Automata, semigroups and recognizability of words on ordinals

Nicolas BEDON
Institut Gaspard Monge
Université de Marne-la-Vallée
2, rue de la Butte Verte
93166 Noisy-le-Grand Cedex
France
Nicolas.Bedon@univ-mlv.fr

ABSTRACT

For a given integer n, we define ω^n-semigroups as a generalization of ω-semigroups for languages of words of length less than ω^{n+1}. When they are finite, those algebraic structures define the same sets as those recognized by Choueka automata. These sets are also equivalent to regular expressions in which an unary ω operator standing for the infinite repetition of a language is free as the Kleene closure operator is. Naturally, the notion of syntactic congruence still works on ω^n-semigroups: among all ω^n-semigroups recognizing a regular language X, there exists an unique ω^n-semigroup of which all others are refinements.

1. Introduction

The theory of finite automata began in the mid 50's with Kleene's theorem, which says that the class of languages recognizable by finite automata is exactly the smallest class of words containing the empty word, the letters and closed under finite union, product and finite iteration. Myhill proved that among all automata recognizing a regular language X there exists a minimal one (in terms of numbers of states). The underlying ideas of the proof were developed by Rabin and Scott, who introduced the notion of syntactic monoid of a language: a language X is regular if and only if it is recognizable by a finite monoid, and among all monoids that recognizes X there exists one, called the syntactic monoid of X, of which all others are refinements. This algebraic approach gives access to more powerful tools than automata for the study of recognizable sets. In particular, it facilitates the classification of regular languages. The first result in this direction is due to Schützenberger ([15]), who proved that the class of languages whose syntactic monoid is aperiodic coincides with the smallest class of words containing the letters, the empty word and closed under finite use of union, product and complementation. This is the beginning of a very rich theory of hierarchy of regular languages. Pin wrote a survey on this subject ([13]).

Finite automata on ω-sequences were first introduced by Büchi ([3]) to prove the decidability of the second order monadic logic of integers. These automata are not deterministic, that is to say that there exist languages accepted by such non deterministic automata which are not accepted by deterministic ones. Independently of Büchi, Muller ([9]) used automata on ω-words to study the behavior of asynchronous circuits. Muller automata are deterministic. McNaughton proved in [8] the equivalence between Muller automata and Büchi's ones acting on ω-words. As for the finite word case, finite automata on ω-words are equivalent to finite semigroups with an adapted structure for the infinite product, called finite ω-semigroups. A first try in this direction was made by J.P. Pécuchet ([11], [12]), but a more satisfying approach is due to T. Wilke ([18]), D. Perrin and J.E. Pin ([10]). Like in the finite words theory, one can associate to any regular language X of ω-words the syntactic ω-semigroup of X, which is finite and unique. This differs from the automata theory: we do not know yet how to attach a canonical automaton to any regular language of infinite words. A lot of results on hierarchies of languages of finite words are extendible to ω-words using this algebraic approach.

Büchi, in [4], generalized his idea to transfinite sequences in order to prove the decidability of the monadic second order theory of $[\alpha, <]$ for α a countable ordinal. His automata acting on transfinite words are build from Kleene's, but in order to deal with the limit ordinals he add a new kind of states: each set of the new kind is a nonempty subset of the set of states of the old kind. These states are only reachable by a second map for transition: if α is a limit ordinal, then the state of the new kind $\{s_1, \ldots, s_p\}$ is reached at position α if and only if the states of the old kind $s_1, \ldots, s_p$ where cofinaly reached in the past. Choueka ([6]) studied, in a classical automata theory point of vue, a generalization of Muller automata on ω-sequences to transfinite sequences of length less than ω^n for a given $n < \omega$. The differences between those automata and [4]'s are the second kind of states and the second map (call it f). For ξ a limit ordinal, $\xi = \beta + \omega^n, n < \omega, \beta = 0$ or $\beta \geq \omega^n$, p belongs to $f(\xi)$ iff there exist an infinity of $k < \omega$ such that $f(\beta + \omega^{n-1} \cdot k) = p$. Note that f is defined only for $n < \omega$, and that $f(\xi)$ depends only on values of $f(\alpha)$ for $\alpha < \xi$. Those automata are deterministic and equivalent to regular expressions, looking like Kleene's regular expressions, but with one more unary operator, ω, standing for the infinite repetition of a language. Choueka automata are equivalent to Büchi's if we restrict the domain of Büchi automata to those of Choueka automata. Under this assumption, Büchi automata are deterministic (see [1]).

In this paper we generalize the idea of semigroups on ω-words to obtain semigroups equivalent to Choueka automata. First of all, we recall the definitions and basic properties of ordinals, automata and regular expressions on words less than ω^n for $n \in \mathbb{N}$. Thus, the algebraic definitions are introduced: we define ω^n-semigroups, morphisms of ω^n-semigroups and a structure of finite signature equivalent to finite ω^n-semigroups, which are of infinite signature, called ω^n-Wilke algebra. Then we prove the equivalence between Choueka automata and ω^n-Wilke algebra. Finally, given a recognizable language X, we show the existence of the syntactic ω^n-semigroup that recognizes X.

The reader is supposed to be familiar with the theory of ordinals and with the classical automata theory (on finite words). Knowledges on the theory of regular languages of ω-words are also required. One can find the classical theory of ordinals in [16] (a more modern approach is in [14]). The traditional theory on finite words is in [7]. Finally, for the theory of automata on ω-words we refer to [10] and to [17].

2. Notations and basic definitions

An ordinal α is a **successor** ordinal if there exists β such that $\alpha = \beta + 1$. Any other ordinal, except for 0, is a **limit** ordinal. We denote by *Succ* the class of successor ordinals, *Lim* the class of limit ordinals and $Ord = Succ \cup Lim \cup \{0\}$. It is well-known that each ordinal α less than ω^ω can be written in an unique way

$$\alpha = \omega^m \cdot a_m + \omega^{m-1} \cdot a_{m-1} + \cdots + \omega^0 \cdot a_0$$

where $a_0, \ldots, a_m$ and m are ordinals less than ω. This decomposition is called the **normal form** of α. For short, we will write $\alpha = \sum_{i=m}^{0} \omega^i \cdot a_i$. When we write $\alpha = \beta + \omega^\gamma$ we are talking about the unique decomposition of α where β is as small as possible and $\beta \geq \omega^\gamma$ or $\beta = 0$. Thus, ω^γ is unique and the **type** of α is the ordinal γ.

Let $\mathcal{S}$ be a set and α an ordinal. The cardinality of $\mathcal{S}$ is denoted by $|\mathcal{S}|$. An **α-sequence** $(a_\beta)_{\beta<\alpha}$ on $\mathcal{S}$ is a function $\varphi : \alpha \to \mathcal{S}$ such that $\forall \beta < \alpha \ \varphi(\beta) = a_\beta$. The **length** of an α-sequence φ, denoted by $|\varphi|$, is the ordinal α. The sequence itself is denoted by $(\varphi(\beta))_{\beta<\alpha}$. We denote by $\mathcal{S}^\alpha$ the class of all α-sequences on $\mathcal{S}$ ($\alpha \in Ord$), $\mathcal{S}^{<\alpha}$ is the class of all sequences on $\mathcal{S}$ of length less than α and $\mathcal{S}^{\geq\beta<\alpha}$ the class of all sequences on $\mathcal{S}$ of length γ such that $\beta \leq \gamma < \alpha$ ($\gamma, \beta, \alpha \in Ord$). We denote by $\overline{\mathcal{S}}^n$ the set of sequences of length less than ω^{n+1} on $\mathcal{S}$. The notations $(\mathcal{S}^{<\omega}, \mathcal{S}^{\geq\omega<\omega^2}, \ldots, \mathcal{S}^{\geq\omega^n<\omega^{n+1}})$ and $(\mathcal{S}_{\omega^0}, \mathcal{S}_{\omega^1}, \ldots, \mathcal{S}_{\omega^n})$, which are equivalent, exhibit a partitioning of $\overline{\mathcal{S}}^n$ according to the length of sequences. Since $\overline{\mathcal{S}}^n$ can always be partitioned in $(\mathcal{S}^{<\omega}, \mathcal{S}^{\geq\omega<\omega^2}, \ldots, \mathcal{S}^{\geq\omega^n<\omega^{n+1}})$, and $\cup_{i=0\ldots n}\mathcal{S}_{\omega^i} = \overline{\mathcal{S}}^n$ we use indiscriminately the notations $\overline{\mathcal{S}}^n$ and $(\mathcal{S}_{\omega^0}, \mathcal{S}_{\omega^1}, \ldots, \mathcal{S}_{\omega^n})$. If φ is an ω-sequence on $\mathcal{S}$, $\mathbf{In}(\varphi)$ will denote the set of elements of $\mathcal{S}$ appearing infinitely often in $(\varphi(\beta))_{\beta<\omega}$.

A **word** of length α is an α-sequence on a set, usually denoted by Σ, called **alphabet**. Except when specified, alphabets are finite. Each element of an alphabet is a **letter**. The word of length 0, also called the *empty word*, is denoted by λ. The set of all finite words on Σ is denoted by Σ^+, and Σ^* stands for $\Sigma^+ \cup \{\lambda\}$.

Let φ be an α-sequence and $\beta, \gamma \in Ord$ such that $\beta < \gamma \leq \alpha$. We denote by $\varphi[\beta, \gamma[$ the $(\gamma - \beta)$-sequence such that $\varphi[\beta, \gamma[(\delta) = \varphi(\beta + \delta)$ for all $\delta < \gamma - \beta$.

Let u be an α-sequence on a finite set Σ_u and v be a β-sequence on a finite set Σ_v. The **product** of u and v, denoted $u \cdot v$, is the $(\alpha + \beta)$-sequence w on $\Sigma_u \cup \Sigma_v$ such that:

$$w_\gamma = \begin{cases} u_\gamma & \text{if } 0 \leq \gamma < \alpha \\ v_{\gamma - \alpha} & \text{if } \alpha \leq \gamma < \alpha + \beta \end{cases}$$

Let $\mathcal{S}$ be a finite set, n and m two integers. We will denote $[\mathcal{S}]^1$ or $[\mathcal{S}]$ the powerset of $\mathcal{S}$ without the empty set $\emptyset$. $[\mathcal{S}]^0$ stands for $\mathcal{S}$, $[\mathcal{S}]^{n+1}$ for $[[\mathcal{S}]^n]$, and $[\mathcal{S}]^n_m$ for $\bigcup\limits_{i=m}^{n} [\mathcal{S}]^i$. Extending this notation, if $s \in \mathcal{S}$, $\{s\}^0$ stands for s and $\{s\}^{n+1} = \{\{s\}^n\}$. We remark that $\{s\}^n \in [\mathcal{S}]^n$. If $x \in [\mathcal{S}]^m$, the **type** of x, $t(x)$, is m.

2.1. Automata

We refer the reader to [7] for the classical automata theory on finite words, and to [10], [17] for the automata theory on ω-words.

When we talk about *automata*, we mean *finite automata*. Infinite automata are not the subject of this paper.

Definition 1 An α-sequence φ on $\mathcal{S}$ is said to be **continuous** if $\forall \beta \in Succ\ \varphi(\beta) \in \mathcal{S}$ and $\forall \beta \in Lim\ \varphi(\beta) = \mathrm{In}(\psi)$ where ψ is the unique ω-sequence such that for all $0 \leq i < \omega\ \psi(i) = \varphi(\gamma + \omega^{n-1} \cdot i)$ if $\beta = \gamma + \omega^n$, $n \in \mathbb{N}$. A continuous α-sequence is then entirely defined by its non-limit values.

Choueka studied these automata in [6]:

Definition 2 Let n be an integer. An **n-automaton** $\mathcal{A}$ is a 5-tuple $<\mathcal{S}, \mathcal{M}, s^\star, \mathcal{F}, \Sigma>$ with:

- $\mathcal{S}$ the finite set of states.
- $\mathcal{M} \subseteq [\mathcal{S}]^n_0 \times \Sigma \times \mathcal{S}$ the relation for transitions.
- $s^\star \in \mathcal{S}$ the initial state.
- $\mathcal{F} \subseteq [\mathcal{S}]^n_0$ the set of final "states" (from now on, we will call, for evident reasons, *state* any element of $[\mathcal{S}]^n_0$, but *set of states* still means $\mathcal{S}$).
- Σ a finite alphabet.

$\mathcal{M}$:$[\mathcal{S}]^n_0 \times \Sigma \times \mathcal{S}$ can naturally be extended from letters to words defining $\overline{\mathcal{M}}$:$[\mathcal{S}]^n_0 \times \Sigma^{<\omega^{n+1}} \times [\mathcal{S}]^n_0$: if u is an α-sequence ($\alpha < \omega^{n+1}$) on Σ, $(s, u, s_f) \in \overline{\mathcal{M}}$ iff there exists a continuous $(\alpha + 1)$-sequence φ on $\mathcal{S}$ such that $\varphi(0) = s$, $\varphi(\alpha) = s_f$ and $\forall \beta < \alpha$, $(\varphi(\beta), u(\beta), \varphi(\beta + 1)) \in \mathcal{M}$.

Definition 3 A **run** of an n-automaton $\mathcal{A} = <\mathcal{S}, \mathcal{M}, s^\star, \mathcal{F}, \Sigma>$ on a word $u \in \Sigma^\alpha$, $\alpha < \omega^{n+1}$, is a continuous $(\alpha + 1)$-sequence φ on $\mathcal{S}$ such that $\varphi(0) = s^\star$ and $\forall \beta < \alpha\ (\varphi(\beta), u(\beta), \varphi(\beta + 1)) \in \mathcal{M}$. If $\mathcal{A}$ is a deterministic automaton this sequence is unique. A run is said to be *accepting*, and u is accepted by $\mathcal{A}$, if $\varphi(\alpha) \in \mathcal{F}$. A **path** from p to q *labeled* with u is a run of $\mathcal{A} = <\mathcal{S}, \mathcal{M}, p, \emptyset, \Sigma>$ on u such that the last element of the sequence is q. Note that a path may not exist, or may be non-unique if the automaton is not deterministic.

The set (or the **language**) $\mathcal{L}(\mathcal{A})$ of words accepted (we also say *defined* or *recognized*) by an automaton $\mathcal{A}$ is also definable in another formalism:

Definition 4 Let $\Sigma = \{\sigma_1, \ldots, \sigma_k\}$ be a finite alphabet and n an integer. An **n-regular expression** (or *Choueka regular expression*) is a finite word on the alphabet $\Sigma \cup \{+, \cdot, ^\omega, ^*, (,), \lambda, \emptyset\}$ such that:

- $\emptyset$ is a 0-regular expression.
- λ is a 0-regular expression.
- $\sigma \in \Sigma$ is a 0-regular expression.
- If e_1 is an n-regular expression and e_2 a m-regular expression, $e_1 + e_2$ is a $max(n, m)$-regular expression.
- If e_1 is an n-regular expression and e_2 a m-regular expression, $e_1 \cdot e_2$ is a $max(n, m)$-regular expression.
- If e is an n-regular expression, then so is (e).
- If e is an n-regular expression, then so is e^*.
- If e is an n-regular expression, then e^ω is a $(n+1)$-regular expression.

If e is an n-regular expression, let $\overline{e}$ be the set of words denoted by e:

- $\overline{\emptyset} = \emptyset$.
- $\overline{\lambda} = \lambda$.
- $\Sigma \ni \overline{\sigma} = \sigma$.
- $\overline{e_1 + e_2} = \overline{e_1} \cup \overline{e_2}$.
- $\overline{e_1 \cdot e_2} = \overline{e_1} \cdot \overline{e_2}$.
- $\overline{(e)} = \overline{e}$.
- $\overline{e^*} = \overline{e}^*$.
- $\overline{e^\omega} = \overline{e}^\omega$.

All the operators have their usual meaning, but unlike in Büchi's regular expressions for ω-words, the ω operator is free (as the $*$ operator is) in n-regular expressions. For convenience, we add a new unary operator, $+$, such that $e^+ = e \cdot e^* = e^* \cdot e$.

Example 5 0-regular expressions are Kleene's regular expressions. $\circ$

Two classes of automata are equivalent if a language accepted by an automaton of one class is recognized by an automaton of the other. The class of languages definable by Choueka automata is closed under complementation:

Theorem 6 ([5],[6]) *Deterministic and non-deterministic Choueka automata are equivalent.*

Theorem 7 ([6]) *Choueka automata are equivalent to Choueka regular expressions.*

A language recognized by an n-automaton is said to be n-regular.

2.2. Algebraic definitions

We refer to [7] for the theory of grammars on finite words. We recall however the following basic definitions:

A **context-free grammar** is a 4-tuple (V, T, P, S) where V and T are disjoint finite sets of respectively *variables* and *terminals*, P is a finite set of equations, each equation is of the form $A = w$ where A is a variable and w is a finite word on $V \cup T$. Finally, S is a special variable called the *start symbol*. A context-free grammar is said to be **right linear** if all the equations are of the form $A = wB$ or $A = w$ where A and B are variables and w is a finite word on T.

The following result is well-known:

Theorem 8 *Solutions of the equations of a right-linear context-free grammar are 0-regular expressions on the set of terminals.*

We refer to [10] and [18] for the semigroup theory on ω-words. We recall the following basic definitions:

A **semigroup** is a set equipped with an internal binary associative operation, usually called a **product** and noted by a dot $(\cdot)$. For short we sometimes omit the dot (xy means $x \cdot y$). A **monoid** is a semigroup with an identity, usually noted 1. We denote by S^1 the monoid $S \cup \{1\}$ if S is not a monoid, S otherwise. A subset I of a semigroup S is an **ideal** of S iff $S^1 I S^1 = I$. A **morphism** between two algebraic structures of the same kind is an application preserving operations. For example, if S and T are two semigroups and φ is a morphism from S to T, then for all x, y in S, $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$. A **congruence** is an equivalence relation preserving operations, usually noted $\sim$. For example, a semigroup congruence $\sim$ verifies $x \sim y \Rightarrow uxv \sim uyv$. This condition ensures that the set of equivalence classes can naturally be equipped with a product and that the mapping which associates to an element its equivalence class is a semigroup morphism. The semigroup of congruence classes of a semigroup S under a semigroup congruence $\sim$ is denoted by $S/\sim$ and is called the **quotient** of S under $\sim$. If $\sim_1$ and $\sim_2$ are two congruences on an algebraic structure S we say that $\sim_1$ is a **refinement** of $\sim_2$ if and only if, for every $x, y \in S$, $x \sim_1 y \Rightarrow x \sim_2 y$. It is well-known that semigroups are equivalent to classical automata on finite words, and that among all semigroups recognizing a regular language X there exists an unique one which is finite and of which all others are refinements. A similar result holds in the theory of ω-words.

This Ramsey based theorem is a classical result in the theory of ω-words:

Theorem 9 *Let Σ be a finite or infinite alphabet, S a finite semigroup and φ a morphism from Σ^+ to S. For every ω-word u on Σ, there exists a pair (s, e) of elements of S such that $se = s$, $e^2 = e$, and a factorization $u = u_0 u_1 \ldots$ of u as a product of words of Σ^+ such that $\varphi(u_0) = s$ and $\varphi(u_n) = e$ for every $n > 0$.*

Now we can generalize the idea of semigroups recognizing ω-words to words of length less than ω^n ($n \in \mathbb{N}$). We will prove later that the following structure is equivalent to Choueka automata, in the definition of languages point of vue. We point out that the empty word will not be considered when we deal with it.

Definition 10 Let n be an integer. An ω^n**-semigroup** S is a semigroup partitioned into $n+1$ subsemigroups $S_{\omega^0}, S_{\omega^1}, \ldots, S_{\omega^n}$ such that for every $j \leq n$, S_{ω^j} is an ideal of $\cup_{i \leq j} S_{\omega^i}$, and equipped with a family of n infinite products $\overline{\pi}_i : (S_{\omega^i})^\omega \rightarrow S_{\omega^{i+1}}$ such that

(1) for every increasing sequence of integers $(k_n)_{n>0}$ and for every $(s_n)_{n \in \mathbb{N}} \in (S_{\omega^i})^\omega$,

$$\overline{\pi}_i(s_0 s_1 \ldots s_{k_1 - 1}, s_{k_1} s_{k_1+1} \ldots s_{k_2 - 1}, \ldots) = \overline{\pi}_i(s_0, s_1, s_2, \ldots)$$

(2) for every $s \in S_{\omega^i}$ and for every $(s_n)_{n \in \mathbb{N}} \in (S_{\omega^i})^\omega$,

$$s \overline{\pi}_i(s_0, s_1, \ldots) = \overline{\pi}_i(s, s_0, s_1, \ldots)$$

For short, we shall omit the subscripts of π's. In the sequel we use indiscriminately the notations S and $(S_{\omega^0}, \ldots, S_{\omega^n})$.

Let s be an element of an ω^n-semigroup. The **type** of s, $ty(s)$, is the only integer m such that $s \in S_{\omega^m}$.

An ω^n-semigroup can be equipped in an unique way of a **generalized infinite product**, mapping $\pi : S^\omega \rightarrow S_{\omega^k}$ where k is the highest type of elements appearing in the infinite serie if the type appears finitely, incremented by 1 otherwise, satisfying the properties:

(3) for every $(s_n)_{n \in \mathbb{N}} \in (S_{\omega^i})^\omega$,

$$\pi(s_0, s_1, s_2, \ldots) = \overline{\pi}(s_0, s_1, s_2, \ldots)$$

(4) for every increasing sequence of integers $(k_n)_{n>0}$ and for every $(s_n)_{n \in \mathbb{N}} \in S^\omega$,

$$\pi(s_0 s_1 \ldots s_{k_1 - 1}, s_{k_1} s_{k_1+1} \ldots s_{k_2 - 1}, \ldots) = \pi(s_0, s_1, s_2, \ldots)$$

(5) for every $s \in S$ and for every $(s_n)_{n \in \mathbb{N}} \in S^\omega$,

$$\pi(s, s_0, s_1, \ldots) = s\pi(s_0, s_1, \ldots)$$

Obviously, the evaluation of finite and infinites products is independent of the factorization, so we shall use the notation $s_0 s_1 \ldots s_n$ (resp. $s_0 s_1 \ldots$).

Proposition 11 *Let n be an integer and S an ω^n-semigroup. S can be equipped in an unique way of an infinite product π verifying properties (3), (4) and (5). Furthermore, for every ω-sequence $(s_n)_{n\in\mathbb{N}} \in S^\omega$, $\pi(s_0, s_1, s_2, \ldots)$ is entirely determined by the restricted infinite products and finite product.*

Proof Let $(s_n)_{n\in\mathbb{N}} \in S^\omega$. Let S_{ω^k} be the subset of S such that k is as great as possible and there exists $s_i \in (s_n)_{n\in\mathbb{N}} \cap S_{\omega^k}$. Let $(k_i)_{i<\alpha}$ be the increasing sequences of integers such that $s_{k_i} \in S_{\omega^k}$. If $\alpha = \omega$ we assume that $k \neq n$, otherwise the infinite product $\pi(s_0, s_1, \ldots)$ can not be defined. We define $\pi(s_0, s_1, \ldots)$ by induction on k as: $\pi(s_0, s_1, \ldots) = \begin{cases} \overline{\pi}(s_0 \ldots s_{k_0}, \ldots, s_{k_i+1} \ldots s_{k_{i+1}}, \ldots) & \text{if } \alpha = \omega, \\ s_0 \ldots s_{k_{\alpha-1}} \pi(s_{k_{\alpha-1}+1}, \ldots) & \text{if } \alpha < \omega \end{cases}$. We show that this product verifies properties (3), (4) and (5). If the sequence $(k_i)_{i<\alpha}$ contains every integer then (3) is verified. Now let $(k'_i)_{i\geq 0}$ be an increasing sequence of integers such that $k'_0 = 0$. Assume first that $\alpha = \omega$. According to the definition of $\overline{\pi}$ we have $\pi(s_0 \ldots s_{k'_1-1}, s_{k'_1} \ldots s_{k'_2-1}, \ldots) = \overline{\pi}(s_0 \ldots s_{k'_{i_1}-1}, s_{k'_{i_1}} \ldots s_{k'_{i_2}-1}, \ldots)$ where the sequence $(i_j)_{j\geq 0}$ is chosen in order to verify $s_{k'_{i_j}} \ldots s_{k'_{i_{j+1}}-1} \in S_{\omega^k}$ and $i_0 = 0$. This is equal, using (1), to $\overline{\pi}(s_0 \ldots s_{k_0}, s_{k_0+1} \ldots s_{k_1}, \ldots) = \pi(s_0, s_1, \ldots)$. Now let us turn to (5). If $s \in S_{\omega^k}$ we have $\pi(s, s_0, s_1, \ldots) = \overline{\pi}(s, s_0 \ldots s_{k_1}, \ldots) = s\overline{\pi}(s_0 \ldots s_{k_1}, \ldots) = s\pi(s_0, s_1, \ldots)$. If $s \notin S_{\omega^k}$ we have $\pi(s, s_0, s_1, \ldots) = \overline{\pi}(ss_0 \ldots s_{k_0}, s_{k_0+1} \ldots s_{k_1}, \ldots) = ss_0 \ldots s_{k_0}\overline{\pi}(s_{k_0+1} \ldots s_{k_1}, \ldots) = s\overline{\pi}(s_0 \ldots s_{k_0}, s_{k_0+1} \ldots s_{k_1}, \ldots) = s\pi(s_0, s_1, \ldots)$. Now we verify (4) and (5) for $\alpha < \omega$, by induction on k. The case $k = 0$ coincides with the restricted infinite product, so (3), (4) and (5) are verified. Let $(k'_i)_{i>0}$ be an increasing sequence of integers. Assume $k_{\alpha-1} \in [k'_i, \ldots, k'_{i+1}[$. According to the definition of π we have $\pi(s_0 \ldots s_{k'_1-1}, s_{k'_1} \ldots s_{k'_2-1}, \ldots) = s_0 \ldots s_{k'_{i+1}-1}\pi(s_{k'_{i+1}} \ldots s_{k'_{i+2}-1}, \ldots) = s_0 \ldots s_{k'_{i+1}-1}\pi(s_{k'_{i+1}}, \ldots) = \pi(s_0, s_1, \ldots)$. So (4) is verified. Let us finally turn to (5). According to the definition of π, we have $\pi(s, s_0, s_1, \ldots) = ss_0 \ldots s_{k_{\alpha-1}}\pi(s_{k_{\alpha-1}+1}, \ldots) = s\pi(s_0 \ldots s_{k_{\alpha-1}}, s_{k_{\alpha-1}+1}, \ldots)$. This is equal to $s\pi(s_0, s_1, \ldots)$ using (4). Now one can easily verify that if π' is an infinite product that verifies (3), (4) and (5) then $\pi = \pi'$.

∇

In order to deal with transfinite sequences of elements of an ω^n-semigroup $(s_\beta)_{\beta<\alpha}$, with $\alpha < \omega^{n+1}$, we must prove that an ω^n-semigroup can be equipped with a **transfinite product** ψ satisfying the properties:

(6)

$$\begin{cases} \psi(s_0, s_1, \ldots, s_{\alpha-1}) = s_0 s_1 \ldots s_{\alpha-1} & \text{if } \alpha < \omega \\ \psi(s_0, s_1, \ldots) = \pi(s_0, s_1, \ldots) & \text{if } \alpha = \omega \end{cases}$$

(7) for every increasing sequence[a] of ordinals $(\gamma_\delta)_{\delta<\delta_l}$, $\delta_l < \alpha$,

$$\psi(s_0, s_1, \ldots) = \psi(\psi(s_0, s_1, \ldots), \psi(s_{\gamma_0}, s_{\gamma_0+1}, \ldots), \psi(s_{\gamma_1}, s_{\gamma_1+1}, \ldots), \ldots)$$

Proposition 12 *Let n be an integer, and S an ω^n-semigroup and α an ordinal less than ω^{n+1}. S can be equipped in an unique way of a transfinite product ψ verifying properties (6) and (7), which is entirely determined by the restricted infinite product and by the finite product.*

Proof Let $(s_\beta)_{\beta<\alpha} \in S^\alpha$. We define

$$\begin{cases} \psi(s_0, s_1, \ldots, s_{\alpha-1}) = s_0 s_1 \ldots s_{\alpha-1} & \text{if } \alpha < \omega \\ \psi(s_0, s_1, \ldots) = \pi(s_0, s_1, \ldots) & \text{if } \alpha = \omega \end{cases}$$

If $\alpha > \omega$ we write it in Cantor normal form $\alpha = \sum_{i=m}^{0} \omega^i a_i$ and we define ψ by transfinite induction: $\psi(s_0, s_1, \ldots) = \psi(\psi(s_0, \ldots), \psi(s_{\omega^m}, \ldots), \psi(s_{\omega^m 2}, \ldots), \ldots)$ where the factorisation follows the normal form of α. We let the reader verifies that this product verifies (7) by induction on α, and that ψ is entirely determined

[a] Observe that here the notation $s_0, s_1, \ldots$ does not mean that the sequence is infinite but that its length is unknown.

by the restricted infinite products and by the finite product. Clearly, any transfinite product ψ' verifying (6) and (7) also verifies $\psi = \psi'$. ∇

Example 13 Let S be a set and n an integer. Then $\overline{S}^n$ has a structure of ω^n-semigroup. $\circ$

Now we generalize morphisms of semigroups to morphisms of ω^n-semigroups. We recall that a morphism of semigroups is an application preserving the product. Thus, a morphism of ω^n-semigroups is an application preserving finite, restricted infinite, generalized infinite and transfinite products. In fact, if an application preserves the finite and restricted infinite products, then so are the generalized infinite and transfinite ones. This remark justifies the following definition:

Definition 14 Let n be an integer, S and T two ω^n-semigroups. A **morphism of ω^n-semigroups** $\varphi : S \to T$ is a morphism of semigroups verifying

$$ty(\varphi(s)) = ty(s)$$

and the conservation of the restricted infinite products: for $(s_n)_{n<\omega} \in (S_{\omega^i})^\omega$,

$$\varphi(s_0)\varphi(s_1)\varphi(s_2)\ldots = \varphi(s_0 s_1 s_2 \ldots)$$

Proposition 15 *A morphism of ω^n-semigroups preserves the generalized infinite product and the transfinite product.*

Proof We start with the generalized infinite product. Let n, S, T and φ be as in definition 14. Let $(s_n)_{n<\omega} \in S^\omega$ be an ω-sequence of elements of S. Let $(s_0 \ldots s_{k_0})(s_{k_0+1} \ldots s_{k_1})\ldots$ be a factorization of $s_0 s_1 s_2 \ldots$ as in the proof of proposition 11. Since the generalized infinite product is independent of the factorization, if $\alpha = \omega$ we have $\varphi(s_0 s_1 \ldots) = \varphi((s_0 \ldots s_{k_0})(s_{k_0+1} \ldots s_{k_1})\ldots) = \varphi(s_0 \ldots s_{k_0})\varphi(s_{k_0+1} \ldots s_{k_1})\ldots = \varphi(s_0)\varphi(s_1)\varphi(s_2)\ldots$. If $\alpha < \omega$ then $\varphi(s_0 s_1 \ldots) = \varphi(s_0)\ldots\varphi(s_{k_\alpha})\varphi(s_{k_\alpha+1}\ldots) = \varphi(s_0)\varphi(s_1)\varphi(s_2)\ldots$ using an easy induction on k. Then, φ preserves the generalized infinite product.
Let us turn to the transfinite product. Let $(s_\beta)_{\beta<\alpha} \in S^\alpha$ be an α-sequence of elements of the ω^n-semigroup. The result directly comes from a factorization of the sequence as in the proof of proposition 12. ∇

Proposition 16 *Let Σ be a set, n an integer, S an ω^n-semigroup, and $\varphi : \Sigma \to S_{\omega^0}$ a mapping. There exists an unique morphism of ω^n-semigroups $\overline{\varphi} : \overline{\Sigma}^n \to S$ such that for each $a \in \Sigma$, $\overline{\varphi}(a) = \varphi(a)$.*

Proof It suffices to define $\overline{\varphi}$ such that it verifies the properties of definition 14. Obviously, $\overline{\varphi}$ is unique. ∇

Definition 17 Let n be an integer. A morphism of ω^n-semigroups $\varphi : S \to T$ **recognizes** (or **defines**) a subset X of S if $\varphi^{-1}\varphi(X) = X$. We say that T **recognizes** X if there exists $\varphi' : S \to T$ such that ${\varphi'}^{-1}\varphi'(X) = X$.

Let n be an integer, $S = (S_{\omega^0}, S_{\omega^1}, \ldots, S_{\omega^n})$ be an ω^n-semigroup, and $\varphi : S_{\omega^0} \to S_{\omega^0}$ a mapping such that $\varphi(s) = s$ for each $s \in S_{\omega^0}$. Proposition 16 defines an unique morphism of ω^n-semigroups $\overline{\varphi} : \overline{S_{\omega^0}}^n \to S$. We say that S *recognizes* (or *defines*) a subset X of $\overline{S_{\omega^0}}^n$ if $\overline{\varphi}$ recognizes X.

Let Σ be an alphabet and n an integer. We say that a subset X of $\overline{\Sigma}^n$ is **recognizable** if there exist a finite ω^n-semigroup S and a morphism of ω^n-semigroups $\varphi : \overline{\Sigma}^n \to S$ such that φ recognizes X.

Since the description of infinite products is infinite, ω^n-semigroups are not very interesting objects, even if the number of elements is finite. Wilke ([18]) proved that when a semigroup S is finite, then the infinite product of elements of S is entirely determined by the infinite products of the form $s^\omega = ssss\ldots$. That is to say, one has only to define explicitly the infinite products of the form $aaaa\ldots$ to have all the infinite products defined. Thus, ω^n-semigroups become really finite objects when they have a finite number of elements (i.e. finite ω^n-semigroups are objects of finite signature). The proof of this result is based on a Ramsey type argument.

Definition 18 Let n be an integer. An ω^n**-Wilke algebra** is a finite ω^n-semigroup where the family of restricted infinite products is replaced by a family of functions from S_{ω^i} to $S_{\omega^{i+1}}$, denoted by $s \to s^{\omega_i}$ and also called Wilke operators, such that, for all $s, t \in S_{\omega^i}$,

$$s(ts)^{\omega_i} = (st)^{\omega_i} \qquad (W_1)$$

$$(s^n)^{\omega_i} = s^{\omega_i} \text{ for all } n > 0 \qquad (W_2)$$

For short, we shall omit the subscripts (s^ω stands for $s^{\omega_{ty(s)}}$).

For a proof of this theorem, which is similar to the ω-word case, we refer to [10] and [18]. It shows that finite ω^n-semigroups and ω^n-Wilke algebras are equivalent.

Theorem 19 *Let n be an integer and S a finite semigroup partitioned into $n+1$ subsemigroups $S_{\omega^0}, \ldots, S_{\omega^n}$ such that for every $j \leq n$, S_{ω^j} is an ideal of $\cup_{i \leq j} S_{\omega^i}$. Assume there exist n unary functions $\omega_i : S_{\omega^i} \to S_{\omega^{i+1}}$ for $0 \leq i < n$ such that, for all $s, t \in S_{\omega^i}$, (W_1) and (W_2) are verified. Then S can be equipped in a unique way of a family of n infinite restricted product $\overline{\pi_i} : S_{\omega^i} \to S_{\omega^{i+1}}$ such that properties (1) and (2) of definition 10 are verified, and $s^\omega = \overline{\pi}(s, s, s, \ldots)$. Conversely, if there exists a family of n infinite restricted product $\overline{\pi_i} : S_{\omega^i} \to S_{\omega^{i+1}}$ such that properties (1) and (2) of definition 10 are verified, then S can be equipped in a unique way of n unary functions $\omega_i : S_{\omega^i} \to S_{\omega^{i+1}}$ for $0 \leq i < n$ such that, for all $s, t \in S_{\omega^i}$, (W_1) and (W_2) are verified, and $s^\omega = \overline{\pi}(s, s, s, \ldots)$.*

Let n be an integer and Σ an alphabet. We say that an ω^n-Wilke algebra S recognizes a subset X of $\overline{\Sigma}^n$ if the ω^n-semigroup associated with S recognizes X.

Example 20 Let $\Sigma = \{a, b\}$.

The ω^1-Wilke algebra $S = (\{a, b, ab, ba, aba, 0, 1\}, \{a^\omega, ab^\omega, ba^\omega, 0', a^\omega a, ab^\omega a, ba^\omega a\})$ equipped with the product

	a	b	ab	ba	aba	0	1	a^ω	ab^ω	ba^ω	$0'$	$a^\omega a$	$ab^\omega a$	$ba^\omega a$
a	1	ab	b	aba	ba	0	a	a^ω	ba^ω	ab^ω	$0'$	$a^\omega a$	$ba^\omega a$	$ab^\omega a$
b	ba	b	0	ba	0	0	b	ba^ω	$0'$	ba^ω	$0'$	$ba^\omega a$	$0'$	$ba^\omega a$
ab	aba	ab	0	aba	0	0	ab	ab^ω	$0'$	ab^ω	$0'$	$ab^\omega a$	$0'$	$ab^\omega a$
ba	b	0	b	0	ba	0	ba	ba^ω	ba^ω	$0'$	$0'$	$ba^\omega a$	$ba^\omega a$	$0'$
aba	ab	0	ab	0	aba	0	aba	ab^ω	ab^ω	$0'$	$0'$	$ab^\omega a$	$ab^\omega a$	$0'$
0	0	0	0	0	0	0	0	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$
1	a	b	ab	ba	aba	0	1	a^ω	ab^ω	ba^ω	$0'$	$a^\omega a$	$ab^\omega a$	$ba^\omega a$
a^ω	$a^\omega a$	a^ω	$0'$	$a^\omega a$	$0'$	$0'$	a^ω	a^ω	$0'$	a^ω	$0'$	$a^\omega a$	$0'$	$a^\omega a$
ab^ω	$ab^\omega a$	ab^ω	$0'$	$ab^\omega a$	$0'$	$0'$	ab^ω	ab^ω	$0'$	ab^ω	$0'$	$ab^\omega a$	$0'$	$ab^\omega a$
ba^ω	$ba^\omega a$	ba^ω	$0'$	$ba^\omega a$	$0'$	$0'$	ba^ω	ba^ω	$0'$	ba^ω	$0'$	$ba^\omega a$	$0'$	$ba^\omega a$
$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$
$a^\omega a$	a^ω	$0'$	a^ω	$0'$	$a^\omega a$	$0'$	$a^\omega a$	a^ω	a^ω	$0'$	$0'$	$a^\omega a$	$a^\omega a$	$0'$
$ab^\omega a$	ab^ω	$0'$	ab^ω	$0'$	$ab^\omega a$	$0'$	$ab^\omega a$	ab^ω	ab^ω	$0'$	$0'$	$ab^\omega a$	$ab^\omega a$	$0'$
$ba^\omega a$	ba^ω	$0'$	ba^ω	$0'$	$ba^\omega a$	$0'$	$ba^\omega a$	ba^ω	ba^ω	$0'$	$0'$	$ba^\omega a$	$ba^\omega a$	$0'$

and with $\omega : S_{\omega^0} \to S_{\omega^1}$

a	b	ab	ba	aba	0	1
a^ω	ba^ω	$0'$	$0'$	ab^ω	$0'$	a^ω

recognizes $X = ((aa+b)^\omega)^+(aa+b)^* + (aa+b)^+$. The morphism $\varphi : \overline{\Sigma}^1 \to S$ of ω^1-semigroups is defined by $\varphi(a) = a$, $\varphi(b) = b$ and using proposition 16. Clearly, $\varphi(X) = \{1, b, ba^\omega, a^\omega\}$. $\circ$

Example 21 An ω-semigroup $S = (S_f, S_\omega)$ is a restricted ω^1-semigroup such that the elements of S_ω are forbidden in the left member of the binary product (that is, S_ω is no more a semigroup but a set, and the product from $S_\omega \times S_f$ is not defined).
Here is an example of ω-semigroup:
Let $\Sigma = \{a, b\}$, $S = (\{s, t\}, \{u, v\})$ with the products: $ss = s$, $st = ts = tt = t$, $s^\omega = u$, $t^\omega = v$, $su = u, sv = tv = tu = v$, and $\varphi : (\Sigma^+, \Sigma^\omega) \to (S_f, S_\omega)$ be the morphism such that $\varphi(a) = s$ and $\varphi(b) = t$. Clearly,

$$\begin{aligned}
\varphi^{-1}(s) &= a^+ = \{\text{finite words without any } b \text{ letter}\} \\
\varphi^{-1}(t) = \Sigma^+ \setminus \varphi^{-1}(s) &= \{\text{finite words with at least a } b \text{ letter}\} \\
\varphi^{-1}(u) &= a^\omega = \{\omega\text{-words without any } b \text{ letter}\} \\
\varphi^{-1}(v) = \Sigma^\omega \setminus \varphi^{-1}(u) &= \{\omega\text{-words with at least a } b \text{ letter}\}
\end{aligned}$$

Thus, φ defines these sets and union of them.
Of course, if a set is recognizable by an ω-semigroup, it is recognizable by an ω^1-semigroup: we only have to define the undefined products to be equal to a new element of S_ω. $\circ$

Example 22 Let Σ be an alphabet, n an integer, X a subset of $\overline{\Sigma}^n$ and $(S_{\omega^0}, \ldots, S_{\omega^n})$ be an ω^n-semigroup recognizing X. For each integer $m \geq n$ any ω^m-semigroup $T = (S_{\omega^0}, \ldots, S_{\omega^n}, T_{\omega^{n+1}}, \ldots, T_{\omega^m})$ recognizes X, whatever are $T_{\omega^{n+1}}, \ldots, T_{\omega^m}$, if of course the intersection between products of T and products of S is equal to the products of S. $\circ$

3. Equivalence between n-automata and finite ω^n-semigroups

In this section we prove that the class of sets defined by finite ω^n-semigroups is the same as the class of sets defined by n-automata.

We first give an effective algorithm to compute an ω^n-Wilke algebra S from an n-automaton $\mathcal{A} = <\mathcal{S}, \mathcal{M}, s^\star, \mathcal{F}, \Sigma>$ such that S recognizes $\mathcal{L}(\mathcal{A})$. We use a classical idea, adapted to our case: the encoding of the automaton by matrices. To every sequence u of length $|u| = \sum_{i=m}^{0} \omega^i \cdot a_i$ less than ω^{n+1} we associate a square matrix $\mu(u)$, of size $t = (\sum_{0 \leq i \leq n} |[S]^i|)^2$, in which

$$\mu(u)_{p,q} = \begin{cases} \emptyset & \text{if } (p, u, q) \notin \overline{\mathcal{M}} \\ \{l_1, l_2, \ldots, l_r\} & \text{the set of all sets } l_i \text{ of elements of } [\mathcal{S}]_0^m \text{ appearing in paths} \\ & \text{from } p \text{ to } q \text{ labeled with } u \text{ otherwise} \end{cases}$$

Let M be the set of such matrices. The **type** of a non-empty component c of a matrix is the lowest integer m such that $c \in [\mathcal{S}]_2^m$. Observe that the components of a matrix that are not $\emptyset$ are all of the same type. Since we want μ to be a morphism of ω^n-semigroup, we split M into $n+1$ subsets denoted by M_{ω^i} $(i = 0 \ldots n)$ such that $m \in M_{\omega^i}$ iff

$$m_{p,q} \in \begin{cases} [\mathcal{S}]_2^{i+2} \cup \{\emptyset\} & \text{if } t(q) \leq i \\ \{\emptyset\} & \text{if } t(q) > i \end{cases}$$

In fact, $\mu(u) \in M_{\omega^i}$ iff $\omega^i \leq |u| < \omega^{i+1}$. The **type** of a matrix m, $t(m)$, is the integer i. For convenience we say that the matrix composed only with $\emptyset$ has all types.

We need the following operation in order to equip M with a product.

Definition 23 Let r and r' be two integers, and l_j (resp. $l'_{j'}$), with $1 \leq j \leq r$ (resp. r'), a set of elements of $[\mathcal{S}]_0^i$ (resp. $[\mathcal{S}]_0^j$) for some $i \leq n$ (resp. $j \leq n$). We define the $\sqcup$ operator as

$$\sqcup \begin{cases} \emptyset \sqcup a = a \sqcup \emptyset = \emptyset \\ \{l_1, \ldots, l_r\} \sqcup \{l'_1, \ldots, l'_{r'}\} = \{l_1 \cup l'_1, \ldots, l_1 \cup l'_{r'}, \ldots, l_r \cup l'_1, \ldots, l_r \cup l'_{r'}\} \end{cases}$$

Proposition 24 *The $\sqcup$ operator is commutative and associative.*

Proof The commutativity is obvious. Let a, b, c be sets of sets. If $a \sqcup (b \sqcup c) = \emptyset$ then one of a, b, c is $\emptyset$, and $(a \sqcup b) \sqcup c = \emptyset$. Assume now that they are all three non-empty sets and $a \sqcup (b \sqcup c) = \{l_1, \ldots, l_m\}$ for an integer m. Then for all i in $1 \ldots m$ $l_i = l_i^a \cup l_i^{bc}$ where $l_i^a \in a$ and $l_i^{bc} \in b \sqcup c$, and $l_i^{bc} = l_i^b \cup l_i^c$ where $l_i^b \in b$ and $l_i^c \in c$. Since $\cup$ is associative it comes $l_i \in (a \sqcup b) \sqcup c$. The proof of the converse is similar. ∇

We can now equip M with a product $\odot$: if a and b are both elements of M,

$$(a \odot b)_{p,q} = \bigcup_{k \in [\mathcal{S}]_0^n} (a_{p,k} \sqcup b_{k,q})$$

Proposition 25 *If $a \in M_{\omega^i}$ and $b \in M_{\omega^j}$ then $a \odot b \in M_{\omega^{max(i,j)}}$.*

Proof Let $k \in [\mathcal{S}]_0^n$, $a_{p,k} = \{l_1, \ldots, l_r\} \in [\mathcal{S}]_2^{i+2}$, $b_{k,q} = \{l'_1, \ldots, l'_{r'}\} \in [\mathcal{S}]_2^{j+2}$. Immediately, $l \cup l' \in [\mathcal{S}]_1^{max(i,j)+1}$ for each $l \in a_{p,k}$ and $l' \in b_{k,q}$. So $a_{p,k} \sqcup b_{k,q} \in [\mathcal{S}]_2^{max(i,j)+2}$ and $\cup_{k \in [\mathcal{S}]_0^n}(a_{p,k} \sqcup b_{k,q}) \in [\mathcal{S}]_2^{max(i,j)+2}$ because obviously the $\cup$ operator is compatible with the type. If $\forall k \in [\mathcal{S}]_0^n$ $a_{p,k} = \emptyset$ or $b_{k,q} = \emptyset$ then $a_{p,k} \sqcup b_{k,q} = \emptyset$ and $\cup_{k \in [\mathcal{S}]_0^n}(a_{p,k} \sqcup b_{k,q}) = \emptyset$. So $\cup_{k \in [\mathcal{S}]_0^n}(a_{p,k} \sqcup b_{k,q}) \in [\mathcal{S}]_2^{max(i,j)+2} \cup \{\emptyset\}$. Now, if $t(q) > max(i,j)$, we have $t(q) > j$ so $\forall k \in [\mathcal{S}]_0^n$ $b_{k,q} = \emptyset$ and $(a \odot b)_{p,q} = \emptyset$. ∇

Proposition 26 *Let m be an integer, a_i for $1 \leq i \leq m$ and b be sets of sets. Then $(\cup_{1 \leq i \leq m} a_i) \sqcup b = \cup_{1 \leq i \leq m}(a_i \sqcup b)$.*

Proof If $b = \emptyset$ then the left and right members of the equation are both equal to $\emptyset$. If $\cup_{1 \leq i \leq m} a_i = \emptyset$ then each $a_i = \emptyset$ and so are both members of the equation. Otherwise, neither the right member of the equation nor the left one can be equal to $\emptyset$. Now in order to prove that the left member is included in the right one, assume that $l \in (\cup_{1 \leq i \leq m} a_i) \sqcup b$. Then $l = l_1 \cup l_2$ such that $l_1 \in a_r$ for $1 \leq r \leq m$ and $l_2 \in b$. So $l_1 \cup l_2 \in a_r \sqcup b$ and $l \in \cup_{1 \leq i \leq m}(a_i \sqcup b)$. The proof of the converse is similar. ∇

Proposition 27 *Let $a \in M_{\omega^{i_1}}$, $b \in M_{\omega^{i_2}}$, $c \in M_{\omega^{i_3}}$. Then $((a \odot b) \odot c)_{p,q} = (a \odot (b \odot c))_{p,q}$.*

Proof

$$\begin{aligned} ((a \odot b) \odot c)_{p,q} &= \bigcup_{k \in [\mathcal{S}]_0^n} ((\bigcup_{k' \in [\mathcal{S}]_0^n} (a_{p,k'} \sqcup b_{k',k})) \sqcup c_{k,q}) = \bigcup_{k \in [\mathcal{S}]_0^n} (\bigcup_{k' \in [\mathcal{S}]_0^n} ((a_{p,k'} \sqcup b_{k',k}) \sqcup c_{k,q})) \\ &= \bigcup_{k \in [\mathcal{S}]_0^n} (\bigcup_{k' \in [\mathcal{S}]_0^n} (a_{p,k'} \sqcup (b_{k',k} \sqcup c_{k,q}))) = \bigcup_{k' \in [\mathcal{S}]_0^n} (\bigcup_{k \in [\mathcal{S}]_0^n} (a_{p,k'} \sqcup (b_{k',k} \sqcup c_{k,q}))) \\ &= \bigcup_{k' \in [\mathcal{S}]_0^n} (a_{p,k'} \sqcup \bigcup_{k \in [\mathcal{S}]_0^n} (b_{k',k} \sqcup c_{k,q})) = (a \odot (b \odot c))_{p,q} \end{aligned}$$

∇

Proposition 28 *M equipped with its product $\odot$ is a finite semigroup.*

Proof It follows from its definition than M is finite and from the previous proposition that M is a semigroup. ∇

For short we now omit the $\odot$ notation.

Lemma 29 *M_{ω^i} is an ideal for $\cup_{j=0\ldots i} M_{\omega^j}$.*

Proof This directly comes from the previous proposition and proposition 25. ∇

Now we give a method to obtain $s^\omega \in M_{\omega^{i+1}}$ from a matrix $s \in M_{\omega^i}$. An **s-path** from p to q is an ω-sequence of states $t_0, t_1, \ldots, t_j, t_{j+1}, \ldots$ such that $t_0 = p$, for all integers j $s_{t_j,t_{j+1}} \neq \emptyset$, and after an integer k', $\forall k > k'$,

$$q \in \bigsqcup_{j>k} \mathcal{P}_i(s_{t_j,t_{j+1}})$$

where $\mathcal{P}_i(\{l_1, \ldots, l_m\}) = \{\mathcal{P}'_i(l_1), \ldots, \mathcal{P}'_i(l_m)\}$, $\mathcal{P}'_i(l) = [\mathcal{S}]^i \cap l$. Clearly, an s-path expresses the notion of infinite repetition of states of $[\mathcal{S}]^i$.

If s is a matrix of M_{ω^i}, we define s^{ω_i} by: a subset l of $[S]_0^i \cup \{q\}$ belongs to $s^{\omega_i}_{p,q}$ if and only if there exists an s-path $t_0, t_1, t_2, \ldots$ from p to q, a sequence $(l_i)_{i<\omega}$ and an integer k' such that

- $l = \{q\} \cup \cup_{j \in \mathbb{N}} l_j$,
- $l_j \in s_{t_j,t_{j+1}}$ for any integer j,
- $q = \cup_{j>k} \mathcal{P}'_i(l_j)$ for any integer $k > k'$.

In order to prove that this defines a Wilke operator, we need a few easy properties.

Proposition 30 *Let l be a non-empty set, k an integer, and $(a_i)_{1\leq i\leq 4}$ a family of components of a matrix. Then*

(1) $l \in a_1 \Rightarrow \mathcal{P}'_k(l) \in \mathcal{P}_k(a_1)$
(2) $\mathcal{P}_k(a_1 \sqcup a_2) = \mathcal{P}_k(a_1) \sqcup \mathcal{P}_k(a_2)$
(3) $a_1 \subseteq a_3$ *and* $a_2 \subseteq a_4 \Rightarrow a_1 \sqcup a_2 \subseteq a_3 \sqcup a_4$
(4) $a_1 \subseteq a_2 \Rightarrow \mathcal{P}_k(a_1) \subseteq \mathcal{P}_k(a_2)$

We use these properties in the proof of the following lemma:

Lemma 31 *The ω operator which associate to every matrix s the matrix s^ω verifies the properties (W_1) and (W_2) of a Wilke ω operator.*

Proof We first prove that for every integer n, $(s^n)^\omega_{p,q} \subseteq s^\omega_{p,q}$. Assume that $l \in (s^n)^\omega_{p,q}$, $l' = l\backslash\{q\}$. There exists an s^n-path from p to q satisfying the conditions above. We write

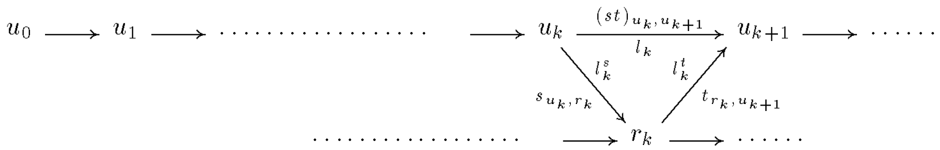
$$t_0 \xrightarrow{l_0} t_1 \xrightarrow{l_1} \cdots \xrightarrow{l_{k-1}} t_k \xrightarrow{l_k} t_{k+1} \xrightarrow{l_{k+1}} \cdots$$

We choose a sequence $(l_j)_{j\geq 0}$ such that $l_j \in s^n_{t_j,t_{j+1}}$, $q = \cup_{j>k} \mathcal{P}'_i(l_j)$, $l' = \cup_{j\geq 0} l_j$. From each l_j we can build a sequence of $n+1$ states $(r^j_m)_{0\leq m\leq n}$ such that for all j $r^j_0 = t_j$, $r^j_n = t_{j+1}$, for all $0 \leq m < n$ $s_{r^j_m, r^j_{m+1}} \neq \emptyset$ and $l_j \in \sqcup_{0\leq m<n} s_{r^j_m, r^j_{m+1}}$. Consider the ω-sequence $r^0_0 r^0_1 \ldots r^0_n r^1_0 r^1_1 \ldots$. Since $l_j \in \sqcup_{0\leq m<n} s_{r^j_m, r^j_{m+1}}$, $l' = \cup_{j\geq 0} l_j \in \sqcup_{j\geq 0} \sqcup_{0\leq m<n} s_{r^j_m, r^j_{m+1}} = \sqcup_{j\geq 0} s_{r_j, r_{j+1}}$. On the other hand, $l_j \in \sqcup_{0\leq m<n} s_{r^j_m, r^j_{m+1}} \Rightarrow \mathcal{P}'_i(l_j) \in \mathcal{P}_i(\sqcup_{0\leq m<n} s_{r^j_m, r^j_{m+1}}) \Rightarrow q = \cup_{j>k} \mathcal{P}'_i(l_j) \in \sqcup_{j>k} \mathcal{P}_i(\sqcup_{0\leq m<n} s_{r^j_m, r^j_{m+1}}) = \sqcup_{j>k} \sqcup_{0\leq m<n} \mathcal{P}_i(s_{r^j_m, r^j_{m+1}}) = \sqcup_{j>k} \mathcal{P}_i(s_{r^j_m, r^j_{m+1}})$. So our sequence is an s-path from p to q and $l \in s^\omega_{p,q}$.

Assume now that $l \in s^\omega_{p,q}$ and $l' = l\backslash\{q\}$. There exists an s-path $t_0 t_1 \ldots$ from p to q. Consider the ω-sequence $t_0, t_n, t_{2n}, \ldots$. Since $\sqcup_{0\leq m<n} s_{t_{nj+m}, t_{nj+m+1}} \subseteq s^n_{t_{nj}, t_{n(j+1)}}$, $l' \in \sqcup_{j\geq 0} s_{t_j, t_{j+1}} = \sqcup_{j\geq 0} \sqcup_{0\leq m<n} s_{t_{nj+m}, t_{nj+m+1}} \subseteq \sqcup_{j\geq 0} s^n_{t_{nj}, t_{n(j+1)}}$. On the other hand, using the same kind of argument, $q \in \sqcup_{j>k} \mathcal{P}_i(s_{t_j, t_{j+1}})$

$= \mathcal{P}_i(\sqcup_{j>k} s_{t_j,t_{j+1}}) = \mathcal{P}_i(\sqcup_{j>k} \sqcup_{0\leq m<n} s_{t_{nj+m},t_{nj+m+1}}) \subseteq \mathcal{P}_i(\sqcup_{j>k} s^n_{t_{nj},t_{n(j+1)}}) = \sqcup_{j>k}\mathcal{P}_i(s^n_{t_{nj},t_{n(j+1)}})$. So our sequence is an s^n-path from p to q and $l \in (s^n)^\omega_{p,q}$.

Now we prove that $(s(ts)^\omega)_{p,q} = (st)^\omega_{p,q}$. First assume that $l \in (st)^\omega_{p,q}$. Let $u_0, u_1, \ldots$ be the st-path from p to q and $l' = l\backslash\{q\}$. There exists an ω-sequence $(r_m)_{m\geq 0}$ such that $s_{u_m,r_m} \neq \emptyset$, $t_{r_m,u_{m+1}} \neq \emptyset$, $s_{u_m,r_m} \sqcup t_{r_m,u_{m+1}} \subseteq (st)_{u_m,u_{m+1}}$, $q \in \sqcup_{j>k}\mathcal{P}_i(s_{u_j,r_j} \sqcup t_{r_j,u_{j+1}})$ and $l' \in \sqcup_{j\geq 0}(s_{u_j,r_j} \sqcup t_{r_j,u_{j+1}})$. Consider the ω-sequence $r_0u_1r_1u_2r_2\ldots$. Since we have $q \in \sqcup_{j>k+1}\mathcal{P}_i(t_{r_j,u_{j+1}} \sqcup s_{u_{j+1},r_{j+1}}) \subseteq \sqcup_{j>k+1}\mathcal{P}_i(ts_{r_j,r_{j+1}})$, this sequence is a ts-path from r_0 to q and $(\sqcup_{j\geq 0}(t_{r_j,u_{j+1}} \sqcup s_{u_{j+1},r_{j+1}})) \sqcup \{\{q\}\} \in (ts)^\omega_{r_0,q}$. Now, since $s_{p,r_0} \neq \emptyset$ it comes $l \in s_{p,r_0} \sqcup ((\sqcup_{j\geq 0}(t_{r_j,u_{j+1}} \sqcup s_{u_{j+1},r_{j+1}})) \sqcup \{\{q\}\}) \subseteq (s(ts)^\omega)_{p,q}$.



Finally assume that $l \in (s(ts)^\omega)_{p,q}$. According to the definition of the product of matrices this implies that there exists a state r such that $s_{p,r} \neq \emptyset$ and $l \in s_{p,r} \sqcup (ts)^\omega_{r,q}$. Using the same argument as in the previous part of this proof one can show that there exists a state r' such that $t_{r,r'} \neq \emptyset$ and $l \in s_{p,r} \sqcup t_{r,r'} \sqcup (st)^\omega_{r',q}$, i.e. there exists an st-path $r', u_1, u_2, \ldots$ from r' to q. So $(st)_{p,r'} \neq \emptyset$, the ω-sequence $p, r', u_1, u_2, \ldots$ is an st-path from p to q and $l \in (st)^\omega_{p,q}$. This finishes the proof of the lemma. ∇

Lemma 32 *$(M_{\omega^0}, \ldots, M_{\omega^n})$ equipped with the $\odot$ product and with a family of functions $\omega_i : M_{\omega^i} \to M_{\omega^{i+1}}$ defined as in lemma 31 is an ω^n-Wilke algebra.*

Proof It is a direct consequence of proposition 28 and lemmas 29 and 31. ∇

Lemma 33 *Let $\mathcal{A} =< \mathcal{S}, \mathcal{M}, i, \mathcal{F}, \Sigma >$ be an n-automaton and $S = \mu(\overline{\Sigma}^n)$. Then S recognizes $\mathcal{L}(\mathcal{A})$.*

Proof Let $P = \{s \in S / \exists f \in \mathcal{F}\ s_{i,f} \neq \emptyset\}$. Then $\mu(u) \in P$ iff $u \in \mathcal{L}(\mathcal{A})$, so S recognizes $\mathcal{L}(\mathcal{A})$. ∇

From a given n-automaton $\mathcal{A}$ we have built an ω^n-Wilke algebra S such that S recognizes $\mathcal{L}(\mathcal{A})$. To finish the proof of the "equivalence" between Choueka automata and finite ω^n-semigroups we show that languages recognized by ω^n-semigroups are n-regular.

Lemma 34 *Let Σ an alphabet, n an integer, S a finite ω^n-semigroup, $\varphi : \overline{\Sigma}^n \to S$ a morphism of ω^n-semigroups and u a word on Σ such that $|u| = \Sigma^0_{i=m}\omega^i \cdot a_i$ for an integer $n \geq m > 0$, with $a_m \neq 0$.*

- *if $|u| = \omega^m$, there exists a factorisation of $u = u_0u_1\ldots$ such that $\varphi(u_0) = s$, for every integer $r > 0$, $\varphi(u_r) = e$ with $s, e \in S_{\omega^{m-1}}$ such that $se = s$ and $e^2 = e$.*
- *if $|u| > \omega^m$, there exists a factorization of $u = u_0u_1\ldots u_\omega$ such that $\varphi(u_0) = s$, for every integer $r > 0$, $\varphi(u_r) = e$ and $\varphi(u_\omega) = t$ with $s, e \in S_{\omega^{m-1}}$ such that $se = s$ and $e^2 = e$ and $t \in S_{\omega^j}$ for $j \leq m$.*

Proof Let u, φ, Σ, S be as in the statement, and $v_0 = u[0, \omega^{m-1}[$, $v_1 = u[\omega^{m-1}, \omega^{m-1}\cdot 2[, \ldots, v_i = u[\omega^{m-1}\cdot i, \omega^{m-1}\cdot(i+1)[\ldots$. The sequence $v_0v_1\ldots$ is an ω-word on an infinite alphabet. Using theorem 9 it comes immediately that $u[0,\omega^m[$ can be factorized as in the statement. If $|u| > \omega^m$ it is clear that $\varphi(u_\omega = u[\omega^m, |u|[) \in S_{\omega^j}$ for some $j \leq m$. ∇

Let n be an integer and S an ω^n-semigroup. In order to clarify the notations we write S^1 when we add a new element noted 1 to S_{ω^0} such that for every $s \in S$, $1 \cdot s = s \cdot 1 = s$. Since it is clear that if there exist another ω^n-semigroup T and a morphism of ω^n-semigroups $\varphi : T \to S$ the reverse image $\varphi^{-1}(1)$ is not defined we do not need to define 1^ω. Finally, if r is a k-regular expression for some integer k, we say that $r \cdot \varphi^{-1}(1) = r$.

Lemma 35 *Let n and i be integers such that $i \leq n$, Σ an alphabet, S a finite ω^n-semigroup, $\varphi : \overline{\Sigma}^n \to S$ a morphism of ω^n-semigroups and X a subset of $(\overline{\Sigma}^n)_i$ recognized by φ. Then X is i-regular.*

Proof Let X, Σ, S, φ, n and i be as in the statement and $P = \varphi(X)$.

If $i = 0$, since there is no ω operator from S_{ω^j} $(j > 0)$ into S_{ω^0} and S_{ω^j} is an ideal for $\cup_{k=0\dots j} S_{\omega^k}$, we can refer to the theory of semigroups recognizing finite words. It is a well-known result of this theory that X is 0-regular.

Let us turn to the case $i > 0$. We prove that

$$X = \bigcup_{(s,e,t)\in Y_P} \varphi^{-1}(s)\varphi^{-1}(e)^{\omega}\varphi^{-1}(t) \qquad (A)$$

where Y_P is the set of triples $(s, e, t) \in S_{\omega^{i-1}} \times S_{\omega^{i-1}} \times S^1$ such that $se = s$, $e^2 = e$, $se^{\omega}t \in P$ and $ty(t) \leq i$. Let $u \in \cup_{(s,e,t)\in Y_P} \varphi^{-1}(s)\varphi^{-1}(e)^{\omega}\varphi^{-1}(t)$, immediately $\varphi(u) = se^{\omega}t \in P$, so $u \in X$.
Now we turn to the converse. Let $u \in X$. Using lemma 34 there exists $(s, e, t) \in Y_P$ such that $\varphi(u) = se^{\omega}t$. It remains to prove that the right member of (A) is i-regular. Since the unions are finite, we just need to show that $\varphi^{-1}(s)\varphi^{-1}(e)^{\omega}\varphi^{-1}(t)$ is i-regular, by induction on i. First of all note that if $(s, e, t) \in Y_P$ and $se^{\omega}t = r$, then $ty(s) = ty(e) < ty(r)$ and $ty(t) \leq ty(r)$. If $i = 1$ then s and e are elements of S_{ω^0} and it is a classical result on theory of semigroups that $\varphi^{-1}(s)$ and $\varphi^{-1}(e)$ are 0-regular. If $t \in S_{\omega^0}$ then $\varphi^{-1}(t)$ is also 0-regular so $\varphi^{-1}(s)\varphi^{-1}(e)^{\omega}\varphi^{-1}(t)$ is 1-regular. If $t = 1$ then $\varphi^{-1}(s)\varphi^{-1}(e)^{\omega}\varphi^{-1}(t) = \varphi^{-1}(s)\varphi^{-1}(e)^{\omega}$ is 1-regular. If $t \in S_{\omega^1}$ then let $x = \varphi^{-1}(s)\varphi^{-1}(e)^{\omega}\varphi^{-1}(t)$. Let $t = t_0$. There exists Y_{t_0} a set of triples $(s_1, e_1, t_1) \in S_{\omega^0} \times S_{\omega^0} \times S^1$ of elements of S such that $s_1e_1 = s_1$, $e_1^2 = e_1$, $s_1e_1^{\omega}t_1 = t_0$ and $ty(t_1) \leq 1$. The following equation holds: $\varphi^{-1}(t_0) = \bigcup_{(s_1,e_1,t_1)\in Y_{t_0}} \varphi^{-1}(s_1)\varphi^{-1}(e_1)^{\omega}\varphi^{-1}(t_1)$. For each $t_1 \in Y_{t_0}$ we use the same argument: $\varphi^{-1}(t_1) = \bigcup_{(s_2,e_2,t_2)\in Y_{t_1}} \varphi^{-1}(s_2)\varphi^{-1}(e_2)^{\omega}\varphi^{-1}(t_2)$ where Y_{t_1} is the set of triples $(s_2, e_2, t_2) \in S_{\omega^0} \times S_{\omega^0} \times S^1$ such that $s_2e_2 = s_2$, $e_2^2 = e_2$, $s_2e_2^{\omega}t_2 = t_1$ and $ty(t_2) \leq 1$, and so on a finite number of time because S_{ω^1} is finite. Since each $\varphi^{-1}(s_i)$ and $\varphi^{-1}(e_i)$ is 0-regular we have a finite system of right linear equations built with $\{\varphi^{-1}(s)\varphi^{-1}(e)^{\omega}$ for every pair (s, e) of each $Y_{t_i}\}$ as the finite set of terminals. According to theorem 8 the solutions of such a system are regular expressions built on the set of terminals, which are 1-regular expressions, so the solutions are 1-regular expressions. The rest of the proof uses the same argument. ∇

We can now give the main result of the paper.

Theorem 36 *Choueka n-automata are effectively equivalent to finite ω^n-semigroups, that is to say, if Σ is an alphabet and n an integer, with any Choueka automaton $\mathcal{A}$ that recognizes a subset of $\overline{\Sigma}^n$ we can effectively associate a finite ω^n-semigroup that recognizes $\mathcal{L}(\mathcal{A})$, and with any finite ω^n-semigroup that recognizes a subset X of $\overline{\Sigma}^n$ we can effectively associate a Choueka automaton $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = X$.*

Proof Let Σ be an alphabet, n an integer and $\mathcal{A}$ an n-automaton that recognizes a subset of $\overline{\Sigma}^n$. Lemma 33 gives an effective method to obtain an ω^n-Wilke algebra S such that S recognizes $\mathcal{L}(\mathcal{A})$. Now let $X = (X_{\omega^0}, \dots, X_{\omega^n})$ be a subset of $\overline{\Sigma}^n$ recognized by a finite ω^n-semigroup. According to lemma 35 each non-empty X_{ω^i}, for $0 \leq i \leq n$, is i-regular, so $\cup_{0\leq i\leq n} X_{\omega^i}$ is n-regular. The obtaining of n-regular expressions is effective. The theorem holds since n-regular expressions are equivalent to n-automata, and one can verify in [6] that this step is also effective. ∇

4. Syntactic ω^n-semigroup

Let n be an integer and S an ω^n-semigroup. In this section we prove that to every recognizable subset X of S one can effectively attach a canonical finite ω^n-semigroup S' and a morphism $\varphi : S \to S'$ such that φ recognizes X. Since S' is the quotient of any ω^n-semigroup that recognizes X under a congruence of ω^n-semigroup, it follows that S' has less elements than the others, and is unique. S' is called the syntactic ω^n-semigroup of X.

First of all, we define a congruence of ω^n-Wilke algebra, that is, an equivalence relation compatible with the finite products and the ω operator.

Lemma 37 *Let n be an integer, S an ω^n-Wilke algebra and P a subset of S. The equivalence relation $\sim_P$ defined by, for every integer i less than $n+1$ and $x, y \in S_{\omega^i}$, $x \sim_P y$ if, for all $r, t \in S^1$,*

$$rxt \in P \iff ryt \in P \tag{A}$$

and, for every integer m and $y_0, y_1, \ldots, y_m \in S^1$ such that $y_0(\ldots(((xy_1)^\omega y_2)^\omega y_3)^\omega \ldots)^\omega y_m \in S$

$$y_0(\ldots(((xy_1)^\omega y_2)^\omega y_3)^\omega \ldots)^\omega y_m \in P \iff y_0(\ldots(((yy_1)^\omega y_2)^\omega y_3)^\omega \ldots)^\omega y_m \in P \tag{B}$$

is a congruence of ω^n-Wilke algebra.

Proof We first show that $x \sim_P y \Rightarrow \forall r, t \in S^1\ rxt \sim_P ryt$. Assume $x \sim_P y$. Let $r, t \in S^1$. According to (A) we have $\forall r', t' \in S^1\ r'xt' \in P \iff r'yt' \in P$. In particular, $\forall r', t' \in S^1\ (r'r)x(tt') \in P \iff (r'r)y(tt') \in P$. Since the product is associative, $\forall r', t' \in S^1\ r'(rxt)t' \in P \iff r'(ryt)t' \in P$. This proves (A). Let us turn to (B). We have, for every integer m and $y_0, y_1, \ldots, y_m \in S^1$ such that $y_0(\ldots(((xy_1)^\omega y_2)^\omega y_3)^\omega \ldots)^\omega y_m \in S$

$$y_0((\ldots(((xy_1)^\omega y_2)^\omega y_3)^\omega \ldots)^\omega y_{m-1})^\omega y_m \in P \iff y_0((\ldots(((yy_1)^\omega y_2)^\omega y_3)^\omega \ldots)^\omega y_{m-1})^\omega y_m \in P$$

substituting y_i by $y_i r$ for every $0 \leq i < m$, and y_1 by ty_1 in the new equivalence

$$y_0 r((\ldots(((xty_1 r)^\omega y_2 r)^\omega y_3 r)^\omega \ldots)^\omega y_{m-1} r)^\omega y_m \in P \iff y_0 r((\ldots(((yty_1 r)^\omega y_2 r)^\omega y_3 r)^\omega \ldots)^\omega y_{m-1} r)^\omega y_m \in P$$

that is to say

$$y_0(r(\ldots(((xty_1 r)^\omega y_2 r)^\omega y_3 r)^\omega \ldots)^\omega y_{m-1})^\omega y_m \in P \iff y_0(r(\ldots(((yty_1 r)^\omega y_2 r)^\omega y_3 r)^\omega \ldots)^\omega y_{m-1})^\omega y_m \in P$$

applying the same argument $m-2$ times

$$y_0((\ldots(((rxty_1)^\omega y_2)^\omega y_3)^\omega \ldots)^\omega y_{m-1})^\omega y_m \in P \iff y_0((\ldots(((ryty_1)^\omega y_2)^\omega y_3)^\omega \ldots)^\omega y_{m-1})^\omega y_m \in P$$

so $x \sim_P y \Rightarrow \forall r, t \in S^1\ rxt \sim_P ryt$.
It remains to verify that if $i < n$ then $x \sim_P y \Rightarrow x^\omega \sim_P y^\omega$. Assume that $x \sim_P y$. Then, for every $y_0, y_1, y_2 \in S^1$ such that $y_0(xy_1)^\omega y_2 \in S$, i.e. $\forall y_0, y_1, y_2 \in S^1$ such that $ty(y_1) < n$, we have $y_0(xy_1)^\omega y_2 \in P \iff y_0(yy_1)^\omega y_2 \in P$. It suffices to take $y_1 = 1$ to have obtain (A). The proof of (B) uses exactly the same argument. ∇

Clearly, the set $S/\sim_P$ of congruence classes of an ω^n-Wilke algebra S has a structure of ω^n-Wilke algebra and the application which associates to every element of S its congruence class is a morphism of ω^n-Wilke algebra. Intuitively, the syntactic ω^n-semigroup $S/\sim_X$ of a recognizable subset X of an ω^n-semigroup S is obtained by an application of the previous congruence of ω^n-Wilke algebra to the ω^n-Wilke algebra associated with any finite ω^n-semigroup that recognizes X. The map which associate to any element of S its congruence class in $S/\sim_X$ is a morphism of ω^n-semigroup that recognizes X. This justifies the following proposition:

Proposition 38 *Let n be an integer, T an ω^n-semigroup, and X a recognizable subset of T. Among all congruences $\sim_X$ such that $T/\sim_X$ recognizes X, there exists an unique one from which all others are refinements. The number of classes of this congruence, which is minimal, is finite. This congruence of ω^n-semigroup, called* **syntactic congruence of** *X, is defined by: for any integer i less than $n+1$ and $x, y \in T_{\omega^i}$, $x \sim_X y$ if, for all $r, t \in T^1$,*

$$rxt \in X \iff ryt \in X \tag{A}$$

and, for every integer m and $y_0, y_1, \ldots, y_m \in T^1$ such that $y_0(\ldots(((xy_1)^\omega y_2)^\omega y_3)^\omega \ldots)^\omega y_m \in T$

$$y_0(\ldots(((xy_1)^\omega y_2)^\omega y_3)^\omega \ldots)^\omega y_m \in X \iff y_0(\ldots(((yy_1)^\omega y_2)^\omega y_3)^\omega \ldots)^\omega y_m \in X \tag{B}$$

Proof We first have to prove that we have really defined a congruence of ω^n-semigroup. Let S and $\varphi : T \to S$ a finite ω^n-semigroup and an onto morphism of ω^n-semigroups such that φ recognizes X. Let $P = \varphi(X)$. In order to prove the proposition easily we show the following lemma:

Lemma 39 *For every $u, v \in T$, $u \sim_X v \iff \varphi(u) \sim_P \varphi(v)$.*

Proof First of all the $\Rightarrow$ direction. For every $r, t \in S^1$, there exist $r', t' \in T^1$ such that $\varphi(r') = r$ and $\varphi(t') = t$ (we assume that $\varphi(1) = 1$). So, $r\varphi(u)t \in P \iff r'ut' \in X \iff r'vt' \in X \iff r\varphi(v)t \in P$. We can prove the second equation using the same argument. Now we turn to the converse. Assume that $\varphi(u) \sim_P \varphi(v)$. This implies that $\forall r, t \in S^1$ $r\varphi(u)t \in P \iff r\varphi(v)t \in P$. For every $r, t \in S^1$,

$$rut \in X \iff \varphi(rut) \in P \iff \varphi(r)\varphi(u)\varphi(t) \in P \iff \varphi(r)\varphi(v)\varphi(t) \in P \iff rvt \in X$$

So the first equivalence is proved. The second one can be proved using the same argument. ∇

We return to the proof of proposition 38. For the proof of $u \sim_X v \Rightarrow \forall r, t \in T^1$ $rut \sim_X rvt$ we refer to the first part of the proof of lemma 37. It remains to prove that $\sim_X$ is compatible with the infinite product. Let $(u_i)_{i\in\mathbb{N}}$ and $(v_i)_{i\in\mathbb{N}}$ be two sequences of elements of T such that for every integer i $u_i \sim_X v_i$ and assume that $u_0u_1\ldots$ is defined (it is clear that if $u_0u_1\ldots$ is defined then so is $v_0v_1\ldots$). It follows from theorem 9 that there exist two pairs (s, e) and (s', e') of elements of S^2 such that $se = s$, $e^2 = e$, $s'e' = s'$, ${e'}^2 = e'$, and $(\varphi \times \varphi)((u_0u_1\ldots, v_0v_1\ldots)) = (s, s')(e, e')^\omega$. Let $r, t \in T^1$. Since $\varphi(u_i) \sim_P \varphi(v_i)$ for every integer i we have

$$ru_0u_1\ldots t \in X \iff \varphi(r)se^\omega\varphi(t) \in P \iff \varphi(r)s'{e'}^\omega\varphi(t) \in P \iff rv_0v_1\ldots t \in X$$

Thus, (A) is proved. The proof of (B) uses the same argument. Finally, we have to prove that there does not exists another congruence of ω^n-semigroup $\sim'_X$ and another morphism of ω^n-semigroup $\varphi' : S \to S/\sim'_X$ such that φ' recognizes X and $S/\sim'_X$ has less elements than $S/\sim_X$. Assume that $\sim'_X$ and φ' exist, and let $\varphi'(X) = P'$. Then, $\varphi'(x) = \varphi'(y) \iff x \sim'_X y$. Assume $x \sim'_X y$ and let $u, v \in T^1$. We can write

$$uxv \in X \iff \varphi'(u)\varphi'(x)\varphi'(v) \in P' \iff \varphi'(u)\varphi'(y)\varphi'(v) \in P' \iff uyv \in X$$

which proves the first equivalence of $x \sim_X y$. The proof of the second equation is similar. Thus $x \sim'_X y \Rightarrow x \sim_X y$, so $\sim'_X$ is a refinement of $\sim_X$. In particular, $\sim_X$ has less equivalence classes than $\sim'_X$ if $\sim_X$ is different from $\sim'_X$. Lemma 39 shows that the number of such classes is finite. This finishes the proof of the proposition. ∇

In example 20, the ω^1-semigroup associated with S is the syntactic ω^1-semigroup of X.

This result can for example be used to give another proof of the following theorem:

Theorem 40 *Let $\mathcal{A}$ and $\mathcal{B}$ be two Choueka n-automata. The question: $\mathcal{L}(\mathcal{A})$=$\mathcal{L}(\mathcal{B})$? is decidable.*

Proof According to theorem 36 we can effectively build two ω^n-Wilke algebras S_A and S_B such that S_A recognizes $\mathcal{L}(\mathcal{A})$ and S_B recognizes $\mathcal{L}(\mathcal{B})$. Let P_A (resp. P_B) be the subset of S_A (resp. S_B) defined as in proof of lemma 33 and P'_A (resp. P'_B) the equivalence classes of P_A (resp. P_B) in $S_A/\sim_{P_A}$ (resp. $S_B/\sim_{P_B}$). Then $\mathcal{L}(\mathcal{A})$=$\mathcal{L}(\mathcal{B})$ iff there exists a bijective morphism of ω^n-Wilke algebras $\varphi : S_A/\sim_{P_A} \to S_B/\sim_{P_B}$ such that $\varphi(P'_A) = P'_B$. ∇

5. Conclusion

The natural developments of this work are the generalization of the results on hierarchies of languages on finite words to words of length less than ω^n. We started this by generalizing Schützenberger's result on finite words in [2]:

Definition 41 Let Σ be an alphabet and n a positive integer. The class $SF(\Sigma,<\omega^n)$ of **star-free sets of transfinite words** on Σ is the smallest set containing all $\{a\}$ for $a \in \Sigma$ and closed under finite union, complement with respect to $\Sigma^{<\omega^n}$ and product.

Theorem 42 *Let Σ be an alphabet, n a positive integer and X a recognizable subset of $\Sigma^{<\omega^n}$. The following conditions are equivalent:*

- $X \in SF(\Sigma,<\omega^n)$,
- $\Sigma^{<\omega^n}/\sim_X$ *is aperiodic,*
- X *is definable by a first order sentence.*

Acknowledgments

The author wishes to express his thanks to Dominique Perrin for suggesting the problem and directing the researches.

References

[1] N. BEDON: Finite automata and ordinals, *Theoretical Computer Science*, **156**, (1996), 119–144.
[2] N. BEDON: Star-free sets of words on ordinals, to appear.
[3] J.R. BÜCHI: On a decision method in the restricted second-order arithmetic, *Logic, Methodology and Philosophy of science: Proc. Intern. Congr.*, Stanford University Press, (1962), 1–11.
[4] J.R. BÜCHI: Transfinite automata recursions and weak second order theory of ordinals, *Proc. Int. Congress Logic, Methodology, and Philosophy of Science*, North-Holland Publ. Co., (1964), 2–23, Invited address.
[5] J.R. BÜCHI: *"The monadic second order theory of all countable ordinals", Lecture notes in mathematics (vol. 328)*, Springer, (1973).
[6] Y. CHOUEKA: Finite automata, definable sets, and regular expressions over ω^n-tapes, *J. Comp. Syst. Sci.*, **17**, (1978), 81–97.
[7] J.E. HOPCROFT, J.D. ULLMAN: *Introduction to automata theory, languages, and computation*, Addison-Wesley, Reading, (1979).
[8] R. McNAUGHTON: Testing and generating infinite sequences by a finite automaton, *Information and control*, **9**, (1966), 521–530.
[9] D. MULLER: Infinite sequences and finite machines, *Switching circuit theory and logical design: Proc. Fourth Annual Symp. IEEE*, (1963), 3–16.
[10] D. PERRIN, J.E. PIN: *Mots infinis*, to appear (LITP report 97-04).
[11] J.P. PÉCUCHET: Étude syntaxique des parties reconnaissables des mots infinis, *Lecture Notes in Computer Science*, **226**, (1986), 294–303.
[12] J.P. PÉCUCHET: Variétés de semigroupes et mots infinis, *Lecture Notes in Computer Science*, **210**, (1986), 180–191.
[13] J.E. PIN: *Syntactic semigroups*, Chapter of "Handbook of formal languages", Springer, 1997.
[14] J.G. ROSENSTEIN: *Linear ordering*, Academic Press, New York, (1982).
[15] M.P. SCHÜTZENBERGER: On finite monoids having only trivial subgroups, *Information and Control*, **8**, (1965), 190–194.

[16] W. SIERPIŃSKI: *Cardinal and ordinal numbers*, Polish Scientific Publishers, Varsovie, (1965).
[17] W. THOMAS: Automata on infinite objects, *Handbook of theoretical computer science*, **B**, Elsevier, (1990), 135–191.
[18] T. WILKE: An Eilenberg theorem for ∞-languages in *Automata, Languages and Programming: Proc. of 18th ICALP Conference*, Springer, Berlin, (1991), 588–599.