \dots, x_{n-1}; \mathbf{a}'),$$

subject to the initial condition

$$(3.1b) \quad f_{\lambda}(-; \mathbf{a}) = \delta_{\lambda, 0}$$

defines a symmetric function. In the above  $\mathbf{a}' = (a'_1, \dots, a'_k) = g(\mathbf{a})$ . Of course, (3.1a) for  $n = 1$  combined with (3.1b) implies that

$$f_{\lambda}(z; \mathbf{a}) = f_{\lambda/0}(z; \mathbf{a}).$$

If one wishes to only consider symmetric functions with the standard property

$$(3.2) \quad f_{\lambda}(x_1, \dots, x_n; \mathbf{a}) = 0 \quad \text{if } l(\lambda) > n$$

then the additional condition

$$f_{\lambda/\mu}(z; \mathbf{a}) = 0 \quad \text{if } l(\lambda) - l(\mu) > 1$$

must be imposed.

Because we assume the branching coefficients to be independent of  $n$ , it may perhaps seem we are excluding interesting classes of symmetric functions such as the interpolation Macdonald polynomials. As will be shown shortly, assuming  $n$ -independence is not actually a restriction, and (1.11) may easily be recovered as a special case of (3.1a).

Now let us assume that (3.1a) yields a symmetric function  $f_\lambda(x_1, \dots, x_n; \mathbf{a})$  for all  $n \leq N$ . (For  $N = 0$  and  $N = 1$  this is obviously not an assumption.) Then,

$$\begin{aligned} f_\lambda(x_1, \dots, x_{n-1}, y, z; \mathbf{a}) &= \sum_{\mu \subseteq \lambda} f_{\lambda/\mu}(z; \mathbf{a}) f_\mu(x_1, \dots, x_{n-1}, y; \mathbf{a}') \\ &= \sum_{\nu \subseteq \mu \subseteq \lambda} f_{\lambda/\mu}(z; \mathbf{a}) f_{\mu/\nu}(y; \mathbf{a}') f_\nu(x_1, \dots, x_{n-1}; \mathbf{a}'') \end{aligned}$$

is a symmetric function in  $x_1, \dots, x_{n-1}, y$  (for  $n \leq N$ ). For it to also be a symmetric function in  $x_1, \dots, x_{n-1}, y, z$  we must have

$$f_\lambda(x_1, \dots, x_{n-1}, y, z; \mathbf{a}) = f_\lambda(x_1, \dots, x_{n-1}, z, y; \mathbf{a}),$$

implying that for fixed  $\lambda$

$$\begin{aligned} \sum_{\nu \subseteq \mu \subseteq \lambda} f_{\lambda/\mu}(z; \mathbf{a}) f_{\mu/\nu}(y; \mathbf{a}') f_\nu(x_1, \dots, x_{n-1}; \mathbf{a}'') \\ = \sum_{\nu \subseteq \mu \subseteq \lambda} f_{\lambda/\mu}(y; \mathbf{a}) f_{\mu/\nu}(z; \mathbf{a}') f_\nu(x_1, \dots, x_{n-1}; \mathbf{a}''), \end{aligned}$$

where  $\mathbf{a}'' := g(\mathbf{a}')$ . Hence a sufficient condition for (3.1a) to yield a symmetric function is

$$(3.3) \quad \sum_{\nu \subseteq \mu \subseteq \lambda} f_{\lambda/\mu}(z; \mathbf{a}) f_{\mu/\nu}(y; \mathbf{a}') = \sum_{\nu \subseteq \mu \subseteq \lambda} f_{\lambda/\mu}(y; \mathbf{a}) f_{\mu/\nu}(z; \mathbf{a}')$$

for partitions  $\lambda, \nu$  such that  $\nu \subseteq \lambda$ .

As a first example let us show how to recover the Macdonald interpolation polynomials of the introduction. To this end we take  $\mathbf{a} = (a)$ ,  $\mathbf{a}' = (a/t)$ , and

$$f_{\lambda/\mu}(z; \mathbf{a}) = f_{\lambda/\mu}(z; a) = \frac{(a/z)_\lambda}{(a/z)_\mu} P_{\lambda/\mu}(z).$$

Clearly, the resulting polynomials  $f_\lambda(x; a)$  correspond to the interpolation polynomials after the specialisation  $a = t^{n-1}$ . To see that (3.3) is indeed satisfied we substitute the above choice for the branching coefficient (recall the convention that  $P_{\lambda/\mu} := 0$  if  $\mu \not\subseteq \lambda$ ) to obtain

$$(3.4) \quad \sum_{\mu} \frac{(a/z)_\lambda (a/ty)_\mu}{(a/z)_\mu (a/ty)_\nu} P_{\lambda/\mu}(z) P_{\mu/\nu}(y) = \sum_{\mu} \frac{(a/y)_\lambda (a/tz)_\mu}{(a/y)_\mu (a/tz)_\nu} P_{\lambda/\mu}(y) P_{\mu/\nu}(z).$$

The identity (3.4) is easily proved using Rains' Sears transformation for skew Macdonald polynomials [20, Corollary 4.9]

$$(3.5) \quad \sum_{\mu} \frac{(aq/b, aq/c)_{\lambda}(d, e)_{\mu}}{(aq/b, aq/c)_{\mu}(d, e)_{\nu}} P_{\lambda/\mu} \left[ \frac{1 - aq/de}{1 - t} \right] P_{\mu/\nu} \left[ \frac{aq/de - a^2q^2/bcde}{1 - t} \right] \\ = \sum_{\mu} \frac{(aq/d, aq/e)_{\lambda}(b, c)_{\mu}}{(aq/d, aq/e)_{\mu}(b, c)_{\nu}} P_{\lambda/\mu} \left[ \frac{1 - aq/bc}{1 - t} \right] P_{\mu/\nu} \left[ \frac{aq/bc - a^2q^2/bcde}{1 - t} \right].$$

After simultaneously replacing  $(a, b, c, d, e) \mapsto (c, a/tz, cqz/a, a/ty, cqy/a)$  and taking the  $c \rightarrow \infty$  limit we obtain (3.4).

If, more generally, we let  $(a, b, c, d, e) \mapsto (c, a/bz, cqz/a, a/by, cqy/a)$  in (3.5) and take the  $c \rightarrow \infty$  limit we find that

$$\sum_{\mu} \frac{(a/z)_{\lambda}(a/by)_{\mu}}{(a/z)_{\mu}(a/by)_{\nu}} P_{\lambda/\mu} \left[ \frac{z - bz}{1 - t} \right] P_{\mu/\nu} \left[ \frac{y - by}{1 - t} \right] \\ = \sum_{\mu} \frac{(a/y)_{\lambda}(a/bz)_{\mu}}{(a/y)_{\mu}(a/bz)_{\nu}} P_{\lambda/\mu} \left[ \frac{y - by}{1 - t} \right] P_{\mu/\nu} \left[ \frac{z - bz}{1 - t} \right].$$

The Macdonald interpolation polynomials may thus be generalised by taking  $\mathbf{a} = (a, b)$ ,  $\mathbf{a}' = (a/b, b)$  and

$$f_{\lambda/\mu}(z; \mathbf{a}) = f_{\lambda/\mu}(z; a, b) = z^{|\lambda - \mu|} \frac{(a/z)_{\lambda}}{(a/z)_{\mu}} P_{\lambda/\mu} \left[ \frac{1 - b}{1 - t} \right].$$

**Proposition 3.1.** *The polynomials  $M_{\lambda}(x_1, \dots, x_n; a, b) = M_{\lambda}(x_1, \dots, x_n; a, b; q, t)$  defined by*

$$M_{\lambda}(x_1, \dots, x_n; a, b) = \sum_{\mu} x_n^{|\lambda - \mu|} \frac{(a/x_n)_{\lambda}}{(a/x_n)_{\mu}} P_{\lambda/\mu} \left[ \frac{1 - b}{1 - t} \right] M_{\mu}(x_1, \dots, x_{n-1}; a/b, b)$$

*subject to  $M_{\lambda}(-; a, b) = \delta_{\lambda, 0}$  are symmetric. Moreover, the interpolation Macdonald polynomials corresponds to*

$$M_{\lambda}(x_1, \dots, x_n) = M_{\lambda}(x_1, \dots, x_n; t^{n-1}, t).$$

The polynomials  $M_{\lambda}(x_1, \dots, x_n; a, b)$  are an example of a class of symmetric functions for which  $l(\lambda) > n$  does not imply vanishing. For example,

$$M_{\lambda}(z; a, b) = z^{|\lambda|} (a/z)_{\lambda} P_{\lambda} \left[ \frac{1 - b}{1 - t} \right] = t^{n(\lambda)} z^{|\lambda|} \frac{(a/z, b)_{\lambda}}{c_{\lambda}}.$$

The next example corresponds to Okounkov's  $BC_n$  symmetric interpolation polynomials [16] (see also [17, 20]).

**Proposition 3.2.** *If we take  $\mathbf{a} = (a, b)$ ,  $\mathbf{a}' = (a/t, b/t)$  and*

$$f_{\lambda/\mu}(z; \mathbf{a}) = f_{\lambda/\mu}(z; a, b) = \frac{(a/z, bz)_{\lambda}}{(a/z, bz)_{\mu}} P_{\lambda/\mu}(1/b)$$

*in (3.1a) then the resulting functions  $f_{\lambda}(x; a, b) = f_{\lambda}(x; a, b; q, t)$  are symmetric.*

Writing  $O_{\lambda}(x; a, b)$  instead of  $f_{\lambda}(x; a, b)$ , the (Laurent) polynomials  $O_{\lambda}(x; a, b)$  satisfy the symmetries

$$O_{\lambda}(x; a, b) = \left(\frac{a}{b}\right)^{|\lambda|} O_{\lambda}(1/x; b, a) = \left(\frac{a}{b}\right)^{|\lambda|} q^{2n(\lambda')} O_{\lambda}(1/x; 1/a, 1/b; 1/q, 1/t).$$

(This follows easily using that  $P_\lambda(X; 1/q, 1/t) = P_\lambda(X, q, t)$ .) Moreover, Okounkov's  $BC_n$  interpolation Macdonald polynomials  $P_\lambda^*(x; q, t, s)$  follow as

$$P_\lambda^*(x_1, tx_2, \dots, t^{n-1}x_n; q, t, s) = q^{-n(\lambda')} O_\lambda(x; 1, s^2 t^{2(n-1)}; q, t).$$

(Since  $O_\lambda(ax; a, b/a) = a^{|\lambda|} O_\lambda(x; 1, b)$  the  $O_\lambda(x; a, b)$  are not more general than the  $P_\lambda^*(x; q, t, s)$ .)

*Proof of Proposition (3.2).* Substituting the claim in (3.3) and using (2.6) gives

$$\begin{aligned} \sum_{\mu} \frac{(a/z, bz)_\lambda (a/yt, byt)_\mu}{(a/z, bz)_\mu (a/yt, byt)_\nu} P_{\lambda/\mu}(1) P_{\mu/\nu}(t) \\ = \sum_{\mu} \frac{(a/y, by)_\lambda (a/zt, bzt)_\mu}{(a/y, by)_\mu (a/zt, bzt)_\nu} P_{\lambda/\mu}(1) P_{\mu/\nu}(t). \end{aligned}$$

This is (3.5) with  $(a, b, c, d, e) \mapsto (ab/qt, bz/t, a/zt, a/yt, by/t)$ .  $\square$

Our final example will (in the limit) lead to the functions studied in the remainder of the paper.

**Proposition 3.3.** *If we take  $\mathbf{a} = (a, b)$ ,  $\mathbf{a}' = (at, b)$  and*

$$f_{\lambda/\mu}(z; \mathbf{a}) = f_{\lambda/\mu}(z; a, b) = \frac{(z/a)_\lambda (bz/t)_\mu}{(z/a)_\mu (bz)_\lambda} P_{\lambda/\mu}(a)$$

*in (3.1a) then the resulting functions  $f_\lambda(x; a, b) = f_\lambda(x; a, b; q, t)$  are symmetric.*

*Proof.* Substituting the claim in (3.3) and using (2.6) gives

$$\begin{aligned} \sum_{\mu} \frac{(z/a, by)_\lambda (bz/t, y/at)_\mu}{(z/a, by)_\mu (bz/t, y/at)_\nu} P_{\lambda/\mu}(1) P_{\mu/\nu}(t) \\ = \sum_{\mu} \frac{(y/a, bz)_\lambda (by/t, z/at)_\mu}{(y/a, bz)_\mu (by/t, z/at)_\nu} P_{\lambda/\mu}(1) P_{\mu/\nu}(t). \end{aligned}$$

This is (3.5) with  $(a, b, c, d, e) \mapsto (byz/aqt, by/t, z/at, bz/t, y/at)$ .  $\square$

If we write  $R_\lambda(x; a, b)$  instead of  $f_\lambda(x; a, b)$  the symmetric functions of Proposition 3.3 correspond to the functions described by the branching rule (1.14) of the introduction. As already mentioned there, the  $R_\lambda(x; a, b)$  are not new, and follow as a special limiting case of much more general functions studied by Rains [19, 21]. More specifically, Rains defined a family of abelian interpolation functions

$$R_\lambda^{*(n)}(x; a, b) = R_\lambda^{*(n)}(x; a, b; q, t; p),$$

where  $x = (x_1, \dots, x_n)$ . The  $R_\lambda^{*(n)}(x)$  are  $BC_n$  symmetric and, apart from parameters  $a, b, q, t$ , depend on an elliptic nome  $p$ . In [21, Theorem 4.16] Rains proved the branching rule

$$R_\lambda^{*(n+1)}(x_1, \dots, x_{n+1}; a, b) = \sum_{\mu \subseteq \lambda} c_{\lambda\mu}^{(n)}(x_{n+1}; a, b) R_\mu^{*(n)}(x_1, \dots, x_n; a, b),$$

where the branching coefficient  $c_{\lambda\mu}^{(n)}(z; a, b) = c_{\lambda\mu}^{(n)}(z; a, b; q, t; p)$  is expressed in terms of the elliptic binomial coefficient  $\langle \lambda \rangle_{[a, b](v_1, \dots, v_k)}$  (see [19, Equation (4.2)]) as

$$(3.6) \quad c_{\lambda\mu}^{(n)}(z; a, b) = \left\langle \begin{matrix} \lambda \\ \mu \end{matrix} \right\rangle_{[at^n/b, t](azt^n, at^n/z, pqa/tb)}.$$

If we define

$$\begin{aligned} R_\lambda(x; a, b) &= R_\lambda(x; a, b; q, t) \\ &= \left(\frac{t^{1-n}}{b}\right)^{|\lambda|} P_\lambda(\langle 0 \rangle) \lim_{p \rightarrow 0} R_\lambda^{*(n)}(p^{1/4}x; p^{1/4}at^{n-1}, p^{3/4}b/q; 1/q, 1/t; p) \end{aligned}$$

and compute the corresponding limit of (3.6) we obtain the branching rule (1.14) with  $n \mapsto n + 1$ .

#### 4. THE SYMMETRIC FUNCTION $R_\lambda(x; b)$

In the remainder of the paper we consider the symmetric function

$$\begin{aligned} R_\lambda(X; b) &= (-1)^{|\lambda|} q^{-n(\lambda')} t^{n(\lambda)} \lim_{a \rightarrow 0} R_\lambda(X; a, b) \\ &= q^{-n(\lambda')} t^{n(\lambda)} \left(-\frac{t^{1-n}}{b}\right)^{|\lambda|} P_\lambda(\langle 0 \rangle) \quad (n := |X|) \\ &\quad \times \lim_{a \rightarrow 0} \lim_{p \rightarrow 0} R_\lambda^{*(n)}(p^{1/4}X; p^{1/4}at^{n-1}, p^{3/4}b/q; 1/q, 1/t; p) \end{aligned}$$

which, alternatively, is defined by the branching rule (1.13). Because  $R_\lambda$  is a limiting case of the abelian interpolation function  $R_\lambda^{*(n)}$  many properties of former follow by taking appropriate limits in the results of [19, 21]. For example, it follows from [21, Proposition 3.9] that the  $R_\lambda(X; b)$  for  $X = \{x_1, \dots, x_n\}$  satisfy a  $q$ -difference equation generalising (2.2). Specifically, with  $D_n(b, c)$  the generalised Macdonald operator

$$D_n(b, c) = \sum_{I \subseteq [n]} (-1)^{|I|} t^{\binom{|I|}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{j \notin I} (1 - bx_j) \prod_{i \in I} (c - bt^{1-n}x_i) T_{q, x_i}$$

we have

$$(4.1) \quad D_n(b, c) R_\lambda(X; b) = R_\lambda(X; bq) \prod_{i=1}^n (1 - cq^{\lambda_i} t^{n-i}).$$

Below we will first prove a number of elementary properties of the functions  $R_\lambda(X; b)$  using only the branching rule (1.13). Like the previous result, most of these can also be obtained by taking appropriate limits in results of Rains for the abelian interpolation functions  $R_\lambda^{*(n)}(X; a, b)$ . Then we give several deeper results for  $R_\lambda(X; b)$  (such as Theorem 5.1 and Corollaries 5.2, 5.4 and 6.2) that, to the best of our knowledge, have no analogues for  $R_\lambda^{*(n)}(X; a, b)$  or  $R_\lambda(X; a, b)$ . First however we restate the branching rule (1.13) in the equivalent form

$$(4.2) \quad R_\lambda(X; b) = \sum_{\mu} \frac{(bz/t)_\mu}{(bz)_\lambda} P_{\lambda/\mu}(z) R_\mu(Y; b),$$

where  $X = Y + z$ .

When  $X = \{z\}$  we find from (4.2) that

$$(4.3) \quad R_{(k)}(z; b) = \frac{z^k}{(bz)_k}.$$

From this it is clear that  $R_{(k)}(cz; b) = c^k R_{(k)}(z; bc)$  and that in the  $c \rightarrow \infty$  limit  $R_{(k)}(cz; b)$  is given by  $(-b)^{-k} q^{-\binom{k}{2}}$ . It also shows that

$$R_{(k+1)}(z; b) = R_{(k)}(z; bq).$$

All three statements easily generalise to arbitrary  $X$ .

**Lemma 4.1.** *For  $c$  a scalar,*

$$R_\lambda(cX; b) = c^{|\lambda|} R_\lambda(X; bc).$$

**Lemma 4.2.** *For  $c$  a scalar and  $n := |X|$ ,*

$$\lim_{c \rightarrow \infty} R_\lambda(cX; b) = \left(-\frac{t^{1-n}}{b}\right)^{|\lambda|} q^{-n(\lambda')} t^{n(\lambda)} P_\lambda(\langle 0 \rangle).$$

**Lemma 4.3.** *Let  $n := |X|$  and  $\lambda$  a partition such that  $l(\lambda) = n$ . Define  $\mu := (\lambda_1 - 1, \dots, \lambda_n - 1)$ . Then*

$$R_\lambda(X; b) = R_\mu(X; bq) \prod_{x \in X} \frac{x}{1 - bx}.$$

This last result allows the definition of  $R_\lambda(X; b)$  to be extended to all weakly decreasing integer sequences  $\lambda = (\lambda_1, \dots, \lambda_n)$ .

*Proof of Lemmas 4.1–4.3.* By (4.3) all three statements are obviously true for  $X$  a single-letter alphabet, and we proceed by induction on  $n$ , the cardinality of  $X$ .

By (4.2),

$$(4.4) \quad R_\lambda(cX; b) = \sum_{\mu} \frac{(bcz/t)_\mu}{(bcz)_\lambda} P_{\lambda/\mu}(cz) R_\mu(cY; b).$$

Using (2.6) and the appropriate induction hypothesis this yields

$$R_\lambda(cX; b) = c^{|\lambda|} \sum_{\mu} \frac{(bcz/t)_\mu}{(bcz)_\lambda} P_{\lambda/\mu}(z) R_\mu(Y; bc) = c^{|\lambda|} R_\lambda(X; bc),$$

establishing the first lemma.

Taking the  $c \rightarrow \infty$  limit on both sides of (4.4) and then using induction we get

$$\begin{aligned} \lim_{c \rightarrow \infty} R_\lambda(cX; b) &= \sum_{\mu} (-b)^{|\mu| - |\lambda|} q^{n(\mu') - n(\lambda')} t^{n(\lambda) - n(\mu) - |\mu|} P_{\lambda/\mu}(1) \lim_{c \rightarrow \infty} R_\mu(cY; b) \\ &= \left(-\frac{t^{1-n}}{b}\right)^{|\lambda|} q^{-n'(\lambda)} t^{n(\lambda)} \sum_{\mu} P_{\lambda/\mu}(t^{n-1}) P_\mu(t^{n-2}, \dots, t, 1) \\ &= \left(-\frac{t^{1-n}}{b}\right)^{|\lambda|} q^{-n'(\lambda)} t^{n(\lambda)} P_\lambda(\langle 0 \rangle), \end{aligned}$$

where the last equality follows from (2.5).

To prove the final lemma we consider (4.4) with  $c = 1$  and, in accordance with the conditions of Lemma 4.3, with  $\lambda_n \geq 1$ . Since  $P_{\lambda/\nu}(a)$  vanishes unless  $\lambda - \nu$  is a horizontal strip this implies that  $\nu_{n-1} \geq 1$ . The summand also vanishes if  $l(\nu) > n-1$  so that we may assume that  $l(\nu) = n-1$ . Defining  $\eta = (\nu_1 - 1, \dots, \nu_{n-1} - 1)$  and  $\mu = (\lambda_1 - 1, \dots, \lambda_n - 1)$  and using induction, as well as

$$\frac{(bz/t)_\nu}{(bz)_\lambda} = \frac{1}{1 - bz} \frac{(bzq/t)_\eta}{(bzq)_\mu} \quad \text{and} \quad P_{\lambda/\nu}(z) = z P_{\mu/\eta}(z),$$

we get

$$\begin{aligned} R_\lambda(X; b) &= \frac{z}{1-bz} \left( \prod_{x \in Y} \frac{x}{1-bx} \right) \sum_{\eta \subseteq \mu} \frac{(bzq/t)_\eta}{(bzq)_\mu} P_{\mu/\eta}(z) R_\eta(Y; bq) \\ &= R_\mu(X; bq) \prod_{x \in X} \frac{x}{1-bx}. \end{aligned}$$

where in the final step we have used (4.2) and  $X = Y + z$ .  $\square$

**Proposition 4.4** (Principal specialisation). *For  $\lambda$  such that  $l(\lambda) \leq n$ ,*

$$R_\lambda(\langle 0 \rangle; b) = \frac{P_\lambda(\langle 0 \rangle)}{(bt^{n-1})_\lambda} = \frac{t^{n(\lambda)}(t^n)_\lambda}{(bt^{n-1})_\lambda c_\lambda}.$$

By Lemma 4.1 this may be stated slightly more generally as

$$(4.5) \quad R_\lambda(a\langle 0 \rangle; b) = \frac{P_\lambda(a\langle 0 \rangle)}{(abt^{n-1})_\lambda}.$$

*Proof.* Iterating (1.13) using

$$\sum_{\nu} P_{\lambda/\nu}(X) P_{\nu/\mu}(Y) = P_{\lambda/\mu}(X + Y),$$

we obtain the generalised branching rule

$$R_\lambda(x_1, \dots, x_m, t^{n-1}, \dots, t, 1; b) = \sum_{\mu} \frac{(b/t)_\mu}{(bt^{n-1})_\lambda} P_{\lambda/\mu}(\langle 0 \rangle) R_\mu(x_1, \dots, x_m; b),$$

for  $l(\lambda) \leq n + m$ . When  $m = 0$  this results in the claim.  $\square$

**Proposition 4.5** (Evaluation symmetry). *For  $\lambda$  such that  $l(\lambda) \leq n$  set*

$$\langle \lambda \rangle = (q^{\lambda_1} t^{n-1}, \dots, q^{\lambda_{n-1}} t, 1).$$

*Then*

$$\frac{R_\lambda(a\langle \mu \rangle; b)}{R_\lambda(a\langle 0 \rangle; b)} = \frac{R_\mu(a\langle \lambda \rangle; b)}{R_\mu(a\langle 0 \rangle; b)}.$$

*Proof.* We may view the evaluation symmetry as a rational function identity in  $b$ . Hence it suffices to give a proof for  $b = q^{1-m}$  where  $m$  runs over all integers such that  $\lambda_1, \mu_1 \leq m$ . But (see (1.12) and (1.13))

$$R_\lambda(X; q^{1-m}) = \frac{M_{m^n - \lambda}(1/x_1, \dots, 1/x_n; 1/q, 1/t)}{M_{m^n}(1/x_1, \dots, 1/x_n; 1/q, 1/t)}$$

and

$$M_{m^n}(1/X; 1/q, 1/t) = \prod_{x \in X} x^{m^n}(x; 1/q)_m$$

so that we need to prove that

$$\frac{M_{m^n - \lambda}(a\langle \mu \rangle)}{M_{m^n - \lambda}(a\langle 0 \rangle)} = \frac{M_{m^n - \mu}(a\langle \lambda \rangle)}{M_{m^n - \mu}(a\langle 0 \rangle)} \frac{(aqt^{n-1})_\mu}{(aqt^{n-1})_\lambda} \frac{(aq^{1-m}t^{n-1})_\lambda}{(aq^{1-m}t^{n-1})_\mu}.$$

Making the substitutions  $\lambda \mapsto m^n - \lambda$ ,  $\mu \mapsto m^n - \mu$  and  $a \mapsto aq^{-m}t^{1-n}$  we get

$$\frac{M_\lambda(a/\langle \mu \rangle)}{M_\lambda(aq^{-m}t^{1-n}\langle 0 \rangle)} = \frac{M_\mu(a/\langle \lambda \rangle)}{M_\mu(aq^{-m}t^{1-n}\langle 0 \rangle)} \frac{(q^m t^{n-1}/a)_\mu}{(q^m t^{n-1}/a)_\lambda} \frac{(t^{n-1}/a)_\lambda}{(t^{n-1}/a)_\mu} q^{m(|\lambda| - |\mu|)}.$$

Finally, by the principal specialisation formula for the interpolation Macdonald polynomials [14],

$$\frac{M_\lambda(aq^{-m}t^{1-n}\langle 0 \rangle)}{M_\lambda(a/\langle 0 \rangle)} = \frac{(q^m t^{n-1}/a)_\lambda}{(t^{n-1}/a)_\lambda} q^{-m|\lambda|}$$

so that we end up with

$$\frac{M_\lambda(a/\langle \mu \rangle)}{M_\lambda(a/\langle 0 \rangle)} = \frac{M_\mu(a/\langle \lambda \rangle)}{M_\mu(a/\langle 0 \rangle)}.$$

This is the known evaluation symmetry of the interpolation Macdonald polynomials [14, Section 2].  $\square$

It is clear from (4.2) that  $R_\lambda(X; 0) = P_\lambda(X)$  with on the right a Macdonald polynomial. The Macdonald polynomials in turn generalise the Jack polynomials  $P_\lambda^{(\alpha)}(X)$ , since  $P_\lambda^{(\alpha)}(X) = \lim_{q \rightarrow 1} P_\lambda(X; q, q^{1/\alpha})$ . Combining the last two equations it thus follows that

$$P_\lambda^{(\alpha)}(X) = \lim_{q \rightarrow 1} R_\lambda(X; 0; q, q^{1/\alpha}).$$

Curiously, there is an alternative path from  $R_\lambda(X; b)$  to the Jack polynomials as follows. For  $X$  an alphabet let

$$\hat{X} := \left\{ \left( \frac{x}{1-x} \right) \mid x \in X \right\}.$$

**Proposition 4.6.** *We have*

$$P_\lambda^{(\alpha)}(\hat{X}) = \lim_{q \rightarrow 1} R_\lambda(X; 1; q, q^{1/\alpha}).$$

*Proof.* Let  $X = Y + z$  be a finite alphabet.

Replacing  $(b, t) \mapsto (1, q^{1/\alpha})$  in (4.2) and taking the  $q \rightarrow 1$  limit yields

$$R_\lambda^{(\alpha)}(X) = \sum_{\mu} (1-z)^{|\mu| - |\lambda|} P_{\lambda/\mu}^{(\alpha)}(z) R_\mu^{(\alpha)}(Y),$$

where  $P_{\lambda/\mu}^{(\alpha)}$  is a skew Jack polynomial and

$$R_\lambda^{(\alpha)}(X) := \lim_{q \rightarrow 1} R_\lambda(X; 1; q, q^{1/\alpha}).$$

Using the homogeneity of  $P_{\lambda/\mu}^{(\alpha)}$  the above can be rewritten as

$$R_\lambda^{(\alpha)}(X) = \sum_{\mu} P_{\lambda/\mu}^{(\alpha)}\left(\frac{z}{1-z}\right) R_\mu^{(\alpha)}(Y).$$

Comparing this with

$$P_\lambda^{(\alpha)}(X) = \sum_{\mu} P_{\lambda/\mu}^{(\alpha)}(z) P_\mu^{(\alpha)}(Y)$$

the proposition follows.  $\square$

5. CAUCHY, PIERI AND GAUSS FORMULAS FOR  $R_\lambda(X; b)$ 

Probably our most important new results for  $R_\lambda(X; b)$  are generalisation of the skew Cauchy identity (2.11), the Pieri formula (2.13) and the  $q$ -Gauss formula (1.15).

Before we get to these result we first need a few more definitions. First of all, in analogy with (2.7), we set

$$(5.1) \quad R_\lambda(X; b) := t^{n(\lambda)} \frac{R_\lambda(X; b)}{c'_\lambda}$$

so that (4.2) becomes

$$R_\lambda(X; b) = \sum_{\mu} \frac{(bz/t)_\mu}{(bz)_\lambda} P_{\lambda/\mu}(z) R_\mu(Y; b).$$

Furthermore, we also define the skew functions  $R_{\lambda/\mu}(X; b)$  by

$$(5.2) \quad R_{\lambda/\mu}(X; b) := \sum_{\nu} \frac{(bz/t)_\nu}{(bz)_\lambda} P_{\lambda/\nu}(z) R_{\mu/\nu}(Y; b)$$

and

$$R_{\lambda/\mu}(-; b) = \delta_{\lambda\mu}.$$

In other words,

$$R_{\lambda/\mu}(X + Y; b) = \sum_{\nu} R_{\lambda/\mu}(X; b) R_{\mu/\nu}(Y; b)$$

and  $R_{\lambda/\mu}(X; 0) = P_{\lambda/\mu}(X)$ .

**Theorem 5.1** (Skew Cauchy-type identity). *Let  $ab = cd$  and  $X$  a finite alphabet. Then*

$$(5.3) \quad \sum_{\lambda} \frac{(b/c)_\lambda}{(b/c)_\nu} Q_{\lambda/\nu} \left[ \frac{a-c}{1-t} \right] R_{\lambda/\mu}(X; b) \\ = \left( \prod_{x \in X} \frac{(cx, dx)_\infty}{(ax, bx)_\infty} \right) \sum_{\lambda} \frac{(b/c)_\mu}{(b/c)_\lambda} Q_{\mu/\lambda} \left[ \frac{a-c}{1-t} \right] R_{\nu/\lambda}(X; d).$$

Note that for  $b = 0$  the theorem simplifies to (2.11). We defer the proof of (5.3) till the end of this section and first list a number of corollaries.

**Corollary 5.2** (Pieri formula). *Let  $ab = cd$  and  $X$  a finite alphabet. Then*

$$(5.4) \quad R_\mu(X; d) \prod_{x \in X} \frac{(cx, dx)_\infty}{(ax, bx)_\infty} = \sum_{\lambda} \frac{(b/c)_\lambda}{(b/c)_\mu} Q_{\lambda/\mu} \left[ \frac{a-c}{1-t} \right] R_\lambda(X; b).$$

This follows from the theorem by taking  $\mu = 0$  and then replacing  $\nu$  by  $\mu$ .

When  $b, d \rightarrow 0$  equation (5.4) yields the Pieri formula (2.13) for Macdonald polynomials. When  $b, c \rightarrow 0$  and  $a \rightarrow 1$  such that  $b/c = d$  equation (5.4) yields (after replacing  $d \mapsto b$ )

$$(5.5) \quad \sum_{\lambda} \frac{(b)_\lambda}{(b)_\mu} \left[ \begin{matrix} \lambda \\ \mu \end{matrix} \right] P_\lambda(X) = R_\mu(X; b) \prod_{x \in X} \frac{(bx)_\infty}{(x)_\infty}.$$

For  $\mu = 0$  this is the  $q$ -binomial formula for Macdonald polynomials (2.12), and for  $b = 0$  it is Lassalle's [9]

$$\sum_{\lambda} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} P_{\lambda}(X) = P_{\mu}(X) \prod_{x \in X} \frac{1}{(x)_{\infty}}.$$

The Jack polynomial limit of (5.5) is of particular interest. To concisely state this we need some more notation. Let

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix}^{(\alpha)} := \lim_{q \rightarrow 1} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q, q^{1/\alpha}}$$

or, alternatively [3, 8, 18],

$$\frac{P_{\lambda}^{(\alpha)}(x_1 + 1, \dots, x_n + 1)}{P_{\lambda}^{(\alpha)}(1^n)} = \sum_{\mu} \begin{pmatrix} \lambda \\ \mu \end{pmatrix}^{(\alpha)} \frac{P_{\mu}^{(\alpha)}(x_1, \dots, x_n)}{P_{\mu}^{(\alpha)}(1^n)},$$

where  $n$  is any integer such that  $n \geq l(\lambda)$ . Further let

$$(b; \alpha)_{\lambda} := \prod_{i \geq 1} (b + (1 - i)/\alpha)_{\lambda_i}$$

with  $(b)_k = b(b + 1) \cdots (b + k - 1)$ ,

$$c'_{\lambda}(\alpha) := \prod_{s \in \lambda} (a(s) + 1 + l(s)/\alpha)$$

and

$$P_{\lambda}^{(\alpha)}(X) := \frac{P_{\lambda}^{(\alpha)}(X)}{c'_{\lambda}(\alpha)}.$$

Using all of the above, replacing  $(b, q, t)$  in (5.5) by  $(q^{\beta}, q, q^{1/\alpha})$  and taking the (formal) limit  $q \rightarrow 1$  with the aid of Proposition 4.6, we arrive at the following identity.

**Corollary 5.3** (Binomial formula for Jack polynomials). *For  $X$  a finite alphabet*

$$\sum_{\lambda} \frac{(\beta; \alpha)_{\lambda}}{(\beta; \alpha)_{\mu}} \begin{pmatrix} \lambda \\ \mu \end{pmatrix}^{(\alpha)} P_{\lambda}^{(\alpha)}(X) = P_{\mu}^{(\alpha)}(\hat{X}) \prod_{x \in X} \frac{1}{(1 - x)^{\beta}}.$$

Another special case of (5.4) worth stating is the following multivariable extension of the  ${}_1\phi_1$  summation [2, II.5], which follows straightforwardly by taking the  $a, d \rightarrow 0$  limit,

$$(5.6) \quad \sum_{\lambda} c^{|\lambda - \mu|} \frac{(b/c)_{\lambda}}{(b/c)_{\mu}} Q_{\lambda/\mu} \left[ \frac{0 - 1}{1 - t} \right] R_{\lambda}(X; b) = P_{\mu}(X) \prod_{x \in X} \frac{(cx)_{\infty}}{(bx)_{\infty}}.$$

This provides an expansion of the right-hand side different from (2.13).

If we let  $\nu = 0$  in Theorem 5.1, use (2.10) and then replace  $(a, b, c) \mapsto (c/ab, c, c/a)$  we obtain

$$\sum_{\lambda} \left( \frac{c}{ab} \right)^{|\lambda - \mu|} \frac{(a, b)_{\lambda}}{(a, b)_{\mu}} R_{\lambda/\mu}(X; c) = \prod_{x \in X} \frac{(cx/a, cx/b)_{\infty}}{(cx, cx/ab)_{\infty}}.$$

For  $\mu = 0$  we state this separately.

**Corollary 5.4** ( $\mathfrak{sl}_n$   $q$ -Gauss sum). *For  $X$  a finite alphabet*

$$(5.7) \quad \sum_{\lambda} \left(\frac{c}{ab}\right)^{|\lambda|} (a, b)_{\lambda} R_{\lambda}(X; c) = \prod_{x \in X} \frac{(cx/a, cx/b)_{\infty}}{(cx, cx/ab)_{\infty}}.$$

As mentioned in the introduction, for  $X = \{1\}$  this simplifies to the standard  $q$ -Gauss sum (1.16) thanks to (4.3). More generally, if we principally specialise  $X = t^{1-n}\langle 0 \rangle = \{1, t^{-1}, \dots, t^{1-n}\}$  and use (4.5) the  $\mathfrak{sl}_n$   $q$ -Gauss sum simplifies to Kaneko's  $q$ -Gauss sum for Macdonald polynomials [4, Proposition 5.4]

$$\sum_{\lambda} \left(\frac{ct^{1-n}}{ab}\right)^{|\lambda|} \frac{(a, b)_{\lambda}}{(c)_{\lambda}} P_{\lambda}(\langle 0 \rangle) = \prod_{i=1}^n \frac{(ct^{1-i}/a, ct^{1-i}/b)_{\infty}}{(ct^{1-i}, ct^{1-i}/ab)_{\infty}}.$$

As another consequence of the theorem we obtain an explicit expression for the Taylor series of  $R_{\mu}(X; b)$  in  $b$ . For  $\mu \subseteq \lambda$  let  $(1)_{\lambda/\mu}$  be defined as  $(1)_{\lambda/\mu} = \lim_{a \rightarrow 1} (a)_{\lambda} / (a)_{\mu}$ . That is

$$(1)_{\lambda/\mu} = \prod_{s \in \lambda - \mu} (1 - q^{a'(s)} t^{-l'(s)}).$$

**Corollary 5.5.** *We have*

$$(5.8) \quad R_{\mu}(X; b) = \sum_{\lambda \supseteq \mu} b^{|\lambda - \mu|} (1)_{\lambda/\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} P_{\lambda}(X),$$

or, equivalently,

$$[b^r] R_{\mu}(X; b) = \sum_{\substack{\lambda \supseteq \mu \\ |\lambda - \mu| = r}} (1)_{\lambda/\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} P_{\lambda}(X).$$

*Proof.* Replacing  $(c, X) \mapsto (a/b, bX)$  in Lemma 4.1 and expressing the resulting identity in terms of the normalised function  $R_{\lambda}(X; b)$  (see (5.1)) we get

$$(5.9) \quad R_{\lambda}(aX; b) = \left(\frac{a}{b}\right)^{|\lambda|} R_{\lambda}(bX; a).$$

Combined with (5.5) this implies that

$$R_{\mu}(aX; b) = \left(\frac{a}{b}\right)^{|\mu|} \left( \prod_{x \in X} \frac{(bx)_{\infty}}{(abx)_{\infty}} \right) \sum_{\lambda} b^{|\lambda|} \frac{(a)_{\lambda}}{(a)_{\mu}} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} P_{\lambda}(X).$$

The summand vanishes unless  $\mu \subseteq \lambda$  and so we may add this as a restriction in the sum over  $\lambda$ . Then the limit  $a \rightarrow 1$  limit may be taken without causing ambiguities, and the claim follows.  $\square$

Corollary 5.5 implies the following simple expressions for  $R_{\lambda}(X; b)$  when  $t = q$  (Schur-like case) or  $t = 1$  (monomial-like case).

**Proposition 5.6.** *Let  $X = \{x_1, \dots, x_n\}$ . Then*

$$(5.10a) \quad R_{\lambda}(X; b) = \frac{1}{\Delta(X)} \det_{1 \leq i, j \leq n} \left( \frac{x_i^{\lambda_j + n - j}}{(bx_i)_{\lambda_j - j + 1}} \right) \quad \text{if } t = q$$

and

$$(5.10b) \quad R_{\lambda}(X; b) = \sum_{u^+ = \lambda} \left( \prod_{i=1}^n \frac{x_i^{u_i}}{(bx_i)_{u_i}} \right) \quad \text{if } t = 1.$$

*Proof.* Since the two claims are proved in almost identical fashion we only present a proof of (5.10a). The only significant difference is that the omitted proof of (5.10b) uses (2.19b) instead of (2.19a).

Assume that  $t = q$ . Let  $\nu = \lambda - (0, 1, \dots, n-1)$  and suppose that  $\nu_n \geq 0$ . Since for any  $k \geq 0$ , one has the expansion

$$\frac{x^{k+n-1}}{(bx)_k} = \sum_{r=0}^{\infty} x^{k+n-1+r} \begin{bmatrix} k+r-1 \\ r \end{bmatrix},$$

the matrix  $(x_i^{\nu_j+n-1}/(bx_i)_{\nu_j})$  factorises into the product of rectangular matrices

$$\left( x_i^{n-1+r} \right)_{\substack{i=1,\dots,n \\ r=0,1,\dots}} \quad \text{and} \quad \left( b^{r-k} \begin{bmatrix} k-1 \\ r-k \end{bmatrix} \right)_{\substack{r=0,1,\dots \\ k=\nu_1,\dots,\nu_n}}.$$

According to Cauchy–Binet theorem, the determinant on the right-hand side of (5.10a) factorises into a sum of products of minors of these two matrices.

On the other hand, by (1.1) and (2.19a), the expansion (5.8) gives

$$R_\lambda(X; b)\Delta(X) = \sum_{\mu \supseteq \lambda} b^{|\mu-\lambda|} (1)_{\mu/\lambda} \det_{1 \leq i, j \leq n} \left( \frac{1}{(q)_{\mu_i - \lambda_j - i + j}} \right) \det_{1 \leq i, j \leq n} (x_i^{\mu_j + n - j}),$$

where we have also used (2.21) and (5.1). Using  $\nu$  instead of  $\lambda$ , and  $\eta = \mu - (0, 1, \dots, n-1)$ , this becomes

$$R_\lambda(X; b)\Delta(X) = \sum_{\eta \supseteq \nu} \det_{1 \leq i, j \leq n} \left( \begin{bmatrix} \eta_i - 1 \\ \nu_j - 1 \end{bmatrix} b^{\eta_i - \nu_j} \right) \det_{1 \leq i, j \leq n} (x_i^{\eta_j}),$$

which is precisely the Cauchy–Binet expansion. The restriction  $\nu_n \geq 0$  is lifted using Lemma 4.3.  $\square$

Recall that the Macdonald polynomials are the eigenfunctions of the operator  $D_n^1$ , see (2.4). Because  $P_\lambda(X; q, t) = P_\lambda(X; q^{-1}, t^{-1})$  this can also be stated as

$$\mathcal{D}_n^1 P_\lambda(X) = \omega_\lambda P_\lambda(X),$$

where  $\mathcal{D}_n^1 := D_n^1(q^{-1}; t^{-1})$  and  $\omega_\lambda$  is given in (2.18).

A second consequence of Corollary 5.5 is a generalisation of this identity as follows. Let

$$A_i(x; t) := \prod_{\substack{j=1 \\ j \neq i}}^n \frac{tx_i - x_j}{x_i - x_j}$$

and

$$\mathcal{D}_n^1(b) := \sum_{i=1}^n A_i(x; t^{-1}) \left( \left(1 - \frac{bx_i}{q}\right) T_{q^{-1}, x_i} + \frac{bx_i}{q} \right),$$

so that  $\mathcal{D}_n^1(0) = \mathcal{D}_n^1$ .

**Theorem 5.7.** *We have*

$$\mathcal{D}_n^1(b) R_\lambda(X; b) = \omega_\lambda R_\lambda(X; b).$$

*Proof.* Define the operator  $\mathcal{E}_n$  as

$$\mathcal{E}_n := \sum_{i=1}^n x_i A_i(x; t^{-1}) (T_{q^{-1}, x_i} - 1).$$

Combining the  $(q, t) \mapsto (q^{-1}, t^{-1})$  instance of [9, Proposition 9] with (2.22) gives

$$\mathcal{E}_n \mathbf{P}_\lambda(X) = (1 - q) \sum_{i=1}^n q^{-\lambda_i} t^{i-n} (1)_{\lambda^{(i)}/\lambda} \begin{bmatrix} \lambda^{(i)} \\ \lambda \end{bmatrix} \mathbf{P}_{\lambda^{(i)}}(X),$$

where  $\lambda^{(i)} := (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots, \lambda_i)$ .

Since

$$\mathcal{D}_n^1(b) = \mathcal{D}_n^1(0) - bq^{-1} \mathcal{E}_n$$

this implies that

$$\mathcal{D}_n^1(b) \mathbf{P}_\lambda(X) = \omega_\lambda \mathbf{P}_\lambda(X) - b(1 - q) \sum_{i=1}^n q^{-\lambda_i - 1} t^{i-n} (1)_{\lambda^{(i)}/\lambda} \begin{bmatrix} \lambda^{(i)} \\ \lambda \end{bmatrix} \mathbf{P}_{\lambda^{(i)}}(X).$$

By Corollary 5.5 we can now compute the action of  $\mathcal{D}_n^1(b)$  on  $\mathbf{R}_\lambda(X; b)$ :

$$\begin{aligned} \mathcal{D}_n^1(b) \mathbf{R}_\mu(X; b) &= \sum_{\lambda \supseteq \mu} b^{|\lambda - \mu|} (1)_{\lambda/\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \mathcal{D}_n^1(b) \mathbf{P}_\lambda(X) \\ &= \sum_{\lambda \supseteq \mu} b^{|\lambda - \mu|} (1)_{\lambda/\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \omega_\lambda \mathbf{P}_\lambda(X) \\ &\quad - (1 - q) \sum_{i=1}^n \sum_{\lambda \supseteq \mu} b^{|\lambda - \mu| + 1} q^{-\lambda_i - 1} t^{i-n} (1)_{\lambda^{(i)}/\mu} \begin{bmatrix} \lambda^{(i)} \\ \lambda \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \mathbf{P}_{\lambda^{(i)}}(X) \\ &= \sum_{\lambda \supseteq \mu} b^{|\lambda - \mu|} (1)_{\lambda/\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \omega_\lambda \mathbf{P}_\lambda(X) \\ &\quad - (1 - q) \sum_{i=1}^n \sum_{\lambda \supseteq \mu} b^{|\lambda - \mu|} q^{-\lambda_i} t^{i-n} (1)_{\lambda/\mu} \begin{bmatrix} \lambda \\ \lambda^{(i)} \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \mathbf{P}_\lambda(X), \end{aligned}$$

where we have also used that  $(1)_{\lambda/\mu} (1)_{\mu/\nu} = (1)_{\lambda/\nu}$  for  $\nu \subseteq \mu \subseteq \lambda$ . Recalling the recursion (2.17), the sum over  $i$  on the right can be performed to give

$$\mathcal{D}_n^1(b) \mathbf{R}_\mu(X; b) = \omega_\mu \sum_{\lambda \supseteq \mu} b^{|\lambda - \mu|} (1)_{\lambda/\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \mathbf{P}_\lambda(X).$$

Again using (5.8) completes the proof.  $\square$

As a third and final application of Corollary 5.5 we derive a simple expression for  $D_n(b, c) P_\mu(x)$ , to be compared with (2.2) (obtained for  $b = 0$ ) or with (4.1).

**Proposition 5.8.** *We have*

$$D_n(b, c) P_\mu(x) = \sum_{\lambda' \supseteq \mu'} (-b)^{|\lambda - \mu|} P_\lambda(x) \psi'_{\lambda/\mu} \prod_{\lambda_i = \mu_i} (1 - cq^{\lambda_i} t^{n-i}) \prod_{\lambda_i \neq \mu_i} (1 - q^{\mu_i} t^{1-i}).$$

*Proof.* First we use (2.7b) and (2.15), as well as the fact that for  $\lambda - \mu$  a vertical strip (i.e.,  $\lambda_i - \mu_i = 0, 1$ )

$$\prod_{\lambda_i = \mu_i} (1 - cq^{\lambda_i} t^{n-i}) \prod_{\lambda_i \neq \mu_i} (1 - q^{\mu_i} t^{1-i}) = \frac{(cqt^{n-1})_\mu}{(cqt^{n-1})_\lambda} (1)_{\lambda/\mu} \prod_{i=1}^n (1 - cq^{\lambda_i} t^{n-i}),$$

to put the proposition in the form

$$D_n(b, c) P_\mu(x) = \left( \prod_{i=1}^n (1 - cq^{\lambda_i} t^{n-i}) \right) \sum_{\lambda \supseteq \mu} b^{|\lambda - \mu|} \frac{(cqt^{n-1})_\mu}{(cqt^{n-1})_\lambda} (1)_{\lambda/\mu} \mathbf{Q}_{\lambda/\mu} \left[ \frac{q-1}{1-t} \right] \mathbf{P}_\lambda(x).$$

Since  $M_{\mu\nu} := \begin{bmatrix} \mu \\ \nu \end{bmatrix}$  is lower-triangular (with  $M_{\mu\mu} = 1$ ), it is invertible (for the explicit inverse see [14, page 540]). Since  $(1)_{\lambda/\mu}(1)_{\mu/\nu} = (1)_{\lambda/\nu}$  for  $\nu \subseteq \mu \subseteq \lambda$  the above equation is thus equivalent to

$$(5.11) \quad D_n(b, c) \sum_{\mu \supseteq \nu} b^{|\mu-\nu|} (1)_{\mu/\nu} \begin{bmatrix} \mu \\ \nu \end{bmatrix} P_\mu(x) \\ = \left( \prod_{i=1}^n (1 - cq^{\lambda_i} t^{n-i}) \right) \sum_{\lambda \supseteq \mu \supseteq \nu} b^{|\lambda-\nu|} \frac{(cqt^{n-1})_\mu}{(cqt^{n-1})_\lambda} (1)_{\lambda/\nu} \begin{bmatrix} \mu \\ \nu \end{bmatrix} Q_{\lambda/\mu} \left[ \frac{q-1}{1-t} \right] P_\lambda(x)$$

for fixed  $\nu$ .

Taking  $(a, b, c, d) \mapsto (cq, c, 0, 1)$  in [20, Corollary 4.9] in Rains' generalised  $q$ -Pfaff–Saalschütz sum [20, Corollary 4.9]

$$(5.12) \quad \sum_{\mu} \frac{(a)_\mu}{(c)_\mu} Q_{\lambda/\mu} \left[ \frac{a-b}{1-t} \right] Q_{\mu/\nu} \left[ \frac{b-c}{1-t} \right] = \frac{(a)_\nu (b)_\lambda}{(b)_\nu (c)_\lambda} Q_{\lambda/\nu} \left[ \frac{a-c}{1-t} \right]$$

yields

$$\sum_{\mu} \frac{(cq)_\mu}{(cq)_\nu} \begin{bmatrix} \mu \\ \nu \end{bmatrix} Q_{\lambda/\mu} \left[ \frac{q-1}{1-t} \right] = q^{|\lambda-\nu|} \frac{(c)_\lambda}{(c)_\nu} \begin{bmatrix} \lambda \\ \nu \end{bmatrix}.$$

This allows the sum over  $\mu$  on the right of (5.11) to be carried out, leading to

$$D_n(b, c) \sum_{\mu \supseteq \nu} b^{|\mu-\nu|} (1)_{\mu/\nu} \begin{bmatrix} \mu \\ \nu \end{bmatrix} P_\mu(x) \\ = \left( \prod_{i=1}^n (1 - cq^{\lambda_i} t^{n-i}) \right) \sum_{\lambda \supseteq \nu} (bq)^{|\lambda-\nu|} \frac{(ct^{n-1})_\lambda (cqt^{n-1})_\nu}{(ct^{n-1})_\nu (cqt^{n-1})_\lambda} (1)_{\lambda/\nu} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} P_\lambda(x) \\ = \left( \prod_{i=1}^n (1 - cq^{\nu_i} t^{n-i}) \right) \sum_{\lambda \supseteq \nu} (bq)^{|\lambda-\nu|} (1)_{\lambda/\nu} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} P_\lambda(x).$$

By Corollary 5.5 this is the same as (4.1).  $\square$

**5.1. Proof of Theorem 5.1.** To prove the theorem we first prepare the following result.

**Proposition 5.9.** *For  $\mu, \nu$  partitions*

$$\sum_{\lambda} \frac{(a)_\lambda}{(c)_\lambda} P_{\lambda/\mu} \left( \frac{c}{ab} \right) Q_{\lambda/\nu} \left[ \frac{1-b}{1-t} \right] \\ = \frac{(a)_\mu}{(c/t)_\mu} \frac{(a)_\nu}{(c/b)_\nu} \frac{(c/a, c/b)_\infty}{(c, c/ab)_\infty} \sum_{\lambda} \frac{(c/bt)_\lambda}{(a)_\lambda} P_{\nu/\lambda} \left( \frac{c}{ab} \right) Q_{\mu/\lambda} \left[ \frac{1-b}{1-t} \right].$$

Note that for  $\nu = 0$  this is

$$\sum_{\lambda} \frac{(a, b)_\lambda}{(c)_\lambda} P_{\lambda/\mu} \left( \frac{c}{ab} \right) = \frac{(c/a, c/b)_\infty}{(c, c/ab)_\infty} \frac{(a, b)_\mu}{(c/t)_\mu},$$

which, for  $\mu = 0$ , simplifies to the  $q$ -Gauss sum (1.16).

*Proof of Proposition 5.9.* Key is the Cauchy-type identity for skew Macdonald polynomials due to Rains [22, Corollary 3.8]:

$$(5.13) \quad \frac{1}{Z} \sum_{\lambda} q^{|\lambda|} \frac{(a, b)_{\lambda}}{(e, f)_{\lambda}} P_{\lambda/\mu} \left[ \frac{1-c}{1-t} \right] Q_{\lambda/\nu} \left[ \frac{1-d}{1-t} \right] \\ = \left( \frac{q}{c} \right)^{|\mu|} \frac{(a, b)_{\mu}}{(e/c, f/c)_{\mu}} \left( \frac{q}{d} \right)^{|\nu|} \frac{(a, b)_{\nu}}{(e/d, f/d)_{\nu}} \\ \times \sum_{\lambda} \left( \frac{cd}{q} \right)^{|\lambda|} \frac{(e/cd, f/cd)_{\lambda}}{(a, b)_{\lambda}} P_{\nu/\lambda} \left[ \frac{1-c}{1-t} \right] Q_{\mu/\lambda} \left[ \frac{1-d}{1-t} \right],$$

provided that the sum on the left terminates and the balancing condition  $abcdq = eft$  holds. The prefactor  $Z$  refers to the sum on the left for  $\mu = \nu = 0$ , i.e., to

$$Z = \sum_{\lambda} q^{|\lambda|} \frac{(a, b)_{\lambda}}{(e, f)_{\lambda}} P_{\lambda} \left[ \frac{1-c}{1-t} \right] Q_{\lambda} \left[ \frac{1-d}{1-t} \right] \\ = \sum_{\lambda} q^{|\lambda|} \frac{(a, b, c, d)_{\lambda}}{(e, f)_{\lambda}} P_{\lambda} \left[ \frac{1}{1-t} \right].$$

Since  $(t)_{\lambda}$  vanishes if  $l(\lambda) > 1$ , and since

$$(t)_{(k)} P_{(k)} \left[ \frac{1}{1-t} \right] = \frac{1}{(q)_k}$$

it follows that for  $(b, c) = (q^{-N}, t)$

$$Z = \sum_{k=0}^N \frac{(a, d, q^{-N})_k q^k}{(q, e, adq^{1-N}/e)_k} = \frac{(e/a, e/d)_N}{(e, e/ad)_N},$$

where the second equality follows from the  $q$ -Pfaff-Saalschütz sum [2, Equation (II.12)].

To make the same  $(b, c) = (q^{-N}, t)$  specialisation in the right-hand side of (5.13) we first replace the sum over  $\lambda$  by  $\lambda \subseteq \nu$  (using the fact that  $P_{\nu/\lambda}(X) = 0$  if  $\lambda \not\subseteq \nu$ ). Then

$$\frac{(b)_{\nu}}{(b)_{\lambda}} = \prod_{s \in \nu - \lambda} (1 - bq^{a'(s)} t^{-l'(s)})$$

is well-defined for  $b = q^{-N}$ . It thus follows that for  $(b, c) = (q^{-N}, t)$  (5.13) simplifies to

$$\sum_{\lambda} q^{|\lambda|} \frac{(a, q^{-N})_{\lambda}}{(e, f)_{\lambda}} P_{\lambda/\mu}(1) Q_{\lambda/\nu} \left[ \frac{1-d}{1-t} \right] \\ = \left( \frac{q}{t} \right)^{|\mu|} \frac{(a, q^{-N})_{\mu}}{(e/t, f/t)_{\mu}} \left( \frac{q}{d} \right)^{|\nu|} \frac{(a, q^{-N})_{\nu}}{(e/d, f/d)_{\nu}} \frac{(e/a, e/d)_N}{(e, e/ad)_N} \\ \times \sum_{\lambda \subseteq \nu} \left( \frac{dt}{q} \right)^{|\lambda|} \frac{(e/dt, f/dt)_{\lambda}}{(a, q^{-N})_{\lambda}} P_{\nu/\lambda}(1) Q_{\mu/\lambda} \left[ \frac{1-d}{1-t} \right]$$

for  $adq^{1-N} = ef$ . Eliminating  $f$ , taking the limit  $N \rightarrow \infty$  and using (2.6) this results in the claim with  $(b, c) \mapsto (d, e)$ .  $\square$

We are now prepared to prove Theorem 5.1.

*Proof.* We proceed by induction on  $n := |X|$ . For  $n = 0$  we get the tautology

$$\frac{(b/c)_\mu}{(b/c)_\nu} \mathbf{Q}_{\mu/\nu} \left[ \frac{a-c}{1-t} \right] = \frac{(b/c)_\mu}{(b/c)_\nu} \mathbf{Q}_{\mu/\nu} \left[ \frac{a-c}{1-t} \right].$$

For  $n \geq 1$  we compute the sum on the right-hand side of (5.3), assuming the formula is true for the alphabet  $Y$ . (Recall that  $X = Y + z$ .) Using the branching rule (5.2) and

$$(5.14) \quad \mathbf{Q}_{\lambda/\nu} \left[ \frac{a-c}{1-t} \right] \mathbf{P}_{\lambda/\eta}(z) = a^{|\eta-\nu|} \mathbf{Q}_{\lambda/\nu} \left[ \frac{1-c/a}{1-t} \right] \mathbf{P}_{\lambda/\eta}(az)$$

we obtain

$$\begin{aligned} S_{\mu,\nu}(X; a, b, c) &:= \sum_{\lambda} \frac{(b/c)_\lambda}{(b/c)_\nu} \mathbf{Q}_{\lambda/\nu} \left[ \frac{a-c}{1-t} \right] \mathbf{R}_{\lambda/\mu}(X; b) \\ &= \sum_{\lambda, \eta} a^{|\eta-\nu|} \frac{(b/c)_\lambda (bz/t)_\eta}{(b/c)_\nu (bz)_\lambda} \mathbf{Q}_{\lambda/\nu} \left[ \frac{1-c/a}{1-t} \right] \mathbf{P}_{\lambda/\eta}(az) \mathbf{R}_{\eta/\mu}(Y; b). \end{aligned}$$

The sum over  $\lambda$  can be transformed by Proposition 5.9 with  $(a, b, c) \mapsto (b/c, c/a, bz)$  and  $\mu \mapsto \eta$ . As a result

$$\begin{aligned} S_{\mu,\nu}(X; a, b, c) &= \frac{(cz, dz)_\infty}{(az, bz)_\infty} \sum_{\lambda, \eta} \frac{(b/c)_\eta (dz/t)_\lambda}{(b/c)_\lambda (dz)_\nu} \mathbf{Q}_{\eta/\lambda} \left[ \frac{a-c}{1-t} \right] \mathbf{P}_{\nu/\lambda}(z) \mathbf{R}_{\eta/\mu}(Y; b) \\ &= \frac{(cz, dz)_\infty}{(az, bz)_\infty} \sum_{\lambda} \frac{(dz/t)_\lambda}{(dz)_\nu} \mathbf{P}_{\nu/\lambda}(z) S_{\mu,\lambda}(Y, a, b, c), \end{aligned}$$

where once again we have used (5.14), and where  $d = ab/c$ . By the induction hypothesis,  $S_{\mu,\lambda}(Y, a, b, c)$  may be replaced by the right-hand side of (5.3) with  $(X, \lambda, \nu) \mapsto (Y, \eta, \lambda)$ , leading to

$$S_{\mu,\nu}(X; a, b, c) = \prod_{x \in X} \frac{(cx, dx)_\infty}{(ax, bx)_\infty} \sum_{\lambda, \eta} \frac{(b/c)_\mu (dz/t)_\lambda}{(b/c)_\eta (dz)_\nu} \mathbf{Q}_{\mu/\eta} \left[ \frac{a-c}{1-t} \right] \mathbf{P}_{\nu/\lambda}(z) \mathbf{R}_{\lambda/\eta}(Y; d).$$

One final application of (5.2) results in

$$S_{\mu,\nu}(X; a, b, c) = \prod_{x \in X} \frac{(cx, dx)_\infty}{(ax, bx)_\infty} \sum_{\eta} \frac{(b/c)_\mu}{(b/c)_\eta} \mathbf{Q}_{\mu/\eta} \left[ \frac{a-c}{1-t} \right] \mathbf{R}_{\nu/\eta}(X; d),$$

which is the right-hand side of (5.3) with  $\lambda \mapsto \eta$ .  $\square$

## 6. TRANSFORMATION FORMULAS FOR $\mathfrak{sl}_n$ BASIC HYPERGEOMETRIC SERIES

In this final section we prove a number of additional identities for basic hypergeometric series involving the function  $\mathbf{R}_\lambda(X; b)$ . For easy comparison with known results for one-variable basic hypergeometric series we define

$$\begin{aligned} {}_r\Phi_s \left[ \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z; X \right] \\ := \sum_{\lambda} \frac{(a_1, \dots, a_r)_\lambda}{(b_1, \dots, b_{s-1})_\lambda} \left( (-1)^{|\lambda|} q^{n(\lambda')} t^{-n(\lambda)} \right)^{s-r+1} z^{|\lambda|} \mathbf{R}_\lambda(X; b_s), \end{aligned}$$

where  $X$  is a finite alphabet. There is some redundancy in the above definition since by Lemma 4.1  $z^{|\lambda|}R_\lambda(X; b_s) = R_\lambda(zX; b_s/z)$ . Since  $R_\lambda(X; b)$  vanishes if  $l(\lambda) > |X|$ , and recalling  $R_{(k)}(z; b) = z^k/(bz)_k$  (see (4.3)), it follows that

$$\begin{aligned} {}_r\Phi_s \left[ \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z; \{1\} \right] &= \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r)_k}{(b_1, \dots, b_s)_k} \left( (-1)^k q^{\binom{k}{2}} \right)^{s-r+1} z^k \\ &= {}_r\phi_s \left[ \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z \right], \end{aligned}$$

with on the right the standard notation for one-variable basic hypergeometric series (with dependence on the base  $q$  suppressed). The reader is warned that only in this degenerate case do the ‘‘lower-parameters’’  $b_1, \dots, b_s$  enjoy full  $\mathfrak{S}_s$  symmetry. Using the above notation the  $q$ -Gauss sum (5.7) may be stated as

$$(6.1) \quad {}_2\Phi_1 \left[ \begin{matrix} a, b \\ c \end{matrix}; \frac{c}{ab}; X \right] = \prod_{x \in X} \frac{(cx/a, cx/b)_\infty}{(cx, cx/ab)_\infty}.$$

To generalise this result we first prove the following transformation formula.

**Theorem 6.1.** *For  $f = de/bc$  and  $\mu$  a partition,*

$$\begin{aligned} \sum_{\lambda} \frac{(a, b)_\lambda}{(d)_\lambda} Q_{\lambda/\mu} \left[ \frac{f/a - cf/a}{1-t} \right] R_\lambda(X; e) \\ = \frac{(b)_\mu}{(d/c)_\mu} \left( \prod_{x \in X} \frac{(fx, ex/a)_\infty}{(ex, fx/a)_\infty} \right) \sum_{\lambda} \frac{(a, d/c)_\lambda}{(d)_\lambda} Q_{\lambda/\mu} \left[ \frac{e/a - cf/a}{1-t} \right] R_\lambda(X; f). \end{aligned}$$

Note that the substitution  $(b, c, e, f) \mapsto (d/c, d/b, f, e)$  interchanges the left- and right-hand sides.

*Proof.* We replace  $\mu \mapsto \nu$  and then rename the summation index  $\lambda$  on the right as  $\mu$ . By the Pieri formula of Theorem 5.2 (with  $(a, b, c) \mapsto (bc/d, e, a)$ ) the term

$$R_\mu(X; f) \prod_{x \in X} \frac{(fx, ex/a)_\infty}{(ex, fx/a)_\infty}$$

on the right-hand side can be expanded as

$$\sum_{\lambda} \frac{(a)_\lambda}{(a)_\mu} Q_{\lambda/\mu} \left[ \frac{f/a - e/a}{1-t} \right] R_\lambda(X; e),$$

resulting in

$$\begin{aligned} \sum_{\lambda} \frac{(a, b)_\lambda}{(d)_\lambda} Q_{\lambda/\nu} \left[ \frac{f/a - cf/a}{1-t} \right] R_\lambda(X; e) \\ = \frac{(b)_\nu}{(d/c)_\nu} \sum_{\lambda, \mu} \frac{(a)_\lambda (d/c)_\mu}{(d)_\mu} Q_{\lambda/\mu} \left[ \frac{f/a - e/a}{1-t} \right] Q_{\mu/\nu} \left[ \frac{e/a - cf/a}{1-t} \right] R_\lambda(X; e). \end{aligned}$$

By (5.12) with  $(a, b, c, d) \mapsto (d/c, b, d, e/ab)$  the sum over  $\mu$  can be carried out, completing the proof.  $\square$

For  $\mu = 0$  Theorem 6.1 simplifies to a multiple analogue of the  $q$ -Kummer–Thomae–Whipple formula [2, Equation (III.10)] (corresponding to the formula below when  $X = \{1\}$ ).

**Corollary 6.2** ( $\mathfrak{sl}_n$   $q$ -Kummer–Thomae–Whipple formula). For  $f = de/bc$ ,

$$(6.2) \quad {}_3\Phi_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix}; \frac{f}{a}; X \right] = \left( \prod_{x \in X} \frac{(fx, ex/a)_\infty}{(ex, fx/a)_\infty} \right) {}_3\Phi_2 \left[ \begin{matrix} a, d/b, d/c \\ d, f \end{matrix}; \frac{e}{a}; X \right].$$

For  $d = c$  this reduces to the  $q$ -Gauss sum (6.1).

If we let  $c, d \rightarrow 0$  in Theorem 6.1 such that  $d/c = bf/e$  and then replace  $(e, f) \mapsto (c, az)$ , we find

$$\begin{aligned} \sum_{\lambda} z^{|\lambda-\mu|} \frac{(a, b)_\lambda}{(a, b)_\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} R_\lambda(X; c) \\ = \left( \prod_{x \in X} \frac{(cx/a, azx)_\infty}{(cx, zx)_\infty} \right) \sum_{\lambda} \left( \frac{c}{a} \right)^{|\lambda-\mu|} \frac{(a, abz/c)_\lambda}{(a, abz/c)_\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} R_\lambda(X; az). \end{aligned}$$

For  $\mu = 0$  this is a multiple analogue of Heine's  ${}_2\phi_1$  transformation [2, Equation (III.2)].

**Corollary 6.3** ( $\mathfrak{sl}_n$  Heine transformation). We have

$${}_2\Phi_1 \left[ \begin{matrix} a, b \\ c \end{matrix}; z; X \right] = \left( \prod_{x \in X} \frac{(cx/a, azx)_\infty}{(cx, zx)_\infty} \right) {}_2\Phi_1 \left[ \begin{matrix} a, abz/c \\ az \end{matrix}; \frac{c}{a}; X \right].$$

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