A hierarchy of shift equivalent sofic shifts

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Abstract

We define new subclasses of the class of sofic shifts. These classes form an infinite hierarchy. For irreducible shifts, the lowest class is the class of almost finite type shifts introduced by B. Marcus. We give effective characterizations of these classes with the syntactic and the Krieger semigroups of the shifts. We prove that these classes define invariants for conjugacy and shift equivalence.

Keywords: Automata and formal languages, symbolic dynamics.

1 Introduction

Sofic shifts [25] are sets of bi-infinite labels in a labeled graph. If the graph can be chosen strongly connected, the sofic shift is said to be irreducible. An irreducible sofic shift has a unique (up to isomorphisms of automata) minimal deterministic presentation called its right Fischer cover. A particular subclass of sofic shifts is the class of shifts of finite type which are defined by a finite set of forbidden blocks. Two sofic shifts X and Y are conjugate if there is a bijective block map from X onto Y. It is an open question to decide whether two sofic shifts are conjugate, even in the particular case of irreducible shifts of finite type. There is a notion weaker than conjugacy, called shift equivalence (see [18, Section 7.3]). Therefore, invariants for shift equivalence are also invariants for conjugacy.

Almost finite type shifts have been introduced in [19] (see also [21]). They constitute a meaningful intermediate class above the class of shifts of finite type for several reasons. For instance, if $\tilde{X}$ is the shift presented by the reversed presentation of a shift X that has almost finite type, then X and $\tilde{X}$ are conjugate [8]. Almost finite type shifts are of practical interest in coding for constrained channels. Sliding block decoding theorems hold in the case of almost finite type constraints while they do not hold beyond this class [13].

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In this article, we first give a characterization of almost finite type shifts based on the syntactic semigroup of the shift. This semigroup S is the transition semigroup of the right Fischer cover of the irreducible sofic shift. The structure of a finite semigroup is determined by the Green's relations (denoted $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}, \mathcal{D}, \mathcal{J}$), see for instance [23]. We show that an irreducible sofic shift has almost finite type if and only if for any regular $\mathcal{H}$-class of S with image I and any $\mathcal{R}$-class of the $\mathcal{D}$-class of rank 1 with domain D, the intersection $D \cap I$ has at most one element. In general, the greatest cardinality of $D \cap I$, where I is an image and D is a domain as above, is called the degree of the shift. This enables the definition of a hierarchy of subclasses of irreducible sofic shifts with respect to this degree, where the lowest class (that with degree 1) is the class of almost finite type shifts. In particular, we prove that conjugate irreducible sofic shifts have the same degree. This degree is thus a conjugacy invariant.

We derive similar results for reducible shifts, where right Krieger covers play the role of right Fisher covers, and Krieger semigroups that of syntactic semigroups. Moreover, we prove that the degree is also a shift equivalence invariant in both cases.

The proof of this invariant uses Nasu's Classification Theorem for sofic shifts [22] that extends William's one for shifts of finite type. This theorem says that two irreducible sofic shifts X, Y are conjugate if and only if there is a sequence of symbolic adjacency matrices of right Fischer covers $A = A_0, A_1, \ldots, A_{l-1}, A_l = B$, such that A_{i-1} and A_i are elementary strong shift equivalent for $1 \leq i \leq l$, where A and B are the adjacency matrices of the right Fischer covers of X and Y, respectively. This means that, for each i, there are two symbolic matrices U_i and V_i such that, after recoding the alphabets of A_{i-1} and A_i, one has $A_{i-1} = U_iV_i$ and $A_i = V_iU_i$. A bipartite shift is associated in a natural way to a pair of elementary strong shift equivalent and irreducible sofic shifts [22]. By means of right Krieger covers, Nasu's Classification Theorem extends to reducible sofic shifts.

Another syntactic conjugacy invariant called the syntactic graph of the sofic shift was defined in [3, 4]. We give an example of two non-almost finite type shifts with different degrees (and therefore not conjugate) that have the same syntactic graph. In [12], N. Jonoska presented an invariant for reducible sofic shifts which is a lattice whose vertices represent the sub-synchronizing subshifts of the shift. In [24], K. Thomsen gives other invariants for sofic shifts as the derived shift spaces and the depth of the shift.

Basic definitions related to symbolic dynamics are given in Section 2.1. We refer to [18] or [16] for more details. See also [19], [18, Section 13.1], [21], [13],[7], [8], [26] and [10] about almost finite type shifts. Basic definitions and properties related to finite semigroups and their structure are given in Section 2.2. We refer to [23, Chapter 3] for a more comprehensive expository. Nasu's Classification Theorems are recalled in Section 2.3 and in Section 2.4. In Section 3.1, we define a hierarchy of irreducible sofic shifts. In Section 3.2, we extend the hierarchy to the case of reducible sofic shifts. In Section 4, we recall the definition of shift equivalence between sofic shifts and we prove that the hierarchy of sofic shifts is also invariant under shift equivalence. Finally, in Section 5, we consider the problem of characterizing classes of shifts (as the class of almost finite type shifts), by algebraic properties of the syntactic semigroup. Part of this paper was presented at the conference MFCS'04 [2].

2 Definitions and background

2.1 Almost finite type shifts and their presentations

Let $\mathcal{A}$ be a finite alphabet, i.e. a finite set of symbols. The shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $\sigma((a_i)_{i\in\mathbb{Z}}) = (a_{i+1})_{i\in\mathbb{Z}}$, for $(a_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$. If $\mathcal{A}^{\mathbb{Z}}$ is endowed with the product topology of the discrete topology on $\mathcal{A}$, a *shift* is a closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$.

If X is a shift of $\mathcal{A}^{\mathbb{Z}}$ and n a positive integer, the nth *higher power* of X is the shift of $(\mathcal{A}^n)^{\mathbb{Z}}$ defined by $X^n = \{(a_{in}, \ldots, a_{in+n-1})_{i\in\mathbb{Z}} \mid (a_i)_{i\in\mathbb{Z}} \in X\}$.

A finite *automaton* is a finite multigraph labeled by $\mathcal{A}$. It is denoted $\mathsf{A} = (Q, E)$, where Q is a finite set of states, and E a finite set of edges labeled by $\mathcal{A}$. It is equivalent to a *symbolic adjacency* $(Q \times Q)$*-matrix* A, where A_{pq} is the finite formal sum of the labels of all the edges from p to q. A *sofic shift* is the set of the labels of all the bi-infinite paths on a finite automaton. If A is a finite automaton, we denote by X_{A} the sofic shift defined by the automaton A. Several automata can define the same sofic shift. They are also called *presentations* or *covers* of the sofic shift. We will assume that all presentations are *essential*: all states have at least one outgoing edge and one incoming edge. An automaton is *deterministic* if for any given state and any given symbol, there is at most one outgoing edge labeled by this given symbol. An automaton is *left closing with delay* D if whenever two paths of length $D + 1$ end at the same state and have the same label, then they have the same final edge. An automaton is *left closing* if it is left-closing with some delay $D \geq 0$. A sofic shift is *irreducible* if it has a presentation with a strongly connected graph. Irreducible sofic shifts have a unique (up to isomorphisms of automata) *minimal deterministic presentation*, that is a deterministic presentation having the fewest states among all deterministic presentations of the shift. This presentation is called the *right Fischer cover* of the shift.

An irreducible sofic shift has *almost finite type* (AFT) if it has a deterministic and left-closing presentation. The class of almost finite type shifts was introduced by B. Marcus in [19], see also [21] and [18, Section 13.1].

Let $\mathsf{A} = (Q, E)$ be a deterministic automaton labeled by $\mathcal{A}$. The *square* of A is the deterministic automaton $(Q \times Q, F)$ where $(p, q) \xrightarrow{a} (p', q') \in F$ if and only if $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q' \in E$. A *diagonal* state of the square of A is a state (p, p), with $p \in Q$.

An almost finite type shift is an irreducible shift whose right Fischer cover is left-closing. Thus the square of its right Fischer cover has no strongly connected component with at least one edge containing a non-diagonal state and admitting a path going from this component to a diagonal state (see for instance [19], [1]). Checking whether an irreducible sofic shift has almost finite type can thus be done in a quadratic time in the number of states of the right Fischer cover of the shift.

2.2 The syntactic semigroup of an irreducible sofic shift

In this section, we recall the definition and the structure of the syntactic semigroup of an irreducible sofic shift [4].

Let $\mathsf{A} = (Q, E)$ be a finite deterministic (essential) automaton on the alphabet $\mathcal{A}$. Each finite word w of $\mathcal{A}^*$ defines a partial function from Q to Q. This function sends the state p to the state q, if w is the label of a path from p to q. The semigroup generated by all these functions is called the *transition semigroup* of the automaton. When X_{A} is not the full shift, the semigroup has a null element, denoted 0, which corresponds to words which are not factors of any bi-infinite word of X_{A}. The *syntactic semigroup* of an irreducible sofic shift is defined as the transition semigroup of its right Fischer cover.

Given a semigroup S, we denote by S^1 the following monoid: if S is a monoid, $S^1 = S$. If S is not a monoid, $S^1 = S \cup \{1\}$ together with the law $*$ defined by $x * y = xy$ if $x, y \in S$ and $1 * x = x * 1 = x$ for every $x \in S^1$.

We recall the *Green's relations* $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{J}$, which are fundamental equivalence relations defined in a semigroup S. They are defined as follows. Let $x, y \in S$,

$$\begin{aligned}
x\mathcal{R}y &\Leftrightarrow xS^1 = yS^1, \\
x\mathcal{L}y &\Leftrightarrow S^1x = S^1y, \\
x\mathcal{J}y &\Leftrightarrow S^1xS^1 = S^1yS^1, \\
x\mathcal{H}y &\Leftrightarrow x\mathcal{R}y \text{ and } x\mathcal{L}y.
\end{aligned}$$

Another relation $\mathcal{D}$ is defined by:

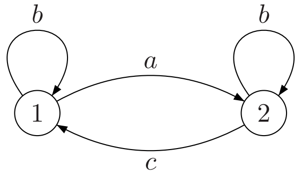
$$x\mathcal{D}y \Leftrightarrow \exists z \in S\ x\mathcal{R}z \text{ and } z\mathcal{L}y.$$

In a finite semigroup $\mathcal{J} = \mathcal{D}$.

An $\mathcal{R}$*-class* is an equivalence class for a relation $\mathcal{R}$ (similar notations hold for the other Green's relations). An *idempotent* is an element $e \in S$ such that $ee = e$. A *regular* class is a class containing an idempotent. In a regular $\mathcal{D}$-class, any $\mathcal{H}$-class containing an idempotent is a maximal subgroup of the semigroup. Moreover, two regular $\mathcal{H}$-classes contained in a same $\mathcal{D}$-class are isomorphic (as groups), see for instance [23, Chapter 3 Proposition 1.8].

We say that two elements $x, y \in S$ are *conjugate* if there are elements $u, v \in S^1$ such that $x = uv$ and $y = vu$.

Let S be a transition semigroup of an automaton $\mathsf{A} = (Q, E)$ and $x \in S$. The *rank* of x is the cardinal of the image of x as a partial function from Q to Q. The *kernel* of x is the partition induced by the equivalence relation $\sim$ over the domain of x where $p \sim q$ if and only p, q have the same image under x. We describe the so called "egg-box" pictures with the sofic shifts of Figure 1 and Figure 2 which have almost finite type and not almost finite type, respectively.

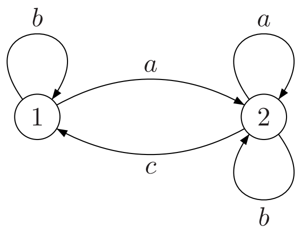


	12
1/2	$^*b$

	1	2
1	$^*ac$	a
2	c	$^*ca$

	–
–	^{*0}

Figure 1: An irreducible sofic shift which has almost finite type. Its syntactic semigroup is represented on the right part of the figure. It is composed of three $\mathcal{D}$-classes of rank 2, 1 and 0, respectively, represented by the above tables from left to right. Each square in a table represents an $\mathcal{H}$-class. Each row represents an $\mathcal{R}$-class and each column an $\mathcal{L}$-class. The common kernel of the elements in each row is written on the left of each row. The common image of the elements in each column is written above each column. Idempotents are marked with the symbol $*$. Each $\mathcal{D}$-class of this semigroup is regular.



	12
1/2	$^*b$

	1	2
12	$^*ac$	a
2	c	$^*ca$

	–
–	^{*0}

Figure 2: An irreducible sofic shift which has not almost finite type. Indeed, there are two distinct left-infinite paths labelled $...bbbbbba$ ending at state 2. Also in this case, each $\mathcal{D}$-class is regular.

The syntactic semigroup of an irreducible sofic shift has a unique $\mathcal{D}$-class of rank 1 which is regular (see for instance [5] or [6], and also [12]). Moreover, if u is a nonnull element of this semigroup, there is a word w such that uw has rank 1.

2.3 Nasu's Classification Theorem for sofic shifts

In this section, we recall Nasu's Classification Theorem for sofic shifts [22] (see also [18, Theorem 7.2.12]), which extends William's Classification Theorem for shifts of finite type (see [18, Theorem 7.2.7]).

Let $X \subset \mathcal{A}^{\mathbb{Z}}$, $Y \subset \mathcal{B}^{\mathbb{Z}}$ be two shifts and m, a be nonnegative integers. A map $\phi : X \to Y$ is a (m,a)-*block map* (or (m,a)-*factor map*) if there is a map $\delta : \mathcal{A}^{m+a+1} \to \mathcal{B}$ such that $\phi((a_i)_{i\in\mathbb{Z}}) = (b_i)_{i\in\mathbb{Z}}$ where $\delta(a_{i-m} \ldots a_{i-1}$ $a_i\ a_{i+1} \ldots a_{i+a}) = b_i$. A *block map* is a (m,a)-*block map* for some nonnegative integers m, a (respectively called its *memory* and *anticipation*). The well known theorem of Curtis, Hedlund, and Lyndon [11] asserts that continuous maps commuting with the shift map σ, are exactly block maps. A *conjugacy* is a one-to-one and onto block map (then, being a shift compact, also its inverse is a block map).

Having almost finite type is a property of shifts which is invariant under conjugacy [19].

We now define the notion of strong shift equivalence between two symbolic adjacency matrices. A *symbolic monomial* is a formal product of several non-commuting variables. In particular, the entries of a symbolic adjacency matrix are integral combinations of symbolic monomials. In this category of matrices, we write $A \leftrightarrow B$ if $A = B$ modulo a bijection of their underlying symbolic monomials. For example we can write

$$\begin{bmatrix} 0 & b \\ b+c & 2a \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & a \\ a+d & 2e \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & bb \\ bb+cc & 2cb \end{bmatrix}.$$

Two symbolic matrices A and B with entries in $\mathcal{A}$ and $\mathcal{B}$ respectively, are *elementary strong shift equivalent* if there is a pair symbolic matrices (U, V) with entries in disjoint alphabets $\mathcal{U}$ and $\mathcal{V}$ respectively, such that $A \leftrightarrow UV$ and $B \leftrightarrow VU$.

Another equivalent formulation of this definition is the following. Let $\mathcal{A}$ and $\mathcal{B}$ be two finite alphabets. We denote by $\mathcal{AB}$ the set of words ab with $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Let f be a map from $\mathcal{A}$ to $\mathcal{B}$. The map f is extended to a morphism from finite formal sums of elements of $\mathcal{A}$ to finite formal sums of elements of $\mathcal{B}$. We say that f *transforms* a symbolic $(Q \times Q)$-matrix A into a symbolic $(Q \times Q)$-matrix B if $B_{pq} = f(A_{pq})$ for each $p, q \in Q$. Two symbolic matrices A and B with entries in $\mathcal{A}$ and $\mathcal{B}$ respectively, are elementary strong shift equivalent if there is a pair of symbolic matrices (U, V) with entries in disjoint alphabets $\mathcal{U}$ and $\mathcal{V}$ respectively, such that there is a one-to-one map from $\mathcal{A}$ to $\mathcal{UV}$ which transforms A into UV, and there is a one-to-one map from $\mathcal{B}$ to $\mathcal{VU}$ which transforms B into VU.

Two symbolic adjacency matrices A and B are *strong shift equivalent within right Fischer covers* if there is a sequence of symbolic adjacency matrices of right Fischer covers

$$A = A_0, A_1, \ldots, A_{l-1}, A_l = B$$

such that for $1 \leq i \leq l$ the matrices A_{i-1} and A_i are elementary strong shift equivalent.

THEOREM 1 (NASU) *Let X and Y be irreducible sofic shifts and let A and B be the symbolic adjacency matrices of the right Fischer covers of X and Y, respectively. Then X and Y are conjugate if and only if A and B are strong shift equivalent within right Fischer covers.*

Let us consider the two irreducible sofic shifts X and Y defined by the right Fischer covers in Figure 3. The symbolic adjacency matrices of these automata

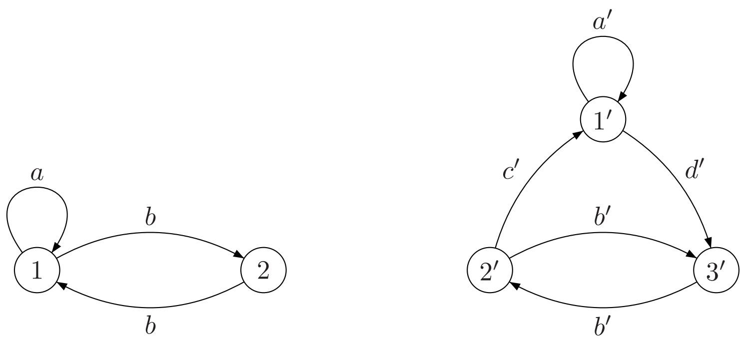


Figure 3: Two conjugate shifts X and Y.

are respectively

$$A = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix}, \quad B = \begin{bmatrix} a' & 0 & d' \\ c' & 0 & b' \\ 0 & b' & 0 \end{bmatrix}.$$

Then A and B are elementary strong shift equivalent with

$$U = \begin{bmatrix} u_1 & 0 & u_2 \\ 0 & u_2 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & 0 \\ v_2 & 0 \\ 0 & v_2 \end{bmatrix}.$$

Indeed,

$$UV = \begin{bmatrix} u_1v_1 & u_2v_2 \\ u_2v_2 & 0 \end{bmatrix}, \quad VU = \begin{bmatrix} v_1u_1 & 0 & v_1u_2 \\ v_2u_1 & 0 & v_2u_2 \\ 0 & v_2u_2 & 0 \end{bmatrix}.$$

The one-to-one maps from $\mathcal{A} = \{a, b\}$ to $\mathcal{UV}$ and from $\mathcal{B} = \{a', b', c', d'\}$ to $\mathcal{VU}$ are described in the tables below.

a	u_1v_1
b	u_2v_2

,

a'	v_1u_1
b'	v_2u_2
c'	v_2u_1
d'	v_1u_2

.

An elementary strong shift equivalence between $\mathsf{A} = (Q, E)$ and $\mathsf{B} = (Q', E')$, enables the construction of an irreducible sofic shift Z on the alphabet $\mathcal{U} \cup \mathcal{V}$ as follows. The sofic shift Z is defined by the automaton $\mathsf{C} = (Q \cup Q', F)$, where the symbolic adjacency matrix C of C is

$$\begin{array}{cc} & \begin{array}{cc} Q & Q' \end{array} \\ \begin{array}{c} Q \\ Q' \end{array} & \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} \end{array}.$$

The shift Z is called the *bipartite shift* defined by U, V (see Figure 4). An edge of C labeled by $\mathcal{U}$ goes from a state in Q to a state in Q'. An edge of C labeled by $\mathcal{V}$ goes from a state in Q' to a state in Q.

Remark that the second higher power of Z is the disjoint union of X and Y since

$$C^2 = \begin{bmatrix} UV & 0 \\ 0 & VU \end{bmatrix}.$$

Remark that C is a right Fischer cover (i.e. is minimal).

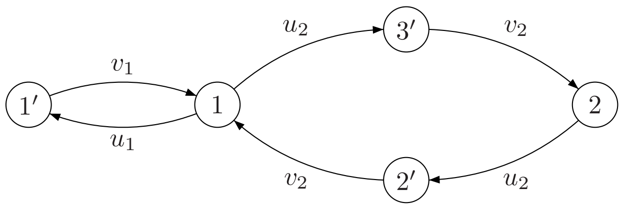


Figure 4: The bipartite shift Z of the shifts X and Y in Figure 3.

2.4 The reducible case

Nasu's Classification Theorem holds for reducible sofic shifts by the use of right Krieger covers instead of right Fischer covers [22]. This enables the extension of our result to the case of reducible sofic shifts.

Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a shift. We define

$$X_- = \{x_- \mid x \in X\},$$

where for $x \in \mathcal{A}^{\mathbb{Z}}$, we denote by x_- the left infinite word $\ldots x_{-2}x_{-1}x_0$. The equivalence relation $\approx$ on X_- is defined as follows. Let $x, y \in X_-$,

$$x \approx y \Leftrightarrow \{u \in \mathcal{A}^+ \mid xu \in X_-\} = \{u \in \mathcal{A}^+ \mid yu \in X_-\}.$$

If X is a sofic shift, the equivalence classes of $\approx$ are finitely many [17]. The *right Krieger cover of* X is defined as the automaton labeled by $\mathcal{A}$ in which the states are the $\approx$-classes $[x]$ with $x \in X_-$, and there is a an edge labeled a from $[x]$ to $[xa]$ if $xa \in X_-$.

The right Krieger cover of X is a deterministic presentation of X and it is unique up to isomorphisms of automata [17].

The analogous of Theorem 1 for (possibly) reducible sofic shifts is the following (see [22, Theorem 3.3]).

THEOREM 2 (NASU) *Let X and Y be sofic shifts and let A and B be the symbolic adjacency matrices of the right Krieger covers of X and Y, respectively. Then X and Y are conjugate if and only if A and B are strong shift equivalent within right Krieger covers.*

Hence we can define the *Krieger semigroup* of a shift as the transition semigroup of its right Krieger cover. Note that the right Krieger cover of a shift is essential.

An effective procedure to construct the right Krieger cover of a sofic shift is described in [22]. First, one constructs the (unique) minimal deterministic automaton with one initial state recognizing the language of finite blocks of the shift. Next, one erases all the states which are not the end of any left-infinite path. This automaton turns out to be the right Krieger cover of the shift. For instance, the right Krieger cover of the first shift in Figure 3, is illustrated in Figure 5.

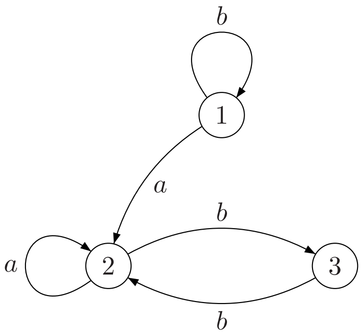


Figure 5: The right Krieger cover of the shift X described in Figure 3. Notice that, although the shift X is irreducible, the right Fisher cover of X does not coincide with its right Krieger cover.

3 A hierarchy of sofic shifts

In this section, we define a hierarchy of sofic shifts. We will distinguish between irreducible and reducible shifts.

3.1 The irreducible case

By means of right Fischer covers, we define a hierarchy of irreducible sofic shifts.

First, we give a syntactic characterization of almost finite type shifts. Recall that an almost finite type shift is irreducible by definition.

PROPOSITION 3 *Let X be an irreducible sofic shift and S its syntactic semigroup. Then X has almost finite type if and only if for any regular $\mathcal{H}$-class of S with image I and any $\mathcal{R}$-class of the $\mathcal{D}$-class of rank* 1 *with domain D, the intersection $D \cap I$ has at most one element.*

PROOF Let us assume that X has not almost finite type. Let A be the right Fischer cover of X. Then there are two states $p \neq q$ and a state r, two words u, v and two paths on A labeled uv as follows.

$$p \xrightarrow{u} p \xrightarrow{v} r$$
$$q \xrightarrow{u} q \xrightarrow{v} r$$

There is a positive integer n that u^n is a nonnull idempotent in S. Let I be the image of u^n. It contains p and q. Let w be a word of rank 1 in S. Since A has a strongly connected graph, there is a word w' such that the domain of $w'w$ contains the state r. The word $vw'w$ has rank 1 and its domain contains p and q. Thus the intersection of the domain of $vw'w$ and I contains at least two elements.

Conversely, let us assume that there is in S a regular $\mathcal{H}$-class H with image I and an $\mathcal{R}$-class R of the $\mathcal{D}$-class of rank 1 with domain D, such that $D \cap I$ has at least two elements. Let e be an idempotent of H. Then e induces the identity map on its image I. Let $p \neq q \in D \cap I$, and $v \in R$. Then there is a state r and two paths on A as follows

$$p \xrightarrow{e} p \xrightarrow{v} r$$
$$q \xrightarrow{e} q \xrightarrow{v} r$$

It follows that X has not almost finite type. $\square$

For instance, the shift presented in Figure 2 has not almost finite type since it has a regular $\mathcal{H}$-class of rank 2 (containing the idempotent b) whose image is $\{1, 2\}$. This image intersects the domain of $\{ac, a\}$ (which is a $\mathcal{R}$-class contained in the $\mathcal{D}$-class of rank 1), with $\{1, 2\}$ as intersection.

We now introduce the following classification of irreducible sofic shifts.

DEFINITION 4 An irreducible sofic shift is *d-non-left closing* if its syntactic semigroup has a regular $\mathcal{H}$-class with image I and a $\mathcal{R}$-class of the $\mathcal{D}$-class of rank 1 with domain D, such that $D \cap I$ has d elements.

DEFINITION 5 An irreducible sofic shift has *degree d* if it is d-non-left closing with $d \geq 0$ and not d'-non-left closing for any $d' > d$.

Notice that the degree of an irreducible sofic shift is always nonnull.

The following proposition states that the class of irreducible sofic shifts with degree d is invariant under conjugacy. In this classification, the class of almost finite type shifts is the class of irreducible sofic shifts with degree 1. Hence we recover the known fact that having almost finite type is a conjugacy invariant [18].

PROPOSITION 6 *Let X and Y be two conjugate irreducible sofic shifts and let d be a positive integer. If X is d-non-left closing, then Y is d-non-left closing.*

Before proving Proposition 6, we recall some results from [4] about the syntactic semigroup of a bipartite shift. Let X (respectively Y) be an irreducible sofic shift whose symbolic adjacency matrices of its right Fischer cover is a $(Q \times Q)$-matrix (respectively $(Q' \times Q')$-matrix) denoted by A (respectively by B). We assume that A and B are elementary strong shift equivalent through a pair of matrices (U, V). The corresponding alphabets are denoted $\mathcal{A}$, $\mathcal{B}$, $\mathcal{U}$, and $\mathcal{V}$ as before. We denote by f a one-to-one map from $\mathcal{A}$ to $\mathcal{UV}$ which transforms A into UV and by g a one-to-one map from $\mathcal{B}$ to $\mathcal{VU}$ which transforms B into VU. Let Z be the bipartite irreducible sofic shift associated to U, V. We denote by S (respectively T, R) the syntactic semigroup of X (respectively Y, Z).

Let $w \in R$. If w is nonnull, the bipartite nature of Z implies that w is a function from $Q \cup Q'$ to $Q \cup Q'$ whose domain is included either in Q or in Q', and whose image is included either in Q or in Q'. If $w \neq 0$ with a domain included in P and an image included in P', we say that w has the *type* (P, P'). Remark that w has type (Q, Q) if and only if $w \neq 0$ and $w \in (f(\mathcal{A}))^*$, and w has type (Q', Q') if and only if $w \neq 0$ and $w \in (g(\mathcal{B}))^*$. Elements of R in a same nonnull $\mathcal{H}$-class have the same type.

Let $w = a_1 \dots a_n$ be an element of S, we define the element $f(w)$ as $f(a_1) \dots f(a_n)$. Note that this definition is consistent since if $a_1 \dots a_n = a'_1 \dots a'_m$ in S, then $f(a_1) \dots f(a_n) = f(a'_1) \dots f(a'_m)$ in R. Similarly we define an element $g(w)$ for any element w of T.

Conversely, let w be an element of R belonging to $f(\mathcal{A})^*$ ($\subset (\mathcal{UV})^*$). Then $w = f(a_1) \dots f(a_n)$, with $a_i \in \mathcal{A}$. We define $f^{-1}(w)$ as $a_1 \dots a_n$. Similarly we define $g^{-1}(w)$. Again these definitions and notations are consistent. Thus f is a semigroup isomorphism from S to the subsemigroup of R of transition functions defined by the words in $(f(\mathcal{A}))^*$. Notice that $f(0) = 0$ if $0 \in S$. Analogously, g is a semigroup isomorphism from T to the subsemigroup of R of transition functions defined by the words in $(g(\mathcal{B}))^*$.

We now prove Proposition 6.

PROOF[of Proposition 6] By Nasu's Theorem [22] we can assume, without loss of generality, that the symbolic adjacency matrices of the right Fischer covers of X and Y are elementary strong shift equivalent. We define the bipartite shift Z as above. We denote by S, T and R the syntactic semigroups of X, Y and Z respectively.

Let us assume that X is d-non-left closing. Thus S has a regular $\mathcal{H}$-class H with image I and an $\mathcal{R}$-class of the $\mathcal{D}$-class of rank 1 with domain D, such that

$D \cap I$ has d elements. Let e be an idempotent of H. It induces the identity map on its image I.

The element $f(e)$ is an idempotent element of type (Q, Q) in R. Let $u_1 v_1 \ldots u_n v_n \in (\mathcal{U}\mathcal{V})^*$ such that $f(e) = u_1 v_1 \ldots u_n v_n$. We define an element $\bar{e}$ as $\bar{e} = v_1 \ldots u_n v_n u_1$. Thus $f(e)u_1 = u_1 \bar{e}$ in R. Remark that $\bar{e}$ depends on the choice of the word $u_1 v_1 \ldots u_n v_n$ representing $f(e)$ in R. Notice that $\bar{e}$ and $f(e)$ are conjugate. Indeed, if $w = v_1 \ldots u_n v_n$, then $f(e) = u_1 w$ and $\bar{e} = w u_1$. Hence, $\bar{e}^3 = w u_1 w u_1 w u_1 = w f(e)^2 u_1 = w f(e) u_1 = w u_1 w u_1 = \bar{e}^2$. We have $\bar{e}^2 \neq 0$ since $f(e) \neq 0$ and $f(e) = f(e)^2 = f(e)^3 = u_1 \bar{e}^2 w$. Thus $\bar{e}^2$ is an idempotent of R of type (Q', Q').

Let $I \cap D = \{x_1, \ldots, x_d\}$. Then there is a word $z \in \mathcal{A}^*$ of rank 1, a state $x \in Q$ and a path on the right Fischer cover of X labeled z from any state in $I \cap D$ to x. Moreover, there are a letter $u \in \mathcal{U}$, a state $y \in Q'$ and an edge $x \xrightarrow{u} y$ in the right Fischer cover of Z. It follows that there are paths as follows in the right Fischer cover of Z.

$$\begin{aligned}
&x_1 \xrightarrow{u_1} y_1 \xrightarrow{w} x_1 \xrightarrow{u_1} y_1 \xrightarrow{w} x_1 \xrightarrow{f(z)} x \xrightarrow{u} y \\
&x_2 \xrightarrow{u_1} y_2 \xrightarrow{w} x_2 \xrightarrow{u_1} y_2 \xrightarrow{w} x_2 \xrightarrow{f(z)} x \xrightarrow{u} y \\
&\vdots \\
&x_d \xrightarrow{u_1} y_d \xrightarrow{w} x_d \xrightarrow{u_1} y_d \xrightarrow{w} x_d \xrightarrow{f(z)} x \xrightarrow{u} y
\end{aligned}$$

The states y_i, for $1 \leq i \leq d$, belong to Q'. Since the states x_i are distinct, also the states y_i are distinct. Indeed, let us assume for instance that $y_1 = y_2$. Then $x_1 = x_2$ by considering the paths labeled w from y_i to x_i for $i = 1, 2$. Thus, in the right Fischer cover of Z there are the following paths, for $1 \leq i \leq d$.

$$y_i \xrightarrow{(wu_1)^2} y_i \xrightarrow{wf(z)u} y$$

Since $\bar{e} = wu_1$ and $wf(z)u$ are contained in $(g(\mathcal{B}))^*$, the elements $e' = g^{-1}(\bar{e}^2) = g^{-1}((wu_1)^2)$ and $w' = g^{-1}(wf(z)u)$ are in T. Hence the following paths are in the right Fischer cover of Y, for $1 \leq i \leq d$.

$$y_i \xrightarrow{e'} y_i \xrightarrow{w'} y$$

Notice that e' is an idempotent of T. Moreover, the element $wf(z)u$ of R has rank 1 because $f(z)$ has rank 1. This implies that w' is an element of rank 1 in T. Hence the domain D' of the $\mathcal{R}$-class of w' and the image I' of the idempotent e', contain $\{y_1, \ldots, y_d\}$. We now prove that $D' \cap I'$ is exactly the set $\{y_1, \ldots, y_d\}$.

Suppose that $\bar{y} \in D' \cap I'$. Hence the following path is in the right Fischer cover of Y.

$$\bar{y} \xrightarrow{e'} \bar{y} \xrightarrow{w'} y$$

Thus, in the right Fischer cover of Z there is the following path.

$$\bar{y} \xrightarrow{(wu_1)^2} \bar{y} \xrightarrow{wf(z)u} y$$

Let $\bar{x}$ be the final state of the path labelled by w and starting at $\bar{y}$. It follows that a path of the kind

$$\bar{y} \xrightarrow{w} \bar{x} \xrightarrow{u_1} . \xrightarrow{w} . \xrightarrow{u_1} \bar{y} \xrightarrow{w} \bar{x} \xrightarrow{f(z)} x$$

is in the right Fischer cover of Z (recall that $f(z)$ has rank 1). Being $\bar{x}$ in the image of $u_1wu_1w = f(e)^2 = f(e)$, we have that $\bar{x}$ is also in the image I of e. Moreover, $\bar{x}$ is in the domain of $f(z)$ and hence it is also in the domain D of z. This implies that $\bar{x}$ is one of the elements x_i and hence $\bar{y}$ is the corresponding y_i. Thus the cardinality of $D' \cap I'$ is d. $\square$

We get the following corollary.

Corollary 7 *Let X be an irreducible sofic shift. Then its degree is invariant under conjugacy. Moreover, the increasing sequence $(d_1, d_2, \ldots, d_n)$ of positive integers such that X is d_i-non-left closing (where d_n is the degree of X), is invariant under conjugacy.*

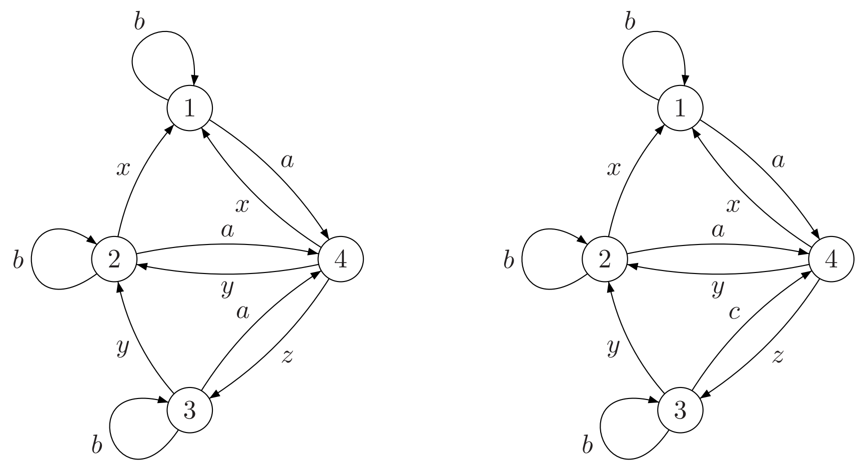


Figure 6: The right Fischer covers of two non conjugate sofic shifts X (on the left) and Y (on the right), with $\mathcal{A} = \{a, b, x, y, z\}$ and $\mathcal{B} = \{a, b, c, x, y, z\}$. The shift X has degree 3 while the shift Y has degree 2.

We show in Figure 6 an example of two sofic shifts X and Y, where X has not almost finite type with degree 3, and Y has not almost finite type with degree 2. Thus these two shifts are not conjugate since their degrees are different. Remark that they have the same syntactic graph, which is another conjugacy invariant defined and described in [4]. In Figure 7 we give an example of two sofic shifts with the same degree, for which the increasing sequences defined in Corollary 7, are different. Hence these two shifts are not conjugate even if they have the same degree.

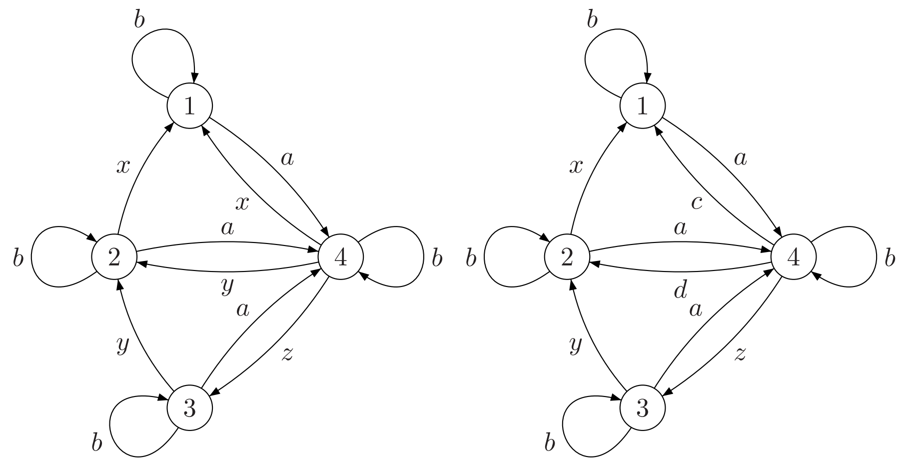


Figure 7: The right Fischer covers of two non conjugate sofic shifts X (on the left) and Y (on the right), with $\mathcal{A} = \{a, b, x, y, z\}$ and $\mathcal{B} = \{a, b, c, d, x, y, z\}$. Both the shifts X and Y have degree 3, but X is also 2-non-left closing and Y is not. Hence the sequence defined in Corollary 7 is $(1, 2, 3)$ for X and $(1, 3)$ for Y.

There are irreducible sofic shifts with degree d for every $d > 1$. For instance, consider the right Fischer cover $\mathsf{A} = (\{1, 2, \ldots, d, d+1\}, E)$ on the alphabet $\mathcal{A} = \{a, b, c\}$, where the set of edges is $E = \{i \xrightarrow{b} i,\ i \xrightarrow{a} d+1 \mid i \neq d+1\} \bigcup \{i \xrightarrow{c} i-1 \mid 2 \leq i \leq d\} \bigcup \{d+1 \xrightarrow{c} d\}$. This right Fischer cover has degree d.

3.2 The reducible case

Nasu's Classification Theorem for reducible sofic shifts (Theorem 2) enables the extension of the results of the previous section to reducible sofic shifts.

We first give the definition of d-non-left closure in the case of reducible shifts.Notice that the Krieger semigroup may have more than one $\mathcal{D}$-class of rank 1.

Definition 8 A reducible sofic shift is *d-non-left closing* if its Krieger semigroup has a regular $\mathcal{H}$-class with image I and a $\mathcal{R}$-class of a $\mathcal{D}$-class of rank 1 with domain D, such that $D \cap I$ has d elements.

With Definition 8, the degree of a reducible sofic shift is defined as in Definition 5. Also in this case the degree of a reducible sofic shift is always nonnull. As pointed out in Section 2.4, the syntactic semigroup and the Krieger semigroup of an irreducible shift are in general different. Hence, the correspondent degrees may also be different.

Again with Definition 8, Proposition 6 and Corollary 7 still hold for reducible shifts. Indeed, the proof of Proposition 6 does not need the irreducibility of the shift but the essentiality of its right Krieger cover. Hence it holds if we use the Krieger semigroup instead of the syntactic semigroup.

Since the right Krieger cover is also defined for irreducible sofic shifts, Corollary 7 used in this new framework defines an invariant which does not coincide in general with the one in the irreducible case.

We give in Figures 8 and 9 an example of two reducible shifts which are not conjugate since they have different degree.

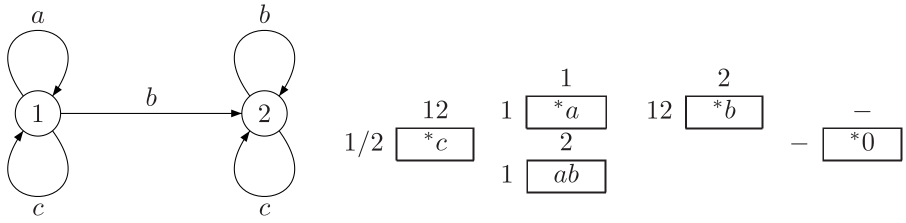


Figure 8: A reducible sofic shift which has degree 2. Its Krieger semigroup is represented on the right part of the figure. It has a regular $\mathcal{D}$-class of rank 2 containing an $\mathcal{H}$-class whose image is $I = \{1, 2\}$. This image is the domain of the $\mathcal{D}$-class of rank 1 containing b.

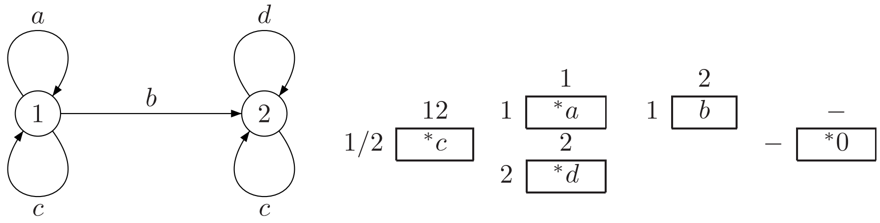


Figure 9: A reducible sofic shift which has degree 1. Its Krieger semigroup is represented on the right part of the figure. It has a regular $\mathcal{H}$-class of rank 2 whose image is $I = \{1, 2\}$, but the domain of any $\mathcal{D}$-class of rank 1 is a singleton.

4 An invariant for shift equivalence

We now prove that the invariants for strong shift equivalence defined in Corollary 7, are also invariants of shift equivalence. Although shift equivalence is decidable, even for sofic shifts [14], the algorithm is quite intricate. Hence invariants for shift equivalence of sofic shifts, which is equivalent to eventual conjugacy, may be useful. Most known conjugacy invariants are also invariants for shift equivalence.

Two symbolic adjacency matrices A and B with entries in $\mathcal{A}$ and $\mathcal{B}$ respectively, are *shift equivalent with lag l*, where l is a positive integer, if there is a pair of symbolic adjacency matrices (U, V) with entries in disjoint alphabets $\mathcal{U}$

and $\mathcal{V}$ respectively, such that (see [9])

$$A^l \leftrightarrow UV, \qquad B^l \leftrightarrow VU,$$
$$AU \leftrightarrow UB, \qquad VA \leftrightarrow BV.$$

Two matrices are *shift equivalent* is there is a positive integer l such that they are shift equivalent with lag l. Strong shift equivalence implies shift equivalence but the converse is false [15].

In the following proposition we extend Proposition 6 to the case of shift equivalence.

Proposition 9 *Let X and Y be two irreducible sofic shifts that are shift equivalent and let d be a positive integer. If X is d-non-left closing, then Y is d-non-left closing.*

Proof Let A and B be the symbolic adjacency matrix of the right Fischer cover of X and Y, respectively. Suppose that A and B are shift equivalent with lag l. Being A^l elementary strong shift equivalent to B^l, by Proposition 6 we have that A^l is d-non-left closing if and only if B^l is d-non-left closing.

Hence it suffices to prove that each symbolic adjacency matrix A is d-non-left closing if and only if A^l is d-non-left closing.

Let S and R be the syntactic semigroup of A and A^l, respectively. First, notice that the words representing elements of R are words labelled in $\mathcal{A}^l$. Thus R is isomorphic to a subsemigroup of S and a Green's relation in R is still a Green's relation in S. Thus, it can be easily seen that if A^l is d-non-left closing then A is d-non-left closing.

For the converse, let us assume that A is d-non-left closing. Thus S has a regular $\mathcal{H}$-class H with image I and a $\mathcal{R}$-class of the $\mathcal{D}$-class of rank 1 with domain D, such that $D \cap I$ has d elements. Let e be an idempotent of H. Since $e^l = e$, the idempotent e is also an idempotent of R. It follows that e can be represented by a word in $(\mathcal{A}^l)^*$. Let H' be the $\mathcal{H}$-class in R containing e. Note that the image of H' is still I.

Let s be an element of rank 1 and domain D in S. Let $\{p\}$ be its image. Let u be a word in $\mathcal{A}^*$ representing s. Let v be the label of a path starting at p such that l divides $|u| + |v|$. The $\mathcal{R}$-class of R containing the element uv has rank 1 and domain D. This implies that A^l is d-non-left closing. $\square$

Remark With Definition 8, Proposition 9 still holds for reducible shifts. Indeed, we only need to use the Krieger semigroup instead of the syntactic semigroup of the shift.

5 Links with semigroup theory

The above propositions have links with some known results in the theory of varieties of semigroups.

A finite biprefix code (see for instance [6]) defines an almost finite type shift in a natural way. It is known from [20] that, if X is a finite biprefix code and S is the syntactic semigroup of X^+, $e\,S\,e$ defines a semigroup of partial injective transformations for any idempotent e. Margolis [20] also showed that every semigroup of partial injective transformations divides a semigroup of partial injective transformations which is the syntactic semigroup of a finite biprefix code.

An equivalent formulation of Proposition 3 is the following.

PROPOSITION 10 *Let X be an irreducible sofic shift and let S be its syntactic semigroup. Then X has almost finite type if and only if for any idempotent $e \in S$, the semigroup $e\,S\,e$ is a semigroup of partial one-to-one transformations.*

Thus, when X has almost finite type, the semigroup $e\,S\,e$ is, for any idempotent e, a subsemigroup of an inverse semigroup. This implies that the semigroup S belongs to the variety of semigroups T such that for each idempotent e, the semigroup $e\,T\,e$ is in the variety generated by inverse semigroups. We do not know whether this condition is sufficient to guarantee that X has almost finite type.

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