Conjugacy of automata

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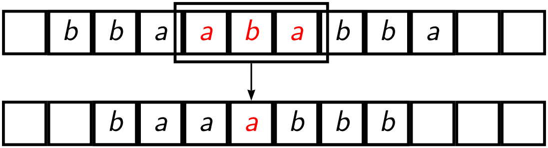
Conjugacy of symbolic dynamical shifts

Subshift

A set $X_{\mathcal{F}}$ of bi-infinite sequences of symbols over a finite alphabet avoiding a set of finite blocks $\mathcal{F}$.

Conjugacy between two subshifts

A bi-continuous bijection commuting with the shift transformation $(\sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}})$.
Equivalently, a one-to-one and onto block map.



Conjugacy of shifts of finite type

Sofic shift

The set of labels of bi-infinite paths of a finite automaton.

Shift of finite type

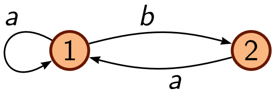
The set $X_{\mathcal{F}}$ of bi-infinite sequences avoiding a finite set of finite blocks $\mathcal{F}$.

Edge shift

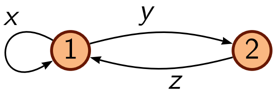
The set of labels of bi-infinite paths of a finite automaton whose labels are distinct. A shift of finite type is conjugate to an edge shift.

Examples

Shift of finite type $\mathcal{F} = \{bb\}$

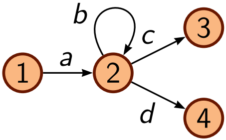


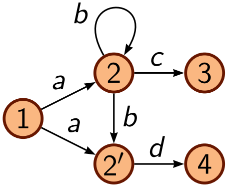
Edge shift



Adjacency matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Output state splitting of the state 2





The edge shifts defined by A and B are conjugate.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$A = DE$ and $ED = B$ with $D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ division matrix

Elementary equivalence

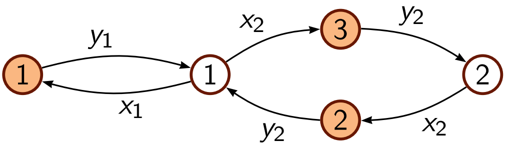
Elementary equivalence of two nonnegative integral square matrices

$$A \approxeq B \text{ iff } A = XY \text{ and } YX = B$$

with X, Y nonnegative integral rectangular matrices.

$$A = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} \leftrightarrow XY \quad YX \leftrightarrow B = \begin{bmatrix} a' & 0 & d' \\ c' & 0 & b' \\ 0 & b' & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 & 0 & x_2 \\ 0 & x_2 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} y_1 & 0 \\ y_2 & 0 \\ 0 & y_2 \end{bmatrix}$$



Strong shift equivalence

Strong shift equivalence = conjugacy

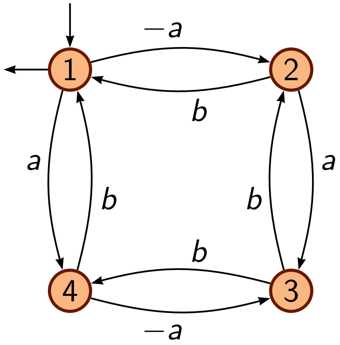
$$A \approx B \text{ iff } A = A_0 \cong A_1 \cong \ldots \cong A_n = B$$

$$A \approx B \text{ iff } A = A_0 \rightarrow A_1 \rightarrow \ldots A_i \leftarrow \ldots \leftarrow A_n = B$$

where $\rightarrow$ is a state splitting, and $\leftarrow$ a state merging

$$? \ A = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} \rightarrow A_1 \rightarrow \ldots \rightarrow A_i \leftarrow \ldots \leftarrow A_n = B = \begin{bmatrix} 1 & 12 \\ 1 & 1 \end{bmatrix}$$

Decidability unknown



$$\mathcal{A} = (I, M, T)$$

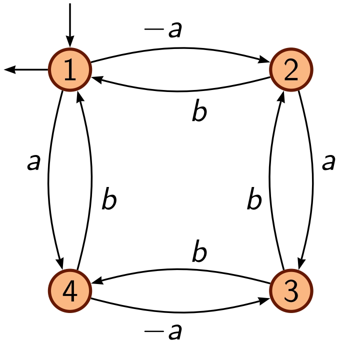
$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & -a & 0 & a \\ b & 0 & a & 0 \\ 0 & b & 0 & b \\ b & 0 & -a & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\langle |\mathcal{A}|, ab \rangle = 1 - 1 = 0$$

Automata with mutliplicities in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{K}$



$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mu(a) = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\mu(b) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\langle |\mathcal{A}|, ab \rangle = 1 - 1 = 0 = I\mu(a)\mu(b)T$$

Conjugacy of $\mathbb{K}$-automata

Let $\mathcal{A} = (I, M, T), \mathcal{B} = (J, N, U)$, We define $\mathcal{A} \stackrel{X}{\Longrightarrow} \mathcal{B}$ iff

$$IX = J, \quad MX = XN, \quad T = XU.$$

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The automata are equivalent.

Equivalence of automata is decidable (Schützenberger reductions).

The conjugacy is not an equivalence relation. It is a pre-order.

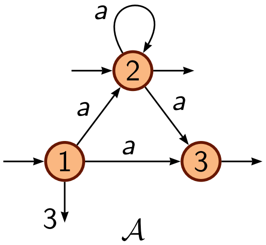
Theorem 1

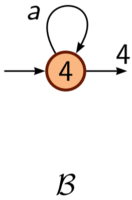
Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{K}$-automata. If $\mathcal{A}$ and $\mathcal{B}$ are equivalent, then there is an automaton $\mathcal{C}$ such that $\mathcal{A} \overset{X}{\Longleftarrow} \mathcal{C} \overset{Y}{\Longrightarrow} \mathcal{B}$.

- Compute a left reduction $\mathcal{C}$ of $\mathcal{A} + \mathcal{B}$ (If $\mathcal{A} + \mathcal{B} = \langle I, \mu, T \rangle$, compute a finite generating set of $\langle I\mu(w) \rangle$).
- One has $\mathcal{C} \overset{[X|Y]}{\Longrightarrow} \mathcal{A} + \mathcal{B}$. If $\mathcal{C} = (J, N, U)$, let $\mathcal{C}' = (J, N, U/2)$. We get $\mathcal{C}' \overset{X}{\Longrightarrow} \mathcal{A}$ and $\mathcal{C}' \overset{Y}{\Longrightarrow} \mathcal{B}$.

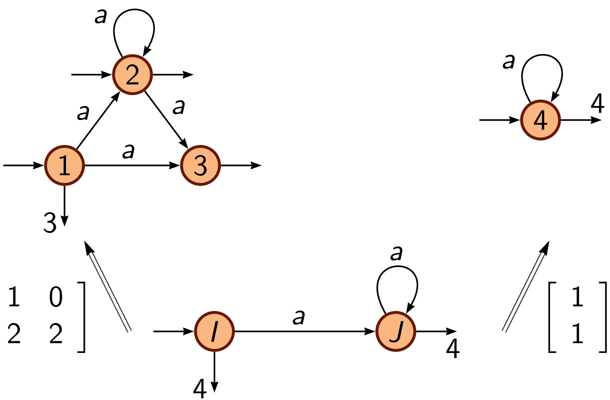
Example





$$I = [1 \; 1 \; 0 | \; 1]$$
$$I\mu(a) = [0 \; 2 \; 2 | \; 1] = J$$
$$J\mu(a) = [0 \; 2 \; 2 | \; 1] = J$$

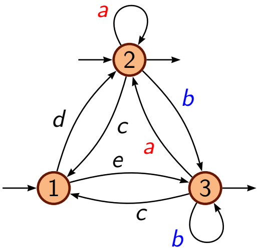
Example

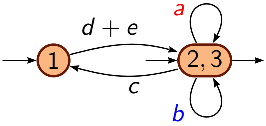


$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

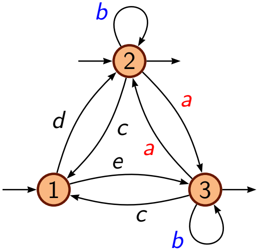
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Input state merging and covering

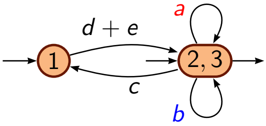




State merging from $\mathcal{A}$ to $\mathcal{B}$



$\mathcal{A} = (I, M, T)$ covering of $\mathcal{B}$



$\mathcal{B} = (J, N, U)$

Definitions

Let $\mathcal{A} = (I, M, T)$ and $\mathcal{B} = (J, N, U)$ be two $\mathbb{Z}$-automata.

There is a covering from $\mathcal{A}$ to $\mathcal{B}$ if $\mathcal{A} \stackrel{X}{\Longrightarrow} \mathcal{B}$, with X an amalgamation matrix $(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix})$

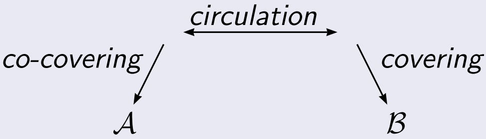
There is a co-covering from $\mathcal{A}$ to $\mathcal{B}$ if $\mathcal{B} \stackrel{X}{\Longrightarrow} \mathcal{A}$, with X a co-amalgamation matrix.

There is a circulation between $\mathcal{A}$ and $\mathcal{B}$ if $\mathcal{A} \stackrel{X}{\Longrightarrow} \mathcal{B}$, with X is diagonal matrix with coefficients 1 or -1.

Theorem 2

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{Z}$-automata. If $\mathcal{A} \xrightarrow{X} \mathcal{B}$, then



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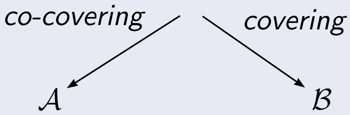
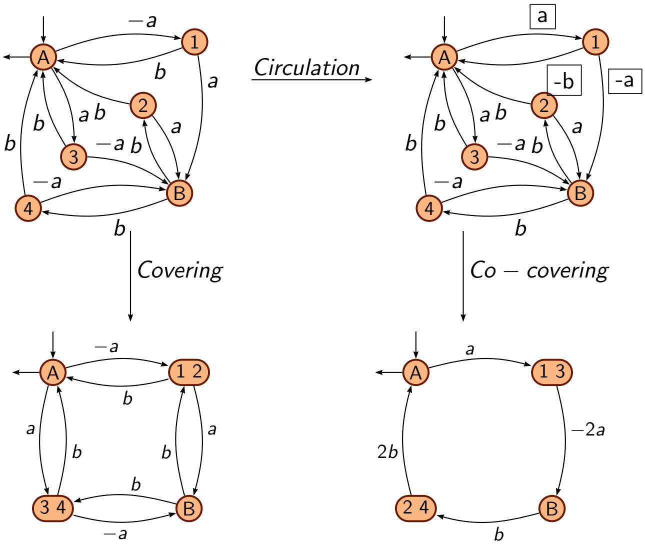


linked with the finite equivalence theorem of W. Parry between sofic shifts of equal entropy.

Example of decomposition of a conjugacy



Theorem 1 + Theorem 2

$|\mathcal{A}| = |\mathcal{B}|$

$$\mathcal{A} \xLeftarrow{\quad X \quad} \mathcal{C} \xRightarrow{\quad Y \quad} \mathcal{B}$$

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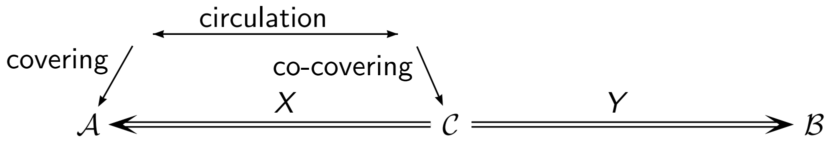

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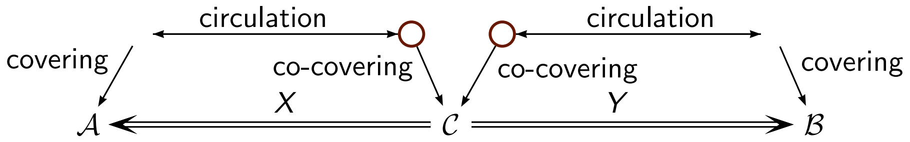

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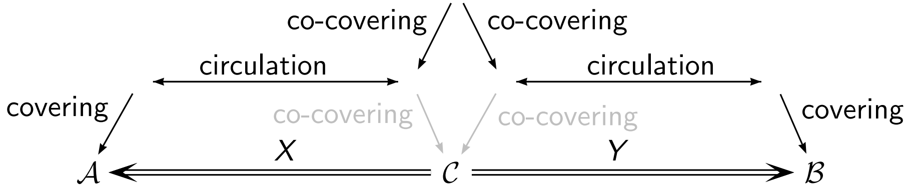

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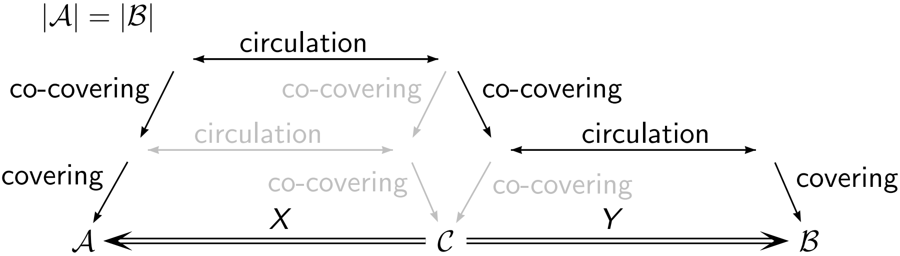

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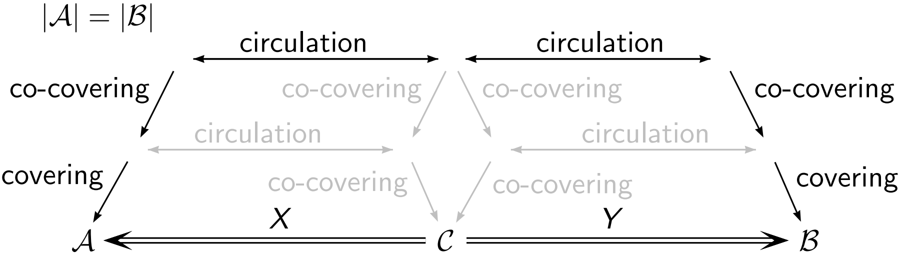

Conclusion

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  - Characterization of the generating sequences of the lengths of words of a regular language on k symbols [B, Perrin].
  - If two regular languages have the same length distribution, there is a rational bijection between them realized by a letter-to-letter transducer.