Conjugacy of automata

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Conjugacy of symbolic dynamical shifts

Subshift

A set $X_\mathcal{F}$ of bi-infinite sequences of symbols over a finite alphabet avoiding a set of finite blocks $\mathcal{F}$.

Conjugacy between two subshifts

A bi-continuous bijection commuting with the shift transformation $(\sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}})$. Equivalently, a one-to-one and onto block map.

\[
\begin{array}{ccccccccc}
\bowtie & b & b & a & a & b & a & b & b & a \\
\downarrow & & & & & & \bowtie & & & \\
\bowtie & b & a & a & a & b & b & b & &
\end{array}
\]
## Conjugacy of shifts of finite type

<table>
<thead>
<tr>
<th><strong>Sofic shift</strong></th>
<th>The set of labels of bi-infinite paths of a finite automaton.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Shift of finite type</strong></td>
<td>The set $X_F$ of bi-infinite sequences avoiding a <em>finite set</em> of finite blocks $F$.</td>
</tr>
<tr>
<td><strong>Edge shift</strong></td>
<td>The set of labels of bi-infinite paths of a finite automaton whose labels are <em>distinct</em>. A shift of finite type is conjugate to an edge shift.</td>
</tr>
</tbody>
</table>
Examples

Shift of finite type $\mathcal{F} = \{bb\}$

Edge shift

Adjacency matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$
Output state splitting of the state 2

The edge shifts defined by $A$ and $B$ are conjugate.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$A = DE$ and $ED = B$ with $D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ division matrix
Elementary equivalence of two nonnegative integral square matrices

\[ A \cong B \text{ iff } A = XY \text{ and } YX = B \]

with \( X, Y \) nonnegative integral rectangular matrices.

\[
A = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} \leftrightarrow XY \quad YX \leftrightarrow B = \begin{bmatrix} a' & 0 & d' \\ c' & 0 & b' \\ 0 & b' & 0 \end{bmatrix}
\]

\[
X = \begin{bmatrix} x_1 & 0 & x_2 \\ 0 & x_2 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} y_1 & 0 \\ y_2 & 0 \\ 0 & y_2 \end{bmatrix}
\]
Strong shift equivalence = conjugacy

\[ A \approx B \text{ iff } A = A_0 \cong A_1 \cong \ldots \cong A_n = B \]

\[ A \approx B \text{ iff } A = A_0 \rightarrow A_1 \rightarrow \ldots A_i \leftarrow \ldots \leftarrow A_n = B \]

where \( \rightarrow \) is a state splitting, and \( \leftarrow \) a state merging

\[ ? \ A = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} \rightarrow A_1 \rightarrow \ldots \rightarrow A_i \leftarrow \ldots \leftarrow A_n = B = \begin{bmatrix} 1 & 12 \\ 1 & 1 \end{bmatrix} \]

Decidability unknown
Automata with multiplicities in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{K}$

\[ A = (I, M, T) \]

\[ I = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \]

\[ M = \begin{bmatrix} 0 & -a & 0 & a \\ b & 0 & a & 0 \\ 0 & b & 0 & b \\ b & 0 & -a & 0 \end{bmatrix} \]

\[ T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \langle |A|, ab \rangle = 1 - 1 = 0 \]
Automata with multiplicities in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{K}$

\[ I = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \]

\[ \mu(a) = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \]

\[ \mu(b) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

\[ T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \langle |A|, ab \rangle = 1 - 1 = 0 = I \mu(a) \mu(b) T \]
Let $A = (I, M, T), B = (J, N, U)$, We define $A \xRightarrow{X} B$ iff

\[
IX = J, \quad MX = XN, \quad T = XU.
\]
Let $A = (I, M, T), B = (J, N, U)$, We define $A \overset{X}{\longrightarrow} B$ iff

\[
IX = J, \quad MX = XN, \quad T = Xu.
\]

$IM^n T =$
Let $\mathcal{A} = (I, M, T), \mathcal{B} = (J, N, U)$, We define $\mathcal{A} \xrightarrow{X} \mathcal{B}$ iff

$$IX = J, \quad MX = XN, \quad T = XU.$$ 

$$IM^n T = IM^n X U$$
Let $A = (I, M, T)$, $B = (J, N, U)$, We define $A \rightarrow^X B$ iff

\[ IX = J, \quad MX = XN, \quad T = XU. \]

\[ IM^n T = IX N^n U \]
Let $\mathcal{A} = (I, M, T), \mathcal{B} = (J, N, U)$, We define $\mathcal{A} \xrightarrow{X} \mathcal{B}$ iff

$$IX = J, \quad MX = XN, \quad T = XU.$$ 

$IM^n T = JN^n U$

The automata are equivalent.

Equivalence of automata is decidable (Schützenberger reductions).

The conjugacy is not an equivalence relation. It is a pre-order.
Theorem 1

Let $A$ and $B$ be two $K$-automata. If $A$ and $B$ are equivalent, then there is an automaton $C$ such that $A \xleftarrow{X} C \xrightarrow{Y} B$.

- Compute a left reduction $C$ of $A + B$ (If $A + B = \langle I, \mu, T \rangle$, compute a finite generating set of $\langle I\mu(w) \rangle$).

- One has $C \xrightarrow{[X\mid Y]} A + B$. If $C = (J, N, U)$, let $C' = (J, N, U/2)$. We get $C' \xrightarrow{X} A$ and $C' \xrightarrow{Y} B$. 

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Conjugacy of automata
Example

\[ I = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \]

\[ I \mu(a) = \begin{bmatrix} 0 & 2 & 2 \end{bmatrix} = J \]

\[ J \mu(a) = \begin{bmatrix} 0 & 2 & 2 \end{bmatrix} = J \]
Example

Conjugacy of automata

\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 2 & 2
\end{bmatrix}
\]
Input state merging and covering

State merging from $\mathcal{A}$ to $\mathcal{B}$

$\mathcal{A} = (I, M, T)$ covering of $\mathcal{B}$

$\mathcal{B} = (J, N, U)$
Let $\mathcal{A} = (I, M, T)$ and $\mathcal{B} = (J, N, U)$ be two $\mathbb{Z}$-automata.

There is a **covering** from $\mathcal{A}$ to $\mathcal{B}$ if $\mathcal{A} \xrightarrow{X} \mathcal{B}$, with $X$ an amalgamation matrix $(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix})$.

There is a **co-covering** from $\mathcal{A}$ to $\mathcal{B}$ if $\mathcal{B} \xrightarrow{X} \mathcal{A}$, with $X$ a co-amalgamation matrix.

There is a **circulation** between $\mathcal{A}$ and $\mathcal{B}$ if $\mathcal{A} \xleftarrow{X} \mathcal{B}$, with $X$ is diagonal matrix with coefficients 1 or $-1$. 
Theorem 2

Let \( A \) and \( B \) be two \( \mathbb{Z} \)-automata. If \( A \xrightarrow{X} B \), then

\[
\begin{align*}
\text{co-covering} & \quad \text{circulation} \quad \text{covering} \\
A & \quad B
\end{align*}
\]
Theorem 2

Let $A$ and $B$ be two $\mathbb{N}$-automata. If $A \xrightarrow{x} B$, then $A$ co-covering $B$ linked with the finite equivalence theorem of W. Parry between sofic shifts of equal entropy.
Example of decomposition of a conjugacy

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Conjugacy of automata
Theorem 1 + Theorem 2

\[ |A| = |B| \]

\[ A \xrightarrow{X} C \xrightarrow{Y} B \]
Theorem 1 + Theorem 2

\[ |A| = |B| \]
Theorem 1 + Theorem 2

|\mathcal{A}| = |\mathcal{B}|

covering
\quad \text{circulation}
\quad \text{co-covering}
\quad X
\quad C
\quad Y
\quad \text{covering}

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Theorem 1 + Theorem 2

\[ |A| = |B| \]
Theorem 1 + Theorem 2

$|A| = |B|$

$A \subseteq B$

covering

co-covering

$X$

$C$

co-covering

co-covering

circulation

co-covering

co-covering

circulation

$Y$

$B$

covering

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Conjugacy of automata
Theorem 1 + Theorem 2

\[ |A| = |B| \]

covering

co-covering

circulation

\[ A \leftarrow X \rightarrow C \rightarrow Y \rightarrow B \]

co-covering

circulation

covering
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Conclusion

- We knew that equivalence of automata with multiplicity was decidable.
- We have proved that there is a finite sequence of elementary transformations (with “graphical” interpretations) between the two equivalent automata.
- Many results about series and rational languages can be obtained by the use of conjugacies of automata. Ex:
  - Characterization of the generating sequences of leaves of regular $k$-ary trees [Bassino, B, Perrin]
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- Characterization of the generating sequences of leaves of regular $k$-ary trees [Bassino, B, Perrin]
- Characterization of the generating sequences of the lengths of words of a regular language on $k$ symbols [B, Perrin].
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- Characterization of the generating sequences of leaves of regular $k$-ary trees [Bassino, B, Perrin]
- Characterization of the generating sequences of the lengths of words of a regular language on $k$ symbols [B, Perrin].
- If two regular languages have the same length distribution, there is a rational bijection between them realized by a letter-to-letter transducer.