Decomposing Cubic Graphs into Connected Subgraphs of Size Three

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International Computing and Combinatorics Conference (COCOON)

August 3rd, 2016

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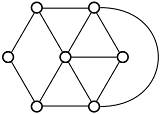

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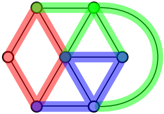

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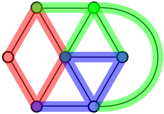


S-DECOMPOSITION

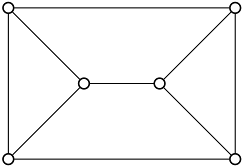
Input: a graph $G = (V, E)$, a set S of graphs.
Question: does G admit an S-decomposition?

S-DECOMPOSITION is NP-complete, even when S contains a single connected graph with at least three edges [Dor and Tarsi, 1997].

Graph decompositions for cubic graphs

We study the S-decomposition problem in the case where G is cubic and S is the set of all connected graphs on three edges.

Example



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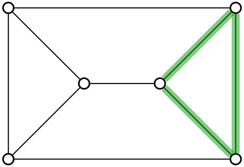

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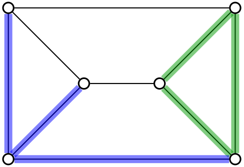

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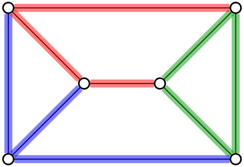

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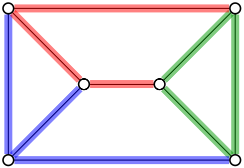

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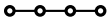
S'-DECOMPOSITION

Input: a cubic graph $G = (V, E)$, a non-empty set $S' \subseteq S$.
Question: does G admit a S'-decomposition?

Our contributions

Here is a summary of what is known about decomposing graphs using subsets of {  ,  ,  }:

Allowed subgraphs			Complexity according to graph class	
			cubic	arbitrary
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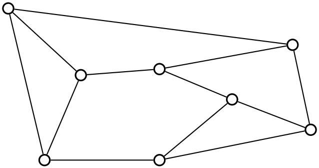
Decomposing cubic graphs without $K_{1,3}$'s

We need the following result:

Proposition ([Kotzig, 1957])

A cubic graph admits a P_4-decomposition if and only if it has a perfect matching.



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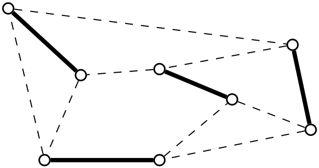

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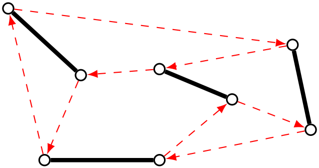

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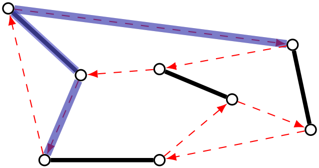

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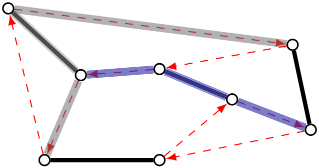

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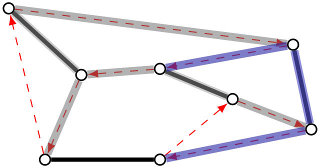

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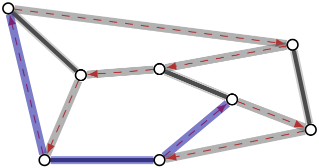

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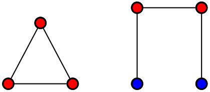

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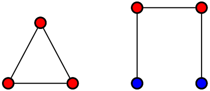

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Use counting argument $\Rightarrow$ no K_3 can be used.

Decomposing cubic graphs without P_4's

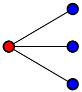
Let us start with $K_{1,3}$-decompositions:

Proposition

A cubic graph admits a $K_{1,3}$-decomposition if and only if it is bipartite.

Proof.

$\boxed{\Rightarrow}$



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$\Rightarrow$ Bipartition: centers – leaves
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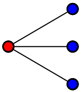

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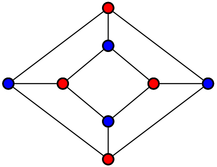

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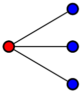

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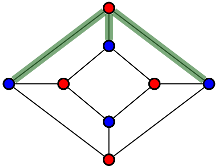

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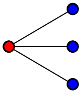

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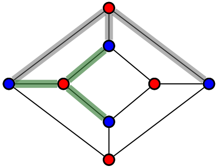

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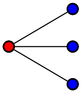

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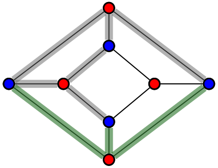

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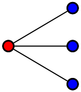

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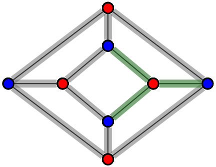

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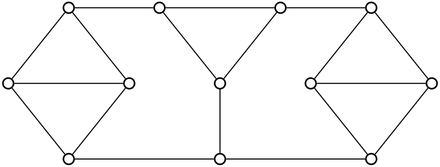
Decomposing cubic graphs without P_4's

What if we also allow K_3's?



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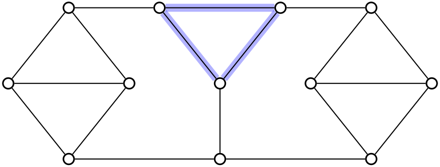

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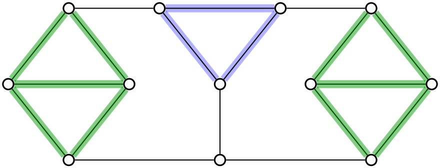

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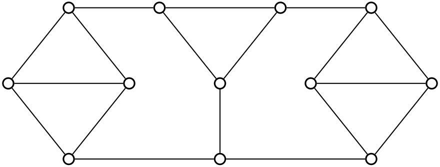

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Lemma

If a cubic graph G admits a $\{K_{1,3}, K_3\}$-decomposition D, then every isolated K_3 in G belongs to D.

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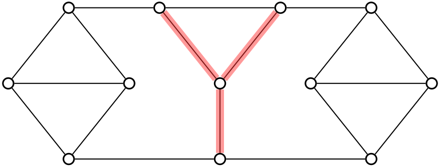

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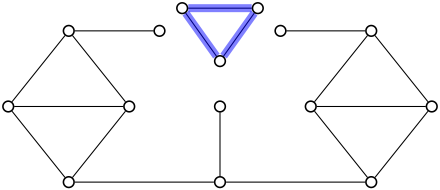

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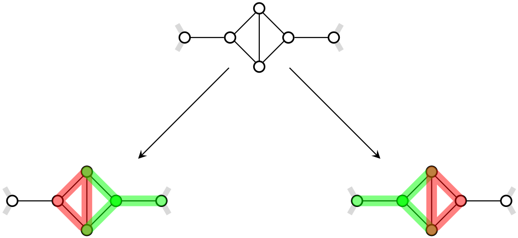
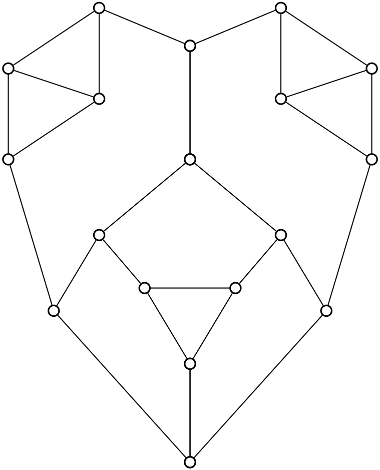
Decomposing cubic graphs without P_4's

If G also contains nonisolated K_3's, then we only have two choices to try:



Summary of algorithm



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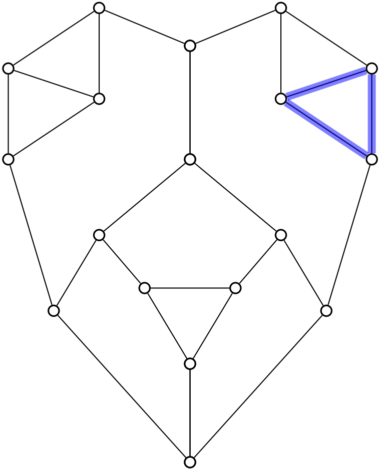

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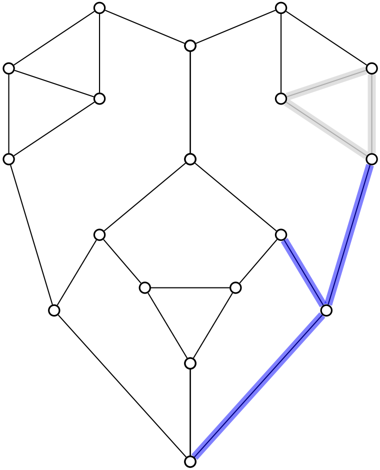

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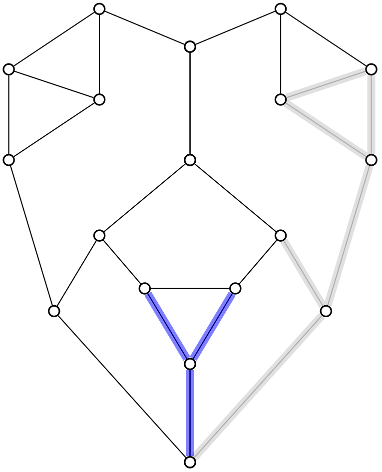

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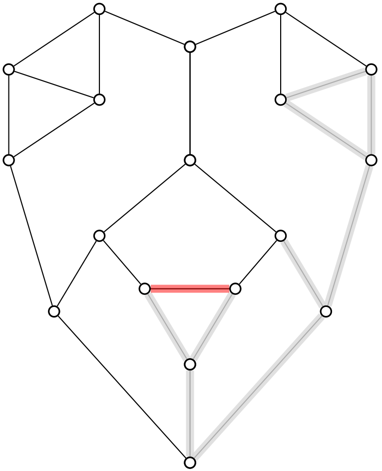

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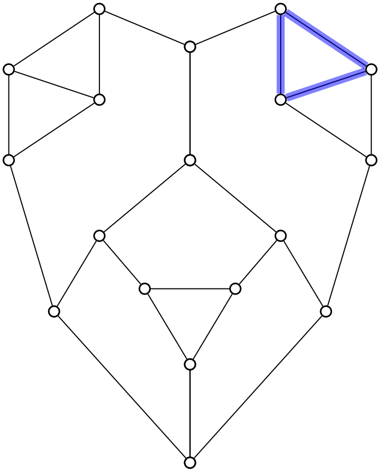

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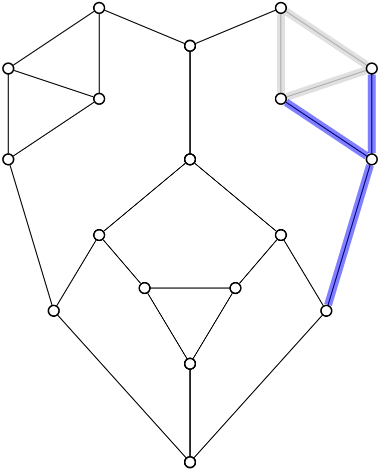

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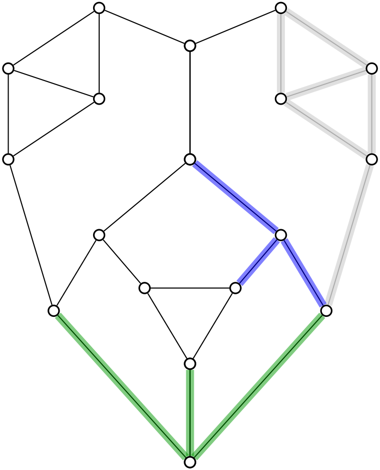

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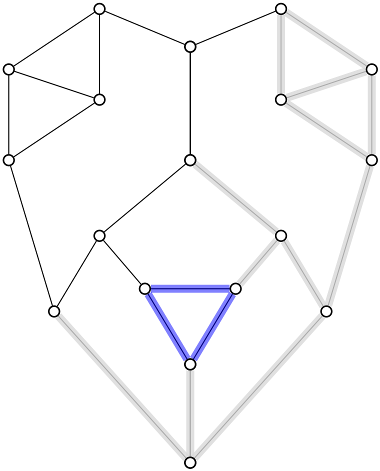
- If it fails, try the other starting K_3

Summary of algorithm



- Select a diamond, pick one K_3
- Follow degree-1,2 nodes:
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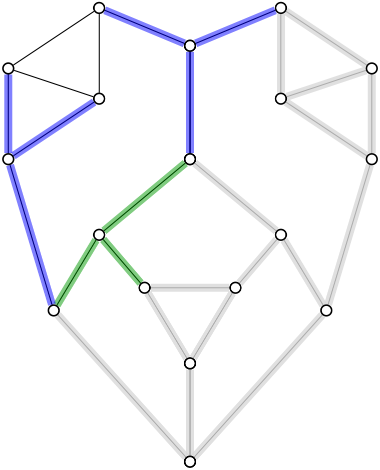

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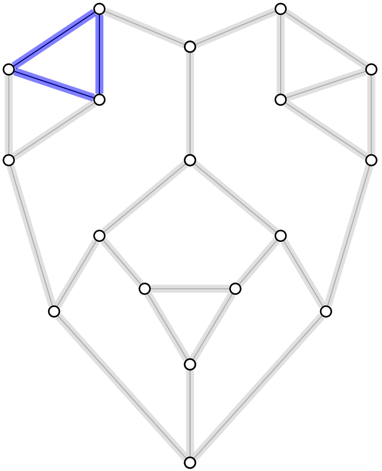

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- If it fails, try the other starting K_3
- Only one branching $\Rightarrow$ polynomial time algorithm

Hardness results

We now show that $\{K_{1,3},\ P_4\}$-DECOMPOSITION is NP-complete, using three reductions:

CUBIC PLANAR MONOTONE 1-IN-3 SATISFIABILITY

$\leq_P$ DEGREE-2,3 $\{K_{1,3},\ K_3,\ P_4\}$-DECOMPOSITION WITH MARKED EDGES

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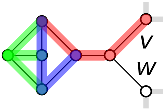
A similar approach can be used to show the NP-completeness of $\{K_{1,3},\ K_3,\ P_4\}$-DECOMPOSITION.

Hardness results 1/3: marked edges

The co-fish gadget



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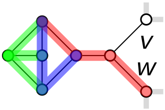

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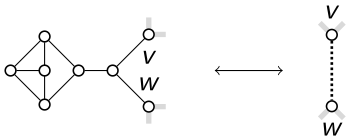

This gadget is equivalent to an edge . . . .

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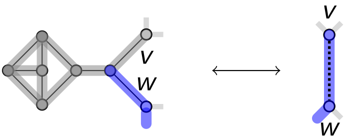
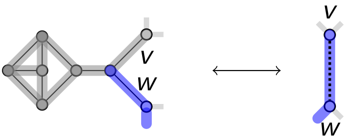


This gadget is equivalent to an edge that cannot be in the middle of a $P_4 \Rightarrow$ **marked** *edges*.

Hardness results 1/3: marked edges

The co-fish gadget



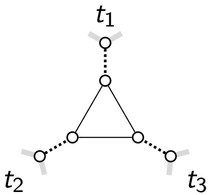
This gadget is equivalent to an edge that cannot be in the middle of a P_4 ⇒ **marked** *edges*.

$\{K_{1,3},\ K_3,\ P_4\}$-DECOMPOSITION WITH MARKED EDGES

Input: a cubic graph $G = (V, E)$ and a subset $M \subseteq E$ of edges.
Question: does G admit a $\{K_{1,3},\ K_3,\ P_4\}$-decomposition D such that no edge in M is the middle edge of a P_4 in D and such that every K_3 in D has either one or two edges in M?

Hardness results 2/3: leafless subcubic graphs

The net gadget



Hardness results 2/3: leafless subcubic graphs

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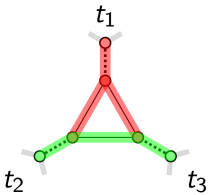

Hardness results 2/3: leafless subcubic graphs

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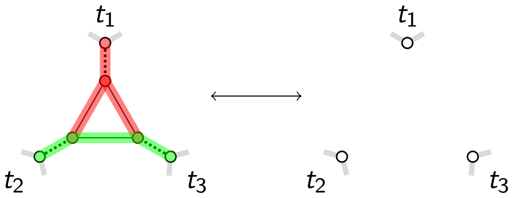
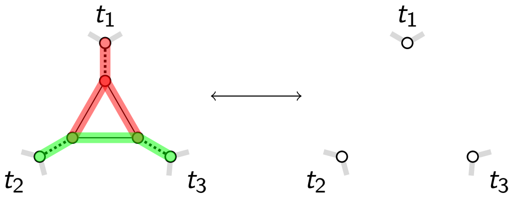


The net gadget is equivalent to 3 degree-2 nodes

Hardness results 2/3: leafless subcubic graphs

The net gadget



The net gadget is equivalent to 3 degree-2 nodes

We can restrict our attention to DEGREE-2,3 $\{K_{1,3}, K_3, P_4\}$-DECOMPOSITION WITH MARKED EDGES, a variant where the input graph contains vertices with degree 2 or 3.

Hardness results 3/3: satisfiability

CUBIC (PLANAR) MONOTONE 1-IN-3 SATISFIABILITY

Input: a Boolean formula $\phi = C_1 \wedge C_2 \wedge \cdots$ without negations; $|C_i| = 3$ for each i and each literal appears in exactly three clauses;

Question: is there an assignment of truth values $f : \Sigma \to \{\text{TRUE}, \text{FALSE}\}$ such that each clause of ϕ contains exactly one TRUE literal?

CUBIC PLANAR MONOTONE 1-IN-3 SATISFIABILITY

$\leq_P$ DEGREE-2,3 $\{K_{1,3}, K_3, P_4\}$-DECOMPOSITION WITH MARKED EDGES

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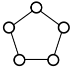
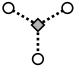
$\leq_P$ $\{K_{1,3}, P_4\}$-DECOMPOSITION

The reduction from CUBIC MONO-1-IN-3-SAT

Clause



Variable

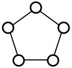


The reduction

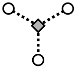
- Map clauses onto C_5's and variables onto marked $K_{1,3}$'s.

The reduction from CUBIC MONO-1-IN-3-SAT

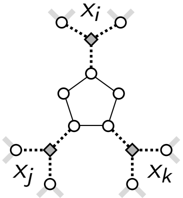
Clause



Variable



$C = x_i \vee x_j \vee x_k$



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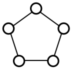

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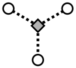

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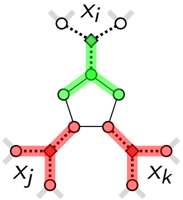
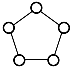


The reduction

- Map clauses onto C_5's and variables onto marked $K_{1,3}$'s.
- From assignments to decompositions: variables set to FALSE yield red $K_{1,3}$'s, those set to TRUE yield green $K_{1,3}$'s.

The reduction from CUBIC MONO-1-IN-3-SAT

Clause



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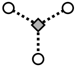

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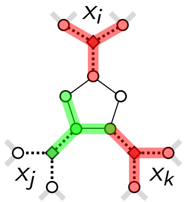

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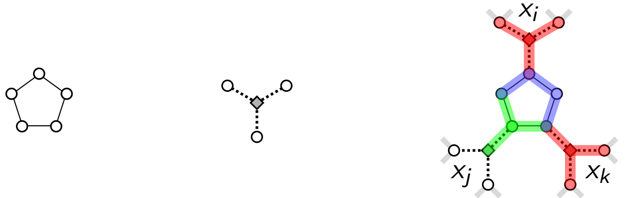
The reduction from CUBIC MONO-1-IN-3-SAT

Clause   Variable $C = x_i \vee x_j \vee x_k$



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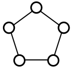

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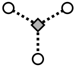

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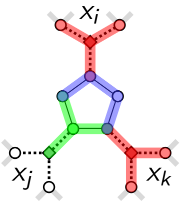

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- Map clauses onto C_5's and variables onto marked $K_{1,3}$'s.
- From assignments to decompositions: variables set to FALSE yield red $K_{1,3}$'s, those set to TRUE yield green $K_{1,3}$'s.
- From decompositions to assignments: show that a decomposable graph **must** conform to the above configuration $\Rightarrow$ truth assignment

Conclusions

- Future work:
  - hardness for *planar* cubic graphs?
  - complexity of those problems for subcubic graphs?
  - generalise positive results to k-regular graphs for $k > 3$;

Thank you!

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