Generalisation of Scott permanent identity

Alain Lascoux*
IGM, Université de Paris-Est
77454 Marne-la-Vallée CEDEX 2
Alain.Lascoux@univ-mlv.fr
http://phalanstere.univ-mlv.fr/∼al

Abstract

Let $\mathbf{x} = \{x_1, \ldots, x_r\}$, $\mathbf{y} = \{y_1, \ldots, y_n\}$, $\mathbf{z} = \{z_1, \ldots, z_n\}$ be three sets of indeterminates. We give the value of the determinant

$$\left| \prod_{x\in\mathbf{x}} (xy - z)^{-1} \right|_{y\in\mathbf{y}, z\in\mathbf{z}}$$

when specializing $\mathbf{y}$ and $\mathbf{z}$ to the set of roots of $y^n - 1$ and $z^n - \xi^n$ respectively.

In the case where $r = 2$ and $\mathbf{x} = \{1, 1\}$ the determinant $\left|(y-z)^{-2}\right|_{y\in\mathbf{y},z\in\mathbf{z}}$ factorizes into the determinant of the Cauchy matrix $[(y - z)^{-1}]$ and its permanent. Scott [10, 8] found the value of this permanent when specializing $\mathbf{y}$ to the roots of $y^n - 1$ and $\mathbf{z}$ to the roots of $z^n + 1$. Han [3] described more generally the case where $\mathbf{z}$ is the set of roots of $z^n + az^k + b$ instead of $z^n + 1$.

Instead of restricting to $r = 2$ and specializing $\mathbf{x}$, we shall consider the determinant

$$\left| \prod_{x\in\mathbf{x}} (xy - z)^{-1} \right|_{y\in\mathbf{y}, z\in\mathbf{z}},$$

and obtain in Th. 2 its value when specializing $\mathbf{y}$ and $\mathbf{z}$. The remarkable feature is that this value is a product of sums of monomial functions in $\mathbf{x}$ without multiplicities, thus extending the factorized expressions of [10, 3].

*partially supported by the ANR project MARS (BLAN06-2 134516).

We first need a few generalities about symmetric functions [5].

Given two sets of indeterminates $\mathbf{x}, \mathbf{z}$ (we say alphabets), the complete functions $S_n(\mathbf{x}-\mathbf{z})$ are the coefficients of the generating function

$$\sum_n \gamma^n S_n(\mathbf{x}-\mathbf{z}) = \prod_{z\in\mathbf{z}}(1-\gamma z)\prod_{x\in\mathbf{x}}(1-\gamma x)^{-1}\,.$$

For any r, any $\lambda\in\mathbb{Z}^r$, $S_\lambda(\mathbf{x}-\mathbf{z}) = \det\big(S_{\lambda_i+j-i}(\mathbf{x}-\mathbf{z})\big)$.

In the case where $\mathbf{z}=0$, and $\mathbf{x}$ of cardinality r, these functions can be obtained by symmetrisation over the symmetric group $\mathfrak{S}_r$. Let π_ω be the following operator on functions in $\mathbf{x}$:

$$f \to f\pi_\omega := \sum_{\sigma\in\mathfrak{S}_r}\left(f\prod_{1\le i<j\le r}(1-x_j/x_i)^{-1}\right)^\sigma\,.$$

Then, when $\lambda \ge [1\text{-}r,\ldots,\text{-}1,0]$ (i.e. $\lambda_1\ge 1\text{-}r,\ \ldots,\ \lambda_r\ge 0$), the monomial $x^\lambda = x_1^{\lambda_1}\cdots x_r^{\lambda_r}$ is sent onto $S_\lambda(\mathbf{x})$ under π_ω. When λ is a partition (i.e. $\lambda_1\le\cdots\lambda_r\le 0$), $S_\lambda(\mathbf{x})$ is the Schur function of index λ.

Let n be a positve integer, ξ and indeterminate and $\mathbf{z}$ be the set of roots of $z^n-\xi^n$. Equivalently, $e_i(\mathbf{z})=0$ for $1\le i\le n\text{-}1$, $e_n(\mathbf{z})=(\text{-}1)^{n-1}\xi^n$. For any integer j, any $\mathbf{x}$, one has

$$S_j(\mathbf{x}-\mathbf{z}) = S_j(\mathbf{x}) - \xi^n S_{j-n}(\mathbf{x})\,,$$

and more generally, from the determinantal expression of Schur functions, for any $\lambda\in\mathbb{N}^r$,

$$S_\lambda(\mathbf{x}-\mathbf{z}) = \sum_{u\in\{0,n\}^r} S_{\lambda-u}(\mathbf{x})(-1)^{|u|/n}\xi^{|u|}\,.$$

In particular, when $\lambda = \underbrace{n\text{-}1,\ldots,n\text{-}1}_{r-1},p$ then the terms with negative last component $n-p$ vanish, and the set $\big\{\lambda-u\big\}$ to consider is $\big\{[v,p]\,:\,v\in\{n\text{-}1,\text{-}1\}^{r-1}\big\}$. Reordering indices, putting $q=p\text{-}r\text{+}1$, one rewrites the sum as

$$S_\lambda(\mathbf{x}-\mathbf{z}) = (-1)^{r-1}\sum_{u\in\{0,n\}^{r-1}} S_{q,u}(\mathbf{x})(-1)^{(|\lambda|-|u|-q)/n}\xi^{|\lambda|-|u|-q}\,. \qquad (1)$$

Since $x^v\pi_\omega = S_v(\mathbf{x})$, for any $v\ge[1\text{-}r,\ldots,\text{-}1,0]$, one can rewrite (1) as a symmetrisation of monomial functions in $\mathbf{x}-x_1=\{x_2,\ldots,x_r\}$:

$$S_\lambda(\mathbf{x}-\mathbf{z}) = (-1)^{r-1}\sum_{j=0}^{r-1} x_1^q m_{n^j}(\mathbf{x}-x_1)(-1)^{r-1-j}\xi^{|\lambda|-jn-q}\,\pi_\omega\,. \qquad (2)$$

From the identity

$$m_{n^j}(\mathbf{x}-x_1) = m_{n^j}(\mathbf{x}) - x_1^n m_{n^{j-1}}(\mathbf{x}) + x_1^{2n} m_{n^{j-2}}(\mathbf{x}) + \cdots + (-x_1^n)^j\,,$$

one sees that $x_1^q m_{n^j}(\mathbf{x}-x_1)\,\pi_\omega$ is equal to

$$S_q(\mathbf{x})m_{n^j}(\mathbf{x}) - S_{q+n}(\mathbf{x})m_{n^{j-1}}(\mathbf{x}) + \cdots + (-1)^j S_{q+jn}(\mathbf{x})\,.$$

Since on the other hand $S_{(n-1)^{r-1}p}(\mathbf{x}-\mathbf{z})$ belongs to the linear span of Schur functions, or monomial functions, indexed by partitions μ such that $\mu_1 \leq n{-}1$, one can restrict this last sum to the term $(-1)^j S_{q+jn}(\mathbf{x})$.

In summary, one has the following expression for the specialisation of the Schur function that we are considering.

Proposition 1 *Let $\mathbf{x}$ be an alphabet of cardinality r, $\mathbf{z}$ be the set of roots $z^n - \xi^n = 0$, $p \leq n{-}1$, $N = (n{-}1)(r{-}1)$. Then*

$$S_{(n-1)^{r-1}p}(\mathbf{x}-\mathbf{z}) = \sum_\mu m_\mu(\mathbf{x})\,\xi^{N+p-|\mu|}\,, \tag{3}$$

sum over all partitions $\mu \in \mathbb{N}^r$, $\mu_1 \leq n{-}1$.

For example, for $n=4$, $r=2$, one has

$$S_{30}(\mathbf{x}-\mathbf{z}) = m_3(\mathbf{x}) + m_{21}(\mathbf{x}),\ \ S_{31}(\mathbf{x}-\mathbf{z}) = m_{31}(\mathbf{x}) + m_{22}(\mathbf{x}) + \xi^4\,,$$

$$S_{32}(\mathbf{x}-\mathbf{z}) = m_{32}(\mathbf{x}) + \xi^4 m_1(\mathbf{x}),\ \ S_{33}(\mathbf{x}-\mathbf{z}) = m_{33}(\mathbf{x}) + \xi^4(m_2(\mathbf{x}) + m_{11}(\mathbf{x}))\,.$$

Let

$$D(\mathbf{x},\mathbf{y},\mathbf{z}) = \Big|\prod_{x\in\mathbf{x}}(xy-z)^{-1}\Big|_{y\in\mathbf{y},z\in\mathbf{z}}\,.$$

In the case $r=2$, this determinant has been obtained by Izergin and Korepin [4] as the partition function of the Heisenberg XXZ-antiferromagnetic model. Gaudin [2] had previously described the partition function of some other model as the determinant $\big|(x-y)^{-1}(x-y+\gamma)^{-1}\big|$ for some parameter γ.

The Izergin-Korepin determinant is used in the enumeration of alternating sign matrices [1]. In that case, one first specializes $\mathbf{x} = \{e^{2i\pi/3}, e^{4i\pi/3}\}$. Okada [9] evaluates more general partition functions corresponding to similar determinants or Pfaffians, and to other roots of unity (see also [6, Th. 7.2]).

We shall take another point of view, keep $\mathbf{x}$ generic, but specialize instead $\mathbf{y}$ and $\mathbf{z}$. In [7, Formula 4], it is shown that the function

$$G(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{D(\mathbf{x},\mathbf{y},\mathbf{z})}{\Delta(\mathbf{z})}\prod_{x\in\mathbf{x}}\prod_{y\in\mathbf{y}}\prod_{z\in\mathbf{z}}(xy-z)$$

is equal to the determinant of the matrix

$$\Big[S_{\square j}(y_i\mathbf{x}-\mathbf{z})\Big]_{j=0\ldots n-1,\, i=1\ldots n}\,, \tag{4}$$

where $\square=(n-1)^{r-1}$, and $\Delta(\mathbf{z})=\prod_{1\le i<j\le n}(z_i-z_j)$.

For any $k\in\mathbb{N}$, let φ_k be the sum of all monomial functions $m_\mu(\mathbf{x})$ of degree k, with $\mu_1\le n{-}1$ (notice that $\varphi_k=0$ when $k>(n-1)r$). From (3), one has that $S_{\square j}(y_i\mathbf{x}-\mathbf{z})$ specializes, when $\mathbf{z}$ is the set of roots of $z^n-\xi^n$, into

$$S_{\square j}(y_i\mathbf{x}-\mathbf{z}) = y_i^{N+j}\varphi_{N+j}+\xi^n y_i^{N+j-n}\varphi_{N+j-n}+\xi^{2n}y_i^{N+j-2n}\varphi_{N+j-2n}+\cdots\,. \tag{5}$$

Specializing further $\mathbf{y}$ into the roots of y^n-1, one sees that the matrix (4) factorizes into the product of the matrix $\Big[y_i^{(N+j)}\Big]$, where $(k)=k \mod n$, and the diagonal matrix

$$diag\Big((\varphi_N+\xi^n\varphi_{N-n}+\xi^{2n}\varphi_{N-2n}+\cdots),(\varphi_{N+1}+\xi^n\varphi_{N+1-n}+\xi^{2n}\varphi_{N+1-2n}+\cdots), \\ \cdots,\ (\varphi_{N+n-1}+\xi^n\varphi_{N+n-1-n}+\xi^{2n}\varphi_{N+1-n}+\cdots)\Big)\,.$$

For example, for $n=3$, $r=3$,

$$S_{220}(y_i\mathbf{x}-\mathbf{z})=y_i^4\varphi_4+y_i\varphi_1\xi^3, S_{221}(y_i\mathbf{x}-\mathbf{z})=y_i^5\varphi_5+y_i^2\varphi_2\xi^3, \\ S_{222}(y_i\mathbf{x}-\mathbf{z})=y_i^6\varphi_6+y_i^3\varphi_3\xi^3+\xi^6\,,$$

and the matrix factorizes into

$$\begin{bmatrix} y_1 & y_1^2 & 1\\ y_2 & y_2^2 & 1\\ y_3 & y_3^2 & 1\end{bmatrix}\begin{bmatrix}\varphi_4+\varphi_1\xi^3 & 0 & 0\\ 0 & \varphi_5+\varphi_2\xi^3 & 0\\ 0 & 0 & \varphi_6+\varphi_3\xi^3+\xi^6\end{bmatrix}$$

Taking into account that $\prod_{x,y,z}(xy-z)$ specializes into $\prod_{y,x}(x^ny^n-\xi^n)=\prod_{x\in\mathbf{x}}(x^n-\xi^n)^n$, and that the determinant of powers of the $y\in\mathbf{y}$ is a permutation of the Vandermonde in $\mathbf{y}$, one obtains the following theorem.

Theorem 2 *Let n,r be two positive integers, $N=(n-1)(r-1)$. Let $\mathbf{x}$ be an alphabet of cardinality r, $\mathbf{y}$ be the set of roots of y^n-1, $\mathbf{z}$ be the set of roots of $z^n-\xi^n$. Then*

$$\Delta(\mathbf{y})\Delta(\mathbf{z})\left|\prod_{k=1}^{r}(x_ky_i-z_j)^{-1}\right|_{i,j=1\ldots n} = \frac{(-1)^{(n-1)(n/2+r-1)}}{\prod_{x\in\mathbf{x}}(x^n-\xi^n)^n}\prod_{i=0}^{n-1}\left(\sum_{j=0}^{\infty}\varphi_{N+i-nj}\,\xi^{nj}\right)\,.$$

For $\mathbf{x} = \{1, 1\}$, this theorem is due to Han[3]. In that case, $\varphi_i = i+1$ and $\varphi_{n-1+i} = n - i$ for $i = 0, \ldots, n-1$, and the product appearing in the theorem is

$$n\,(n-1+\xi^n)\,(n-2+2\xi^n)\cdots(1+(n-1)\xi^n)\,.$$

For $r = 5$, $n = 3$, as a further example, the theorem furnishes the expression

$$\prod_{k=1}^{5}(x_k^3-\xi^3)^{-3}\,\left(\varphi_8+\varphi_5\xi^3+\varphi_2\xi^6\right)\left(\varphi_9+\varphi_6\xi^3+\varphi_3\xi^6+\xi^9\right)$$
$$\left(\varphi_{10}+\varphi_7\xi^3+\varphi_4\xi^6+\varphi_1\xi^9\right)\;,$$

which specializes, for $\mathbf{x} = \{1, 1, 1, 1, 1\}$, into

$$(1-\xi^3)^{-15}(15+51\xi^3+15\xi^6)(5+45\xi^3+30\xi^6+\xi^9)(1+30\xi^3+45\xi^6+5\xi^9)\,.$$

References

[1] D. Bressoud. *Proofs and Confirmations: the Story of Alternating Sign Matrix Conjecture*, Cambridge University Press (1999).

[2] M. Gaudin. *La fonction d'onde de Bethe*, Masson (1983).

[3] Han GuoNiu. *Généralisation de l'identité de Scott sur les permanents*, Linear Alg. Appl. **311** (2000) 25–34.

[4] A.G. Izergin. *Partition function of the six-vertex model in a finite volume*, Soviet Phys. Dokl. 32 (1987), 878–879.

[5] A. Lascoux. *Symmetric functions & Combinatorial operators on polynomials*, CBMS/AMS Lectures Notes **99**, (2003).

[6] A. Lascoux. *Pfaffians and Representations of the Symmetric Group* Acta Mathematica Sinica **25** (2009) 1929–1950.

[7] A. Lascoux. *Gaudin functions, and Euler-Poincaré characteristics*, arXiv:0709.1635 (2007)

[8] H. Minc. *Permanents*, Encyclopedia of mathematics and its applications, vol. 6, Addison-Wesley (1978).

[9] S. Okada. *Enumeration of symmetry classes of alternating sign matrices and characters of classical groups*, J. Alg. Comb. **23** (2006) 43-69.

[10] R. F. Scott. *Mathematical notes*, Messenger of Math. **10 (1881) 142–149.**