Schubert and Macdonald polynomials:
A Parallel

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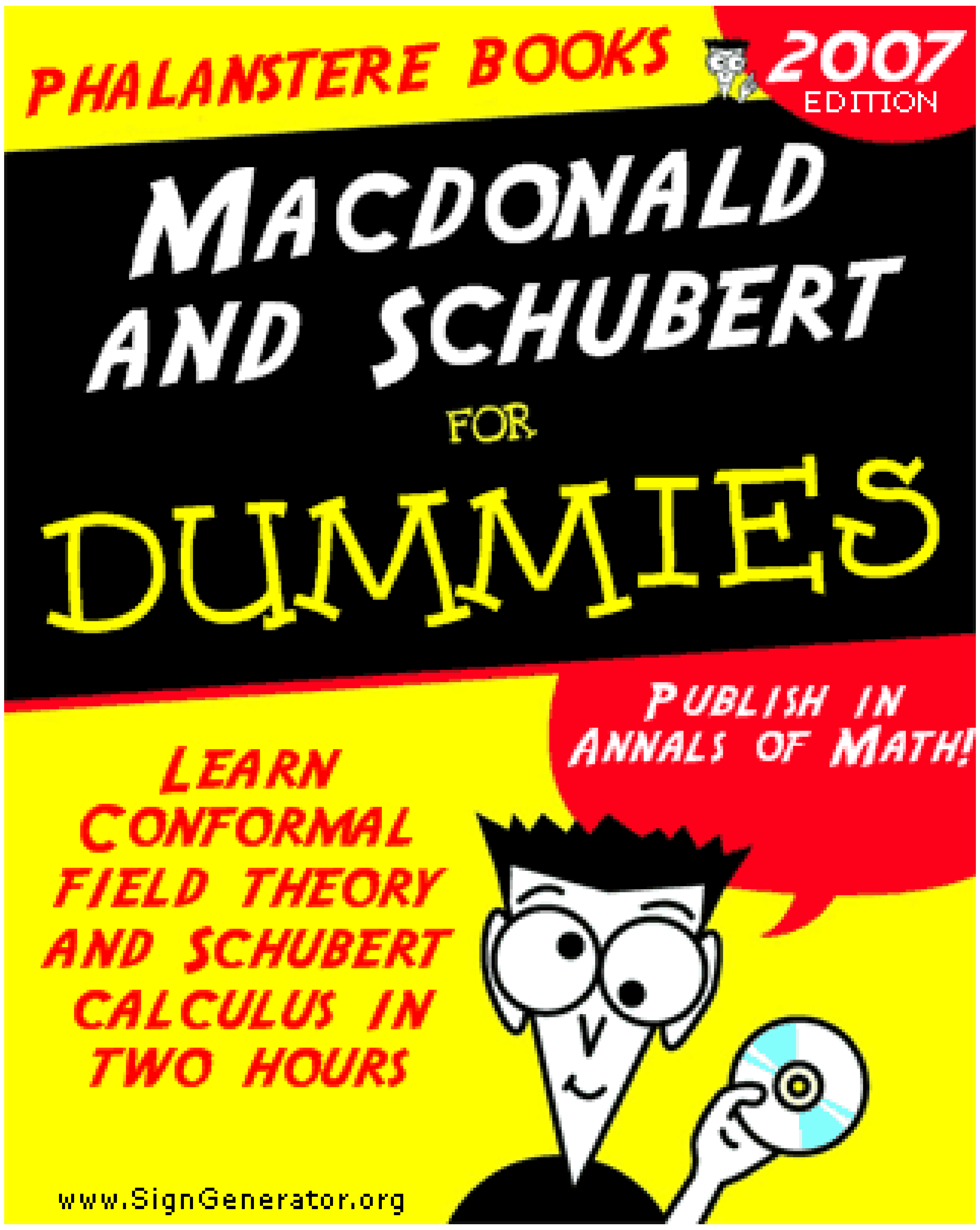
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Schubert and (non-symmetric) Macdonald polynomials are two linear bases of the ring of polynomials which can be characterized by vanishing conditions. These conditions are such that they entail an easy recursion, using essentially divided differences (plus an affine operation for Macdonald polynomials).

Having a linear basis of a ring is not enough. One must recover the multiplicative structure. I shall only describe multiplication by a single variable. This will for example give without any further work a combinatorial description of both Schubert and Macdonald polynomials.

Space: polynomials in $x_1, \ldots, x_n$, with basis monomials

$$\{x^v : v \in \mathbb{N}^n\}.$$

Two other bases, triangular, (but not for the same order):

Schubert $\{Y_v : v \in \mathbb{N}^n\}$ & Macdonald $\{M_v : v \in \mathbb{N}^n\}$.

Each v gives rise to two spectral vectors:

$\langle v \rangle$ (components are some extra variables y_j),

$\langle v \rangle$ (components are of the type $t^i q^j$).

Definition: Y_v and M_v are the only polynomials such that

$$Y_v(\langle u\rangle) = 0 \text{ and } M_v(\langle u\rangle) = 0,$$

for every $u : |u| \leq |v|,\ u \neq v$,

plus normalisation conditions

In the case of polynomials in 1 variable, it is very easy to use vanishing properties (or specialisations). Each point allows to decrease the degree of the unknown polynomial.

In fact, the reasoning of Galileo to determine the quadratic law which describes a ball falling from the Pisa tower has been somehow different:

Quando, dunque, osservo che una pietra, che discende dall'alto a partire dalla quiete, acquista via nuovi incrementi di velocità, perché non dovrei credere che tali aumenti avvengano secondo la pi semplice e pi ovvia proporzione? Ora, se consideriamo attentamente la cosa, non troveremo nessun aumento o incremento più semplice di quello che aumenta *sempre nel medesimo modo. Il che facilmente intenderemo considerando la stretta connessione tra tempo e moto: come infatti la equabilità e uniformità del moto si definisce e si concepisce sulla base della eguaglianza dei tempi e degli spazi (infatti chiamiamo equabile il moto, allorché in tempi eguali vengono percorsi spazi eguali), così, mediante una medesima suddivisione uniforme del tempo, possiamo concepire che gli incrementi di velocit avvengano con [altrettanta] semplicità; [lo possiamo] in quanto stabiliamo in astratto che risulti uniformemente e, nel medesimo modo, continuamente accelerato, quel moto che in tempi eguali, comunque presi, acquista eguali aumenti di velocità.*

In other words, if it is not the increment of space which is uniform, it must be the increment of speed.

Galileo's reasoning can be iterated to attain any polynomial law, corresponding to observations at equal intervals of time.

But what to do with comets, which are not likely to appear at regular intervals of time ?

Answer by Newton : normalize differences of positions by differences of times, i.e. use divided differences.

Thanks to them, Newton was able to write an interpolation formula for the position $f(t)$ of a comet at times $t_0, t_1, \ldots$:

$$f(t) = f(t_0) + f^{\partial}(t - t_0) + f^{\partial\partial}(t - t_0)(t - t_1) + \cdots$$

where the coefficients f^{∂}, $f^{\partial\partial}$ are the successive divided differences of the positions at time $t_0, t_1, t_2, \ldots$.

The Newton's polynomials 1, $(t - t_0)$, $(t - t_0)(t - t_1), \ldots$ are easy to build and can be characterized by their zeroes.

Schubert and Macdonald are not much more complicated, though involving several variables, when using an interpolation point of view !!
Method due to Sahi, Knop, Okounkov.

Divided differences-like operators: operators T_i operating on x_i, x_{i+1}, commuting with $\mathfrak{Sym}(x_i, x_{i+1})$. They are therefore defined by their action on $\{1, x_{i+1}\}$.

Here are 4 classical examples :

1	0	1	1	t
x_{i+1}	-1	0	$x_i+x_{i+1}-1$	x_i
	Schubert	*Demazure*	*Grothendieck*	*Macdonald*

This is linear algebra in dimension 2 !

Back to Schubert.

To define the spectral vector $\langle v \rangle$, we need a bijection between permutations and integral vectors, the code v of a permutation σ:

$$\mathfrak{S}_N \ni \sigma \Rightarrow v : \; v_i = \#\{j : j > i \,\&\, \sigma_i > \sigma_j\}$$

Inversion polynomial:

$$\Cap(v) := \prod_{i<j,\, \sigma_i > \sigma_j} (y_{\sigma_i} - y_{\sigma_j})$$

Spectral vector:

$$v \Rightarrow \sigma \Rightarrow \langle v \rangle = [y_{\sigma_1}, y_{\sigma_2}, y_{\sigma_3}, \ldots]$$

Schubert polynomials :

Polynomials Y_v, $v \in \mathbb{N}^n$, in $x_1, \ldots, x_n$, with coefficients in $y_1, y_2, \ldots$, uniquely determined by the conditions:

$$\begin{aligned} Y_v(\langle u \rangle) &= 0\,,\; u \neq v,\, |u| \leq |v| \\ Y_v(\langle v \rangle) &= \Cap(v)\,. \end{aligned}$$

Supposing known the Schubert polynomial Y_v, with $v : v_i > v_{i+1}$, then from the values $Y_v(\langle v\rangle) = \Cap(v)$, $Y_v(\langle vs_i\rangle) = 0$, one deduces that

$$(Y_v - (Y_v)^{s_i})(x_i - x_{i+1})^{-1}$$

is the Schubert polynomial Y_{vs_i}. Division by $(x_i - x_{i+1})$ has eliminated one inversion, which is the relation between $\Cap(v)$ and $\Cap(vs_i)$.

Starting point of the recursion: Dominant weights $v :\ v_1 \geq v_2 \geq \cdots \geq v_n$. Then

$$Y_v = \prod_{i=1}^{n} \prod_{j=1}^{v_i} (x_i - y_j)$$

One has to recover multiplication. The same notion will be used, for Schubert and Macdonald.

$v \in \mathbb{N}^n$ is a *successor* of u if $|v| = |u| + 1$ & $Y_u(\langle v \rangle) \neq 0$.

But this can be easily traced on the recursion $v \to vs_i$ on Schubert polynomials. The answer is that v is a successor of u iff the corresponding permutations are consecutive in the Bruhat order.

Now, if one starts with the unknown expansion

$$\big(x_i - \langle v \rangle_i\big) Y_v = \sum_u c_u^v Y_u$$

one sees that the LHS vanishes in all points $\langle u \rangle$: $|u| \leq |v|$. Therefore, the RHS is limited to $u:\ |u| = |v|{+}1$, and moreover all the successors of v and only them appear. In fact the descending recursion also shows the coefficients c_u^v are ±1.

One notices that in the case where i is the righmost such that $u_i > 0$, then there is only one positive term in the RHS. Rewriting the equation, this gives a transition formula :

$$Y_u = (x_k - y_j)Y_v + \sum\nolimits_w Y_w,$$

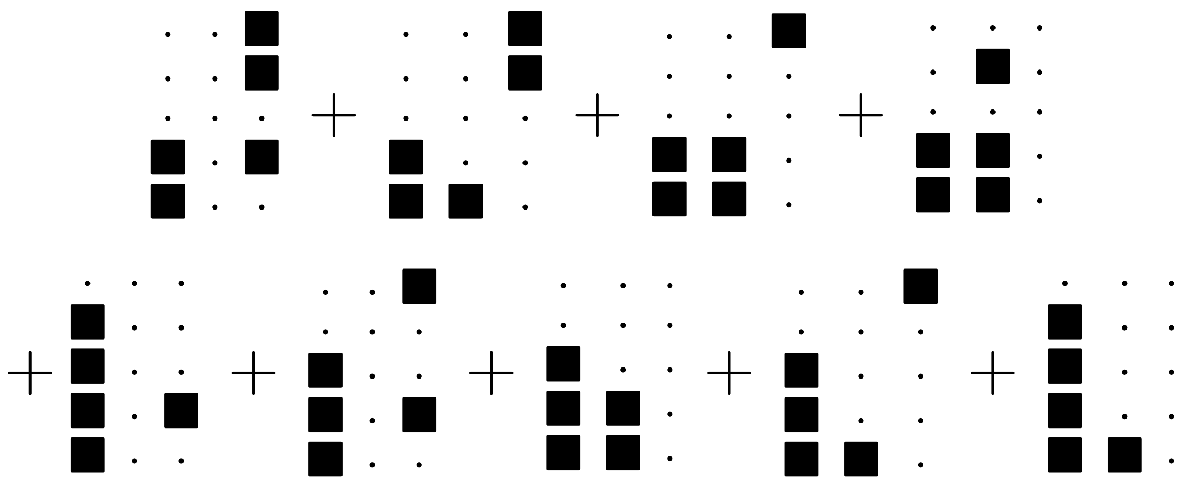
with some precise j and $u = [v_1, \ldots, v_i{+}1, 0, \ldots, 0]$.

Iterating, one gets a decomposition of Y_u into a sum of shifted monomials $\prod(x_i - y_j)$.

Here is an exemple of a sequence of transitions:

$$\begin{aligned}
Y_{203} &= (x_3 - y_5)Y_{202} + Y_{230} + Y_{401}\,, \\
Y_{230} &= (x_2 - y_4)Y_{220} + Y_{320}\,, \\
Y_{401} &= (x_3 - y_2)Y_{400} + Y_{410}\,, \cdots
\end{aligned}$$

Representing each factor $x_i - y_j$ by a box of coordinates i, j, one gets in final that Y_{203} is equal to



We adapt all the preceding constructions to Macdonald, passing from the symmetric group to the affine symmetric group.

First, it is convenient to use an infinite number of variables x_i, with periodicity $x_{i+rn} = q^r x_i$, q an extra parameter, apart from t. Similarly, $v \in \mathbb{N}^n$ is embedded into an infinite vector such that $v_{i+rn} = v_i + r$, $r \in \mathbb{Z}$. This gives us a translation of indices, which amounts to

$$\begin{aligned} [x_1, \ldots, x_{n-1}, x_n] &\xrightarrow{\tau} [x_2, \ldots, x_n, qx_1]\,, \\ [v_1, \ldots, v_{n-1}, v_n] &\xrightarrow{\tau} [v_2, \ldots, v_n, v_1 + 1]\,. \end{aligned}$$

We shall start with the spectral vector

$\langle 0 \dots 0 \rangle = [t^{n-1}, \dots, t^0]$, prolongated by $\langle 0 \dots 0 \rangle_{i+kn} = t^{n-i} q^k$, and define the general $\langle v \rangle$ by performing the same operations on v and $\langle v \rangle$, starting from $v = [0, \dots, 0]$.

Definition. M_v, $v \in \mathbb{N}^n$ is the only polynomial of degree $|v|$ such that

$$\begin{aligned} M_v(\langle u \rangle) &= 0\,,\ u \neq v,\ |u| \leq |v| \\ M_v &= x^v q^{-\sum_i \binom{v_i}{2}} + \text{ lower terms} \end{aligned}$$

Existence of the polynomials is provided by the study of $v \to v s_i$ and $v \to v\tau$. Indeed, let i be such that $v_i < v_{i+1}$.

One can write $M_v = f + x_{i+1}g$, with f, g symmetrical in x_i, x_{i+1}. Having that $M_v(\langle v\rangle s_i) = 0$, $M_v(\langle v\rangle) \neq 0$, one sees that one can find a (unique) constant c such that the image F of M_v by $T_i + c$, which is $F = tf + x_i g + c(f + x_{i+1}g)$, is such that the conditions are swapped: $F(\langle v\rangle s_i) \neq 0$, $F(\langle v\rangle) = 0$. Therefore, F is a good candidate to be M_{vs_i}, and indeed the other required vanishing conditions have not been perturbated by the action of T_i.

$$\mathit{Answer}\ :$$

$$\boxed{M_{vs_i} = M_v\left(T_i + \tfrac{t-1}{\langle v\rangle_{i+1}{\langle v\rangle_i}^{-1}-1}\right)}$$

The affine operation $v \to v\tau$ is not much more complicated to follow. Since $|v\tau| = |v|{+}1$, one needs an operation which increases the degree by 1. Let Φ be $\bar{\tau}\,(x_n - 1)$. Then $M_v\Phi$ inherits the vanishing points of M_v, because $\bar{\tau}$ is a shift of indices, and acquire new vanishing points because the factor $x_n - 1$ vanishes on all $u : u_n = 0$ (this implies $\langle u\rangle_n = 1$).

As for Schubert, one has to recover multiplication. Given any polynomial $f(x)$ of degree k, then $f(x)M_u$ belongs to the span of $M_v : |v| \le |u| + k$. Because of the vanishing conditions, no term $v : |v| \le |u|$ can appear, except u. One eliminates u by requiring $f(\langle u\rangle) = 0$. In particular,

$$\big(x_1 + \cdots + x_n - \langle u\rangle_1 - \cdots - \langle u\rangle_n\big)\, M_u = \sum_v c_u^v M_v$$

with v characterized by

$$|v| = |u| + 1 \ \&\ M_u(\langle v\rangle) \neq 0.$$

Let us call such v the successors of u.

Notice that the coefficients c_u^v are

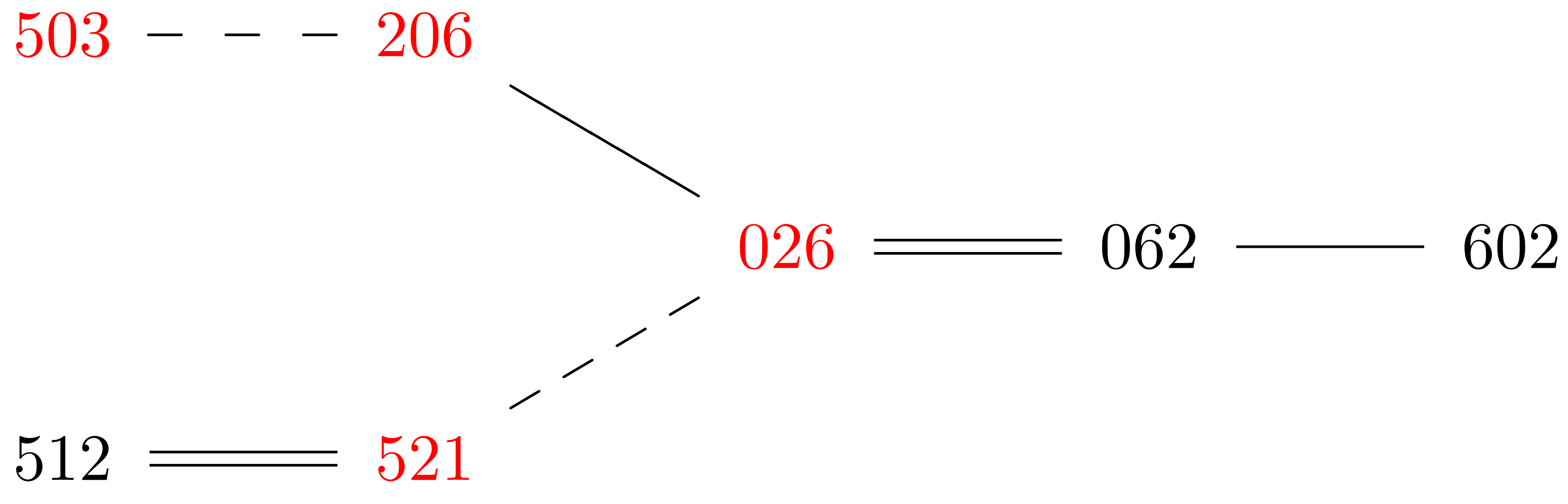
$$c_u^v = \big(\langle v\rangle_1 + \cdots + \langle v\rangle_n - \langle u\rangle_1 - \cdots - \langle u\rangle_n\big)\frac{M_u(\langle v\rangle)}{M_v(\langle v\rangle)}$$

Testing $M_u(\langle v\rangle) \neq 0$ is too costly, compared to the number of terms which survive.

For example, for $u = [5, 0, 2]$, the successors are much fewer than the 45 compositions of 8 in three parts, being

$$[5, 0, 3], [0, 2, 6], [5, 1, 2], [6, 0, 2], [2, 0, 6], [0, 6, 2], [5, 2, 1]$$

This list can be structured by connecting its elements by the simple transpositions s_0 − −, s_1 ——, s_2 ══ .



This graph describes all the images of [0,2,6] under a subword of $\{s_2 s_1,\ s_1 s_0,\ s_0 s_2\}$.

The equation defining the successors is compatible with T_i (one has multiplied by a factor which is symmetrical), one can follow the evolution of coefficients. One has to contributions, the first one coming from the non-zero specializations of the linear factors $x_i - \langle u \rangle_i$:

$$\eth(u,v) := \frac{1}{1-t} \widehat{\prod}_i \frac{1-t}{\langle v \rangle_i \langle u \rangle_i^{-1} - 1}$$

The second one is due to products of the type

$$\left(T_i + \frac{t-1}{\frac{\beta}{\alpha} - 1}\right)\left(T_i + \frac{t-1}{\frac{\alpha}{\beta} - 1}\right) = \frac{(t\beta - \alpha)(\beta - t\alpha)}{(\beta - \alpha)^2}$$

These last factors corresponds to the non-inversion of v relative to $u\tau$. Let $\mathbb{U}(u, v)$ be their products. Then one has the theorem:

Let $u, v \in \mathbb{N}^n$, $|v| = |u| + 1$*. Then* v *is a successor of* u *iff there exists a subword* $s_i \cdots s_j$ *of one of the decreasing words* $s_{n-1} \cdots s_1$, $s_{n-2} \cdots s_1 s_0, \ldots,$ $s_0 s_{n-1} \cdots s_2$*, such that* $v = u\tau s_i \cdots s_j$*. In that case,*

$$\frac{M_u(\langle v \rangle)}{M_v(\langle v \rangle)} = t^{1-n}\, \eth(u, v)\, \mathbb{U}(u, v)\,.$$

Knowing how to multiply by $x_1 + \cdots + x_n$ is sufficient to determine the multiplication by x_i :

$$M_u \left(\frac{x_i}{\langle u \rangle_i} - 1 \right) = \sum_v \frac{M_u(\langle v \rangle)}{M_v(\langle v \rangle)} \left(\frac{\langle v \rangle_i}{\langle u \rangle_i} - 1 \right) M_v \,,$$

summed over all the successors v of u such that $\langle v \rangle_i \neq \langle u \rangle_i$.

As for Schubert polynomials, choosing a particular i (here the leftmost position of the biggest value of v) allows to decompose M_v into smaller polynomials (transition formula) :

$$M_v = (x_i q^{-a} - t^b) M_u + \sum_w \frac{M_u(\langle w\rangle)}{M_w(\langle w\rangle)} \left(\frac{\langle w\rangle_i}{\langle u\rangle_i} - 1 \right) M_w \,,$$

with $u = [v_1, \ldots, v_{i-1}, v_i - 1, v_{i+1}, \ldots, v_n]$, $\langle u\rangle_i = q^a t^b$.

Iterating, one gets a decomposition of M_v into a sum of *shifted monomials* $\prod (x_i q^{-a} - t^b)$, the coefficients being products of $(t^i q^j - 1)^{\pm 1}$ (denoted $[i, j]$).

For example, writing $[i,j]$ for a factor $t^i q^j - 1$, starting with $v = [2,0,2]$, one has $u = [1,0,2]$, $\langle u \rangle = [tq, 1, t^2q^2]$ and the following sequence of transitions :

$$\begin{aligned}
M_{202} &= \big(x_1 q^{-1} - t\big)\, M_{102} + \frac{[1,0]}{[2,2]} M_{022}\,, \\
M_{022} &= \big(x_2 q^{-1} - t\big)\, M_{012} + \frac{tq[1,0]^2}{[1,1][2,1]} M_{121} + \frac{[1,0][3,1]}{[2,1]^2} M_{112}\,, \\
M_{121} &= \big(x_2 q^{-1} - t\big)\, M_{111} + \frac{[1,0]}{[2,1]} M_{112}\,, \\
M_{112} &= \big(x_3 q^{-1} - 1\big)\, M_{111}\,.
\end{aligned}$$

e.g. $M_{012}(\langle 112 \rangle)/M_{112}(\langle 112 \rangle) = [31]/[21]^2$ and $x_2 q^{-1} - t \rightarrow$ $1-t$ give the coefficient of M_{112} in the second transition.

In fact, the powers of t in the factors $(x_i q^{-a} - t^b)$ are determined by i, a, u, and therefore the shifted monomials can be represented by planar objects, taking i, a as coordinates.

In final, M_{202} is equal to

$$\begin{smallmatrix} \cdot & \cdot & \cdot \\ \blacksquare & \cdot & \cdot \\ \blacksquare & \blacksquare & \blacksquare \end{smallmatrix} \frac{[1,0]}{[1,1]} + \begin{smallmatrix} \cdot & \cdot & \cdot \\ \cdot & \blacksquare & \cdot \\ \blacksquare & \blacksquare & \blacksquare \end{smallmatrix} \frac{[1,0]^2}{[1,1][2,2]} + \begin{smallmatrix} \cdot & \cdot & \cdot \\ \blacksquare & \cdot & \blacksquare \\ \cdot & \blacksquare & \blacksquare \end{smallmatrix} \frac{[1,0]}{[1,1]}$$

$$+ \begin{smallmatrix} \cdot & \cdot & \cdot \\ \cdot & \blacksquare & \blacksquare \\ \cdot & \blacksquare & \blacksquare \end{smallmatrix} \frac{[1,0]}{[2,2]} + \begin{smallmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{smallmatrix} \frac{[1,0]^2}{[1,1][2,2]} + \begin{smallmatrix} \cdot & \cdot & \cdot \\ \blacksquare & \cdot & \blacksquare \\ \blacksquare & \cdot & \blacksquare \end{smallmatrix}$$

with leading term

$\begin{smallmatrix} \cdot & \cdot & \cdot \\ \blacksquare & \cdot & \blacksquare \\ \blacksquare & \cdot & \blacksquare \end{smallmatrix} = (x_1 q^{-1} - t)(x_1 - t)(x_3 q^{-1} - t)(x_3 - 1).$

As in the theory of symmetric functions, there is a principal specialization. Here it is the evaluation $\theta_z : x_1 = zt^{n-1}, \ldots, x_n = zt^0$. Since θ_z induces

$$\begin{tikzcd}
\bullet \arrow[r, "T_i"] \arrow[d, "\theta_z"'] & \bullet \arrow[d, "\theta_z"] \\
\bullet \arrow[r, "t"] & \bullet
\end{tikzcd}$$

(proof. Check on $1, x_{i+1}$!), one can easily follow the principal specialization on $M_v \to M_{vs_i}$.

More complicated to control $M_v \to M_{v\tau}$.

Fortunately, multiplication by x_1 involves $v \to v\tau$, and thus one can get $M_v \theta_z$ by recursion. One has again two types of factors, those coming from the affine steps, the others from the action of $T_1, \ldots, T_{n-1}$. More precisely, let

$$\Cap(v) := \prod_{i=1}^{n} \prod_{v_i > v_j}^{j=i+1\ldots\infty} \frac{t\langle v\rangle_i \langle v\rangle_j^{-1} - 1}{\langle v\rangle_i \langle v\rangle_j^{-1} - 1} \,,$$

and

$$\varphi_z(v) := \prod_{i=1}^{n} \left(\frac{z t^{n-1} q^1}{\langle v\rangle_i} - 1 \right) \cdots \left(\frac{z t^{n-1} q^{v_i}}{\langle v\rangle_i} - 1 \right) .$$

In final

$$M_v(zt^{n-1}, \dots, z) = \varphi_z(v)\, \Cap(v)\,.$$

In particular,

$$M_v(0, \dots, 0) = (-1)^{|v|}\, \Cap(v)\,,$$

and

$$E_v(zt^{n-1}, \dots, z) = z^{|v|} t^{\sum (i-1)\lambda_i} q^{-\sum \binom{v_i}{2}}\, \Cap(v)\,.$$

where E_v is the homogeneous non-symmetric Macdonald polynomial (=component of degree $|v|$ of M_v), and λ is the decreasing reordering of v.

To connect with the talk of Di-Francesco:

```
ACE> factor(subs(q=1/t^3,Macd([2,1,0,2,1,0])));
t^6 (x5 t - x4) (x6 t - x4) (x6 t - x5) (x3 t - x2)
    (x3 t - x1) (x2 t - x1)
```

The action of the Hecke algebra $\mathcal{H}_6$ on this element generates the space of link patterns, with its two distinguished bases : the Kazhdan-Lusztig basis, and the basis of non-symmetric Macdonald polynomials.

```
ACE> factor(subs(x1=1,x2=1,x3=1,
                 x4=1,q=1/t^3,Macd([1,1,0,0])));
             2                    2
           (t  + 3 t + 1) (t - 1)  / (t + 1)
```

and, of course, after the change of variables $T = -\sqrt{t} - 1/\sqrt{t}$, one recognizes $T + 1$ (up to a normalizing factor), which indeed specializes into $1 + 1$, which is the number of alternating sign-matrices of order 2.

Another connection :

```
ACE>factor(subs(q=1/t^3,subs(t=t^2,Macd([2,0,2,0]))));
  4            2             2          4       3
-t  (-x3 + x4 t ) (-x1 + x2 t ) (x4 x3 t  + x3 t  x4
              2       2             2          2
     - x4 x2 t  - x2 t  x3 - x1 x3 t  - x4 x1 t
     + x2 t x1 + x2 x1)
```

Next case, $n = 3$, to make it more evident :

```
ACE>factor(subs(q=1/t^3,subs(t=t^2,Macd([4,2,0,4,2,0]))));
```

= Gaudin-Izergin-Korepin determinant, in the variables $x_1, x_2, x_3, x_4, x_5, x_6$, multiplied by the t-Vandermonde in x_1, x_2, x_3 and the t-Vandermonde in x_4, x_5, x_6.

Lives in a 18564-dimensional space. Difficult to obtain without the vanishing properties !