

# A Question of Muir and Littlewood's Products

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**Abstract** We answer a question of Muir, relating it to different determinantal expressions for the products  $\prod_{i<j}(y-x_ix_j)$  and  $\prod_{i\leq j}(y-x_ix_j)$ , and for the products of these functions by an arbitrary Schur function.

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## 1 Introduction

Muir [6] writes determinantal expressions for  $\prod_{1\leq i<j\leq 3}(y-x_ix_j)$  and for  $\prod_{1\leq i\leq j\leq 3}(y-x_ix_j)$  but makes the comment that the method followed is not consistent.<sup>1</sup>

In this paper, we give determinantal expressions for the products  $\prod_{i<j}(y-x_ix_j)$  (Theorem 2.5) and  $\prod_{i\leq j}(y-x_ix_j)$  (Theorem 2.6), and give another type of determinants for the the product of these functions by any Schur function (Theorem 2.7). The first determinants are related to the expression of a Schur function in terms of hook Schur functions, the last ones are related to the symplectic or orthogonal characters.

For example, for  $n = 3$ , we write

$$-\prod_{1\leq i<j\leq 3}(y-x_ix_j) = \begin{vmatrix} s_{11} - y & s_{21} \\ s_{111} & s_{211} + y^2 \end{vmatrix},$$
$$\prod_{1\leq i\leq j\leq 3}(y-x_ix_j) = \begin{vmatrix} s_2 - y & s_3 & s_4 \\ s_{21} & s_{31} + y^2 & s_{41} \\ s_{211} & s_{311} & s_{411} - y^3 \end{vmatrix}$$

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<sup>1</sup> [6]: *Still more curious is the fact that a similar quest, – the search for the equation whose roots are the binary products without repetition, namely,*

$$x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3$$

*– has actually been successful. Unfortunately the method followed – a quasi-dalytic – is not logically consistent ...*

and

$$s_{221} \prod_{1 \leq i < j \leq 3} (y - x_i x_j) = \begin{vmatrix} h_6 - y h_4 & h_7 - y^2 h_3 & h_8 - y^3 h_2 \\ h_5 - y h_3 & h_6 - y^2 h_2 & h_7 - y^3 h_1 \\ h_3 - y h_1 & h_4 - y^2 & h_5 \end{vmatrix}.$$

We first recall some basic definitions about symmetric functions, following the conventions of Macdonald [4].

Let  $\mathbb{X} = \{x_1, x_2, \dots, x_n\}$  be a set of  $n$  indeterminates. The complete symmetric functions  $h_k$  and the elementary symmetric functions  $e_k$  are defined by the generating functions:

$$\sum_{k=0}^{\infty} h_k t^k = \prod_{i=1}^n (1 - x_i t)^{-1}, \quad \sum_{k=0}^{\infty} e_k t^k = \prod_{i=1}^n (1 + x_i t).$$

It follows that

$$\sum_{i=0}^n (-1)^i e_i h_{n-i} = 0.$$

Given a partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  and its conjugate  $\lambda'$ , the Schur function  $s_\lambda$  has the following two expressions:

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i, j \leq n} = \det(e_{\lambda'_i - i + j})_{1 \leq i, j \leq n}.$$

Hook Schur functions  $s_{j, 1^i}$  can also be written as

$$s_{j, 1^i} = \sum_{k=1}^j (-1)^{k+1} e_{i+k} h_{j-k}. \quad (1)$$

Another notation, due to Frobenius, can be used for a partition  $\lambda$ . If  $r$  be the biggest number such that  $\lambda_i \geq i$  then

$$F(\lambda) = (\alpha_1, \dots, \alpha_r | \beta_1, \dots, \beta_r),$$

where  $\alpha_i = \lambda_i - i$  and  $\beta_i = \lambda'_i - i$ ,  $1 \leq i \leq r$ .

This notation is associated to still another determinantal expression of a Schur function, as a determinant of hook Schur functions ([4, Ex.9 p.47]):

$$s_\lambda = \det(s_{\alpha_i | \beta_j})_{1 \leq i, j \leq r}. \quad (2)$$

In this paper, we use the following two generalizations of the Vandermonde determinant:

$$M(\mathbb{X}, y) = \begin{vmatrix} x_1^{n-1} & x_1^n + x_1^{n-2}y & \dots & x_1^{2n-2} + y^{n-1} \\ x_2^{n-1} & x_2^n + x_2^{n-2}y & \dots & x_2^{2n-2} + y^{n-1} \\ \dots & \dots & \dots & \dots \\ x_n^{n-1} & x_n^n + x_n^{n-2}y & \dots & x_n^{2n-2} + y^{n-1} \end{vmatrix}$$

and

$$N(\mathbb{X}, y) = \begin{vmatrix} x_1^{n+1} - x_1^{n-1}y & x_1^{n+2} - x_1^{n-2}y^2 & \cdots & x_1^{2n} - y^n \\ x_2^{n+1} - x_2^{n-1}y & x_2^{n+2} - x_2^{n-2}y^2 & \cdots & x_2^{2n} - y^n \\ \cdots & \cdots & \cdots & \cdots \\ x_n^{n+1} - x_n^{n-1}y & x_n^{n+2} - x_n^{n-2}y^2 & \cdots & x_n^{2n} - y^n \end{vmatrix}.$$

One can factor out from these two determinants the Vandermonde  $\Delta(\mathbb{X}) = \prod_{i < j} (x_j - x_i)$ , obtaining two symmetric functions:

$$SY_{(n-1)^n} = M(\mathbb{X}, y) \Delta(\mathbb{X})^{-1}, \quad OY_{(n+1)^n} = N(\mathbb{X}, y) \Delta(\mathbb{X})^{-1},$$

where, more generally, for any partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ , the determinants  $SY_{(n-1)^{n+\lambda}}$  and  $OY_{(n+1)^{n+\lambda}}$  are the following expressions:

$$\begin{vmatrix} h_{n-1+\lambda_1} & h_{n+\lambda_1} + yh_{n-2+\lambda_1} & \cdots & h_{2n-2+\lambda_1} + y^{n-1}h_{\lambda_1} \\ h_{n-2+\lambda_2} & h_{n-1+\lambda_2} + yh_{n-3+\lambda_2} & \cdots & h_{2n-3+\lambda_2} + y^{n-1}h_{\lambda_2-1} \\ \cdots & \cdots & \cdots & \cdots \\ h_{\lambda_n} & h_{1+\lambda_n} + yh_{\lambda_n-1} & \cdots & h_{n-1+\lambda_n} + y^{n-1}h_{\lambda_n+1-n} \end{vmatrix} \quad (3)$$

and

$$\begin{vmatrix} h_{n+1+\lambda_1} - yh_{n-1+\lambda_1} & h_{n+2+\lambda_1} - y^2h_{n-2+\lambda_1} & \cdots & h_{2n+\lambda_1} - y^n h_{\lambda_1} \\ h_{n+\lambda_2} - yh_{n-2+\lambda_2} & h_{n+1+\lambda_2} - y^2h_{n-3+\lambda_2} & \cdots & h_{2n-1+\lambda_2} - y^n h_{\lambda_2-1} \\ \cdots & \cdots & \cdots & \cdots \\ h_{2+\lambda_n} - yh_{\lambda_n} & h_{3+\lambda_n} - y^2h_{\lambda_n-1} & \cdots & h_{n+1+\lambda_n} - y^n h_{\lambda_n+1-n} \end{vmatrix}. \quad (4)$$

These last two functions are homogeneous versions of Weyl's determinants for the symplectic and orthogonal characters of respective index  $(n-1)^n + \lambda$  and  $(n+1)^n + \lambda$  ([8, p.219, p.228], [1]).

In fact, as the referee points out, the Vandermonde in the variables  $x_i + y/x_i$ ,  $i = 1 \dots n$ , which is equal to

$$\prod_{i < j} (x_j - x_i + y/x_j - y/x_i) = \prod_{i < j} (x_j - x_i)(1 - y/x_i x_j)$$

is represented by the determinant  $\det((x_i + yx_i^{-1})^{j-1})_{1 \leq i, j \leq n}$ . By linear combination of columns, one transforms it into the determinant

$$\begin{vmatrix} 1 & x_1 + yx_1^{-1} & x_1^2 + y^2x_1^{-2} & \cdots & x_1^{n-1} + y^{n-1}x_1^{1-n} \\ 1 & x_2 + yx_2^{-1} & x_2^2 + y^2x_2^{-2} & \cdots & x_2^{n-1} + y^{n-1}x_2^{1-n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n + yx_n^{-1} & x_n^2 + y^2x_n^{-2} & \cdots & x_n^{n-1} + y^{n-1}x_n^{1-n} \end{vmatrix}.$$

This new determinant is proportional to  $M(\mathbb{X}, y)$ , and therefore  $M(\mathbb{X}, y)$  can already be considered as an answer to the question of Muir, except for the parasitic factor  $(-1)^{\binom{n}{2}} \Delta(\mathbb{X})$ . A similar manipulation shows that  $N(\mathbb{X}, y)$  is equal to  $(-1)^{\binom{n+1}{2}} \Delta(\mathbb{X}) \prod_{i < j} (y - x_i x_j)$ .

## 2 Littlewood's Products

Littlewood has given the expansion, in terms of Schur functions, of the two products  $\prod_{1 \leq i < j \leq n} (1 - x_i x_j)$  and  $\prod_{1 \leq i \leq j \leq n} (1 - x_i x_j)$ , which appear in the Weyl character formula for the types  $B$  and  $C$ .

Littlewood's formulas [4, Ex. 9 p.78] read as follows:

$$\prod_{1 \leq i < j \leq n} (1 - x_i x_j) = \sum_{\lambda} (-1)^{|\lambda|/2} s_{\lambda}(\mathbb{X}), \quad (5)$$

summed over all partitions  $\lambda : F(\lambda) = \{\alpha_1 - 1, \dots, \alpha_r - 1 | \alpha_1, \dots, \alpha_r\}$  with  $\alpha_1 \leq n - 1$ ;

$$\prod_{1 \leq i \leq j \leq n} (1 - x_i x_j) = \sum_{\lambda} (-1)^{|\lambda|/2} s_{\lambda}(\mathbb{X}), \quad (6)$$

summed over all partitions  $\lambda : F(\lambda) = \{\alpha_1 + 1, \dots, \alpha_r + 1 | \alpha_1, \dots, \alpha_r\}$  with  $\alpha_1 \leq n - 1$ .

In this section, we obtain new determinantal expressions for these two products, and show that they imply Littlewood's sums. For considerations about generalizations of Littlewood's formulas, see [2].

Let  $\delta_{i,j}$  denote the usual Kronecker delta function ( $= 1$  if  $i = j$  and  $0$  otherwise).

The following easy lemma is a key ingredient.

**Lemma 2.1** *We have*

$$SY_{(n-1)^n} = \det(s_{j,1^i} + (-y)^i \delta_{i,j})_{1 \leq i, j \leq n-1}, \quad (7)$$

$$OY_{(n+1)^n} = \det(s_{j+1,1^{i-1}} + (-y)^i \delta_{i,j})_{1 \leq i, j \leq n}. \quad (8)$$

*Proof.* Multiplying to the left of the matrix corresponding to the determinant  $SY_{(n-1)^n}$  by the triangular matrix  $[(-1)^{j-i} e_{j-i}]_{1 \leq i, j \leq n}$  with determinant 1, we get a matrix with first column  $[0, \dots, 0, 1]$ , the cofactor of its non-zero entry being the determinant

$$\det \left( \sum_{k=1}^{j-1} (-1)^{n-i+k+1} e_{n-i+k} h_{j-1-k} + y^{j-1} \delta_{i, n-j+1} \right)_{1 \leq i, j-1 \leq n-1}.$$

Reordering rows and columns, using (1), we obtain (7).

Similarly, the determinant  $OY_{(n+1)^n}$  equals

$$\det \left( \sum_{k=0}^j (-1)^k e_{i+k} h_{j-k} + (-y)^i \delta_{i,j} \right)_{1 \leq i, j \leq n} = \det (s_{j+1,1^{i-1}} + (-y)^i \delta_{i,j})_{1 \leq i, j \leq n}.$$

■

**Lemma 2.2** *We have*

$$SY_{(n-1)^n} = \sum_{\mu} (-y)^{(n^2-n-|\mu|)/2} s_{\mu}, \quad (9)$$

sum over all  $\mu$  such that  $F(\mu) = \{\alpha_1 - 1, \dots, \alpha_r - 1 | \alpha_1, \dots, \alpha_r\}$  with  $\alpha_1 \leq n - 1$ ;

$$OY_{(n+1)^n} = \sum_{\mu} (-y)^{(n^2+n-|\mu|)/2} s_{\mu}, \quad (10)$$

sum over all  $\mu$  such that  $F(\mu) = \{\alpha_1 + 1, \dots, \alpha_r + 1 | \alpha_1, \dots, \alpha_r\}$ , where  $\alpha_1 \leq n - 1$ .

*Proof.* Expanding the determinants (7) and (8) according to all possible choices of the occurrences of powers of  $y$ , we get, as cofactors, determinants of hook Schur functions, in which we recognize, thanks to formula (2), the two expressions in the lemma. ■

By multilinearity of the determinants (3) and (4), we obtain the following sums of  $2^{n-1}$  (resp.  $2^n$ ) Schur functions (notice that the indexing of a Schur function can be any integral vector, and not only a partition).

**Lemma 2.3** *We have*

$$SY_{(n-1)^n} = \sum_{\epsilon} y^{k(\epsilon)} s_{0, 2\epsilon_2, \dots, (2n-2)\epsilon_n}, \quad (11)$$

$$OY_{(n+1)^n} = \sum_{\epsilon} (-y)^{k(\epsilon)} s_{2\epsilon_1, \dots, 2n\epsilon_n} \quad (12)$$

sum over all  $\epsilon_i$  in  $\{0, 1\}$ , the power  $k(\epsilon)$  being such that the total degree be  $n(n-1)$  in the first formula,  $n(n+1)$  in the second,  $y$  being of degree 2.

The following lemma will allow us to extract the factor  $\prod_{i < j} (y - x_i x_j)$  or  $\prod_{i \leq j} (y - x_i x_j)$  from the determinants that we are considering.

**Lemma 2.4** *Let  $f, g$  be functions of one variable. We have*

$$\det (f(x_i)^{n-j} g(x_i)^{j-1})_{1 \leq i, j \leq n} = \prod_{i=1}^n f(x_i)^{n-1} \Delta \left( \frac{g(x_1)}{f(x_1)}, \dots, \frac{g(x_n)}{f(x_n)} \right). \quad (13)$$

*Proof.* This is the Vandermonde determinant in the variables  $g(x_i)/f(x_i)$ ,  $1 \leq i \leq n$ . ■

Combining Lemmas 2.1, 2.2, 2.4, we obtain the following two theorems.

**Theorem 2.5** *We have*

$$\begin{aligned} \prod_{1 \leq i < j \leq n} (y - x_i x_j) &= (-1)^{\binom{n}{2}} \det (s_{j, 1^i} + (-y)^i \delta_{i, j})_{1 \leq i, j \leq n-1} \\ &= y^{\binom{n}{2}} \sum_{\lambda} (-y)^{-|\lambda|/2} s_{\lambda} \end{aligned} \quad (14)$$

summed over all partitions  $F(\lambda) = \{\alpha_1 - 1, \dots, \alpha_r - 1 | \alpha_1, \dots, \alpha_r\}$  with  $\alpha_1 \leq n - 1$ .

**Theorem 2.6** *We have*

$$\begin{aligned} \prod_{1 \leq i < j \leq n} (y - x_i x_j) &= (-1)^{\binom{n+1}{2}} \det(s_{j+1, 1^{i-1}} + (-y)^i \delta_{i,j})_{1 \leq i, j \leq n} \\ &= y^{\binom{n+1}{2}} \sum_{\lambda} (-y)^{-|\lambda|/2} s_{\lambda} \end{aligned} \quad (15)$$

summed over all partitions  $F(\lambda) = \{\alpha_1 + 1, \dots, \alpha_r + 1 | \alpha_1, \dots, \alpha_r\}$  with  $\alpha_1 \leq n - 1$ .

*Proof.* As already used at the end of the introduction, take  $f(x) = x$ ,  $g(x) = y + x^2$  in (13). We have

$$g(x_j)/f(x_j) - g(x_i)/f(x_i) = \frac{x_i - x_j}{x_i x_j} (y - x_i x_j).$$

Lemma 2.4 specializes into

$$\det \left( x_i^{n-j} (y + x_i^2)^{j-1} \right)_{1 \leq i, j \leq n} = (-1)^{\binom{n}{2}} \Delta(\mathbb{X}) \prod_{i < j} (y - x_i x_j).$$

This leads to the following identity:

$$\Delta(\mathbb{X}) SY_{(n-1)^n} = M(\mathbb{X}, y) = (-1)^{\binom{n}{2}} \Delta(\mathbb{X}) \prod_{i < j} (y - x_i x_j).$$

Multiplying both sides of above equation by  $\prod_i (y - x_i^2)$ , one gets

$$(-1)^n \Delta(\mathbb{X}) OY_{(n+1)^n} = (-1)^n N(\mathbb{X}, y) = (-1)^{\binom{n}{2}} \Delta(\mathbb{X}) \prod_{i \leq j} (y - x_i x_j).$$

By Lemma 2.1 and Lemma 2.2, we complete the proof. ■

Setting  $y = 1$  in (14) and (15), we get Littlewood's formulas (5) and (6).

Let us now show that, having recourse to (3) and (4), we can obtain a determinantal expression for the product of the functions in the preceding two theorems by an arbitrary Schur function.

**Theorem 2.7** *Let  $n$  be an integer,  $\lambda$  be a partition of length  $\leq n$ . Then*

$$SY_{(n-1)^{n+\lambda}} = (-1)^{\binom{n}{2}} s_{\lambda} \prod_{1 \leq i < j \leq n} (y - x_i x_j)$$

and

$$OY_{(n+1)^{n+\lambda}} = (-1)^{\binom{n+1}{2}} s_{\lambda} \prod_{1 \leq i \leq j \leq n} (y - x_i x_j)$$

*Proof.* The function  $SY_{(n-1)^n+\lambda}$  is a weighted sum of skew Schur functions  $s_{\clubsuit/\mu}$  with  $\clubsuit = (n-1)^n + \lambda$  and  $F(\mu) = (\alpha_1 - 1, \dots, \alpha_r - 1 | \alpha_1, \dots, \alpha_r)$  with  $\alpha_1 \leq n-1$ . Interpreting a Schur function in terms of Young tableaux, we remark that the tableaux of shape  $\clubsuit/\mu$  in  $n$  letters are the concatenation of an arbitrary skew tableau of shape  $(n-1)^n/\mu$  and an arbitrary tableau of shape  $\lambda$ . Hence we have the factor  $s_\lambda$ .

For example, for  $n = 4$ , and a tableau of shape  $([3, 3, 3, 3] + [2, 2, 1, 1])/[2, 1, 1]$ , one has the decomposition

$$\begin{array}{|c|c|c|c|} \hline a & b & c & \alpha \\ \hline & d & e & \beta \\ \hline & f & g & \gamma & \delta \\ \hline & & h & \epsilon & \iota \\ \hline \end{array} \Leftrightarrow \left( \begin{array}{|c|c|c|} \hline a & b & c \\ \hline & d & e \\ \hline & f & g \\ \hline & & h \\ \hline \end{array} \& \begin{array}{|c|c|} \hline \alpha \\ \hline \beta \\ \hline \gamma & \delta \\ \hline \epsilon & \iota \\ \hline \end{array} \right).$$

The remaining factor corresponds to the case  $\lambda = 0$  which has been described in Theorem 2.5. This proves the first assertion of the theorem. The second part can be proved in the same manner with a summation over partitions contained in  $(n+1)^n$ . ■

For example, for  $n = 3$ , we have

$$\begin{aligned} SY_{332} &= \begin{vmatrix} h_3 & h_4 + yh_2 & h_5 + y^2h_1 \\ h_2 & h_3 + yh_1 & h_4 + y^2 \\ 1 & h_1 & h_2 \end{vmatrix} = s_{11} SY_{222} \\ &= -(x_3x_1 + x_3x_2 + x_2x_1)(y - x_2x_1)(y - x_3x_1)(y - x_3x_2). \end{aligned}$$

Notice that the theorem implies, for  $y = 1$ , a factorization property of symplectic and orthogonal Schur functions in  $n$  variables, indexed by partitions with parts  $\geq n-1$  or  $\geq n+1$  respectively. For more informations about symplectic and orthogonal characters, we refer to [1], and for relevant tableau considerations, to [7].

### 3 Remarks

Given a polynomial of degree 3, with roots  $x_1, x_2, x_3$ , the polynomial  $(y - x_1^2)(y - x_2^2)(y - x_3^2)$  can be interpreted as the resultant of  $(x^2 - y)$  and  $(x - x_1)(x - x_2)(x - x_3)$ . Using the determinantal expression of the resultant ascribed to Sylvester, one can therefore write

$$-(y - x_1^2)(y - x_2^2)(y - x_3^2) = \begin{vmatrix} 1 & 0 & -y & 0 & 0 \\ 0 & 1 & 0 & -y & 0 \\ 0 & 0 & 1 & 0 & -y \\ 1 & -e_1 & e_2 & -e_3 & 0 \\ 0 & 1 & -e_1 & e_2 & -e_3 \end{vmatrix}.$$

Farkas (1881) [5, p. 335] found that one could derive similar determinants for expressing the polynomial whose roots are  $x_i x_j : 1 \leq i < j \leq 3$ , or  $x_i x_j : 1 \leq$

$i \leq j \leq 3$ . Returning to this problem in 1927, Muir [6] was still puzzled by these determinants, finding no other proof than direct verification for degree 3. Farkas determinants are

$$-(y - x_1x_2)(y - x_1x_3)(y - x_2x_3) = \begin{vmatrix} 1 & 0 & 0 & 0 & y^2 \\ 0 & 1 & 0 & y & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & e_1 & e_2 & e_3 & 0 \\ 0 & 1 & e_1 & e_2 & e_3 \end{vmatrix}$$

and

$$\prod_{1 \leq i < j \leq 3} (y - x_i x_j) = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -y^3 \\ 0 & 1 & 0 & 0 & 0 & -y^2 & 0 \\ 0 & 0 & 1 & 0 & -y & 0 & 0 \\ 1 & e_1 & e_2 & e_3 & 0 & 0 & 0 \\ 0 & 1 & e_1 & e_2 & e_3 & 0 & 0 \\ 0 & 0 & 1 & e_1 & e_2 & e_3 & 0 \\ 0 & 0 & 0 & 1 & e_1 & e_2 & e_3 \end{vmatrix}.$$

However, if we expand the following determinant of order  $2n - 1$  along the first  $n$  rows,

$$\begin{vmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & y^{n-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & 0 \\ 0 & & 1 & 0 & y & & 0 & \vdots \\ 0 & & 0 & 1 & 0 & & 0 & 0 \\ 1 & \cdots & e_{n-2} & e_{n-1} & e_n & \cdots & 0 & 0 \\ 0 & \cdots & e_{n-3} & e_{n-2} & e_{n-1} & \ddots & 0 & 0 \\ \vdots & & & & & & \ddots & \vdots \\ 0 & \cdots & 1 & e_1 & e_2 & \cdots & e_{n-1} & e_n \end{vmatrix},$$

the cofactors are Schur functions, up to sign.

More precisely, we have  $2^{n-1}$  choices of non vanishing  $n \times n$  minors in the first  $n$  rows, which are equal to some power of  $y$ , up to sign. Writing the bottom part

$$\begin{vmatrix} e_0 & e_1 & e_2 & \cdots & e_{2n-4} & e_{2n-3} & e_{2n-2} \\ 0 & e_0 & e_1 & \cdots & e_{2n-5} & e_{2n-4} & e_{2n-3} \\ \vdots & & \ddots & \ddots & & & \ddots \end{vmatrix},$$

one recognizes that the cofactors are Schur functions (expressed in the  $e_i$ ) of indices  $[2\epsilon_1, 4\epsilon_2, \dots, (2n - 2)\epsilon_{n-1}]$ , with  $\epsilon_i \in \{0, 1\}$ . These are the indices occurring in Lemma 2.3, taking into account that the involution exchanging elementary and symmetric functions conjugate partitions.

Therefore, the above determinant equals  $(-1)^{\binom{n}{2}} \prod_{1 \leq i < j \leq n} (y - x_i x_j)$ .

Similarly, we obtain

$$(-1)^{\binom{n+1}{2}} \prod_{1 \leq i < j \leq n} (y - x_i x_j) = \begin{vmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & -y^{n-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & 0 \\ 0 & & 1 & 0 & -y & & 0 & \vdots \\ 1 & \cdots & e_{n-1} & e_n & 0 & \cdots & 0 & 0 \\ 0 & \cdots & e_{n-2} & e_{n-1} & e_n & \ddots & 0 & 0 \\ \vdots & & & & & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & e_1 & \cdots & e_{n-1} & e_n \end{vmatrix}.$$

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