A Characterization of Sturmian Morphisms *

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Abstract. A morphism is called *Sturmian* if it preserves all Sturmian (infinite) words. It is *weakly Sturmian* if it preserves at least one Sturmian word. We prove that a morphism is Sturmian if and only if it keeps the word $ba^2ba^2baba^2bab$ balanced. As a consequence, weakly Sturmian morphisms are Sturmian. An application to infinite words associated to irrational numbers is given.

1 Introduction

A one-sided infinite word is *balanced* if the difference of the number of occurrences of a letter in two factors of the same length never exceeds one. It is *Sturmian* if it is balanced and not ultimately periodic.

Sturmian words have a long history. A clear exposition of early work by J. Bernoulli, Christoffel, and A. A. Markov is given in the book by Venkov [22]. The term "Sturmian" has been used by Hedlund and Morse in their development of symbolic dynamics [9, 10, 11]. These words are also known as Beatty sequences, cutting sequences, or characteristic sequences. There is a large literature about properties of these sequences (see for example Coven, Hedlund [6], Series [20], Fraenkel *et al.* [8], Stolarsky [21]). From a combinatorial point of view, they have been considered by S. Dulucq and D. Gouyou-Beauchamps [7], Rauzy [16, 17, 18], Brown [3], Ito, Yasutomi [12], Crisp *et al.* [5] in particular in relation with iterated morphisms, and by Séébold [19], Mignosi [13]. Sturmian words appear in ergodic theory [15], in computer graphics [2], in crystallography [1], and in pattern recognition.

A morphism is *Sturmian* if the image of every Sturmian word is a Sturmian word. Sturmian morphisms appear in number theory in connection with the so-called substitutions of characteristic sequences. A recent account of results in this direction is given by T. C. Brown in [4]. In this paper, we show that in order to test whether a morphism f is Sturmian, it suffices to check whether the single word $f(ba^2ba^2baba^2bab)$ is balanced. This is in fact a strengthening of a result by Mignosi, Séébold [14]. The decidability is an immediate consequence. We also get a simpler proof of a theorem by Crisp *et al.* [5] characterizing those irrational numbers for which the characteristic sequence is a fixed point of a (Sturmian) morphism.

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2 Definitions

Let $A = \{a, b\}$ be a two letter ***alphabet.*** A^* is the set of (finite) ***words*** on A and ε is the ***empty word.*** A^ω is the set of ***infinite words*** on A and $A^\infty = A^* \cup A^\omega$.

A word $w \in A^*$ is ***primitive*** if it is not a power of another word, i.e. if $w = u^p$ for $u \in A^*$ and $p \in \mathbb{N}$ implies $w = u$.

For any $u \in A^*$, $|u|$ denotes the length of u and $|u|_x$ denotes the number of occurrences of the letter x in the word u.

A ***morphism*** h is a mapping from A^* into itself such that $h(uv) = h(u)h(v)$ for all words u, v. A morphism is ***nonerasing*** if neither $h(a)$ nor $h(b)$ is the empty word. For any morphism f, $||f||$ denotes the ***length*** of f which is $|f(a)| + |f(b)|$. In the sequel, all morphisms f will be supposed to be distinct from the ***null*** morphism which maps all letters into the empty word (thus $||f|| \geq 1$). Consider the morphism ϕ defined by

$$\phi(a) = ab, \qquad \phi(b) = a$$

Setting, for $n \geq 1$,

$$u_n = \phi^n(a), \qquad v_n = \phi^n(b)$$

it is easily seen that $u_{n+1} = u_n u_{n-1}$, $v_{n+1} = u_n$. The morphism ϕ can be extended to infinite words ; it has a unique fixed point

$$\mathbf{F} = abaababaabaababaababa \ldots = \phi(\mathbf{F})$$

For any $w \in A^\infty$, $Fact(w)$ denotes the set of ***finite factors*** of w. Setting, for any $u, v \in A^*$ such that $|u| = |v|$, $\delta(u, v) = \Big| |u|_a - |v|_a \Big|$, we call ***balanced*** a word $w \in A^\infty$ such that $\delta(u, v) \leq 1$ for any $u, v \in Fact(w)$ with $|u| = |v|$.

A word $\mathbf{x} \in A^\omega$ is ***Sturmian*** if it is a non ultimately periodic balanced word. It is a well-known property that

Property 1. *The word* $\mathbf{F}$ *is Sturmian.*

Sturmian words are intimately related to cutting sequences in the plane (also known as Beatty sequences). Let α, ρ be real numbers with $0 \leq \alpha < 1$. Then the infinite word $\mathbf{f}_{\alpha,\rho} = a_0 a_1 \cdots a_n \cdots$ defined by

$$a_n = \begin{cases} a & \text{if } \lfloor \alpha(n+1) + \rho \rfloor = \lfloor \alpha n + \rho \rfloor \\ b & \text{otherwise} \end{cases}$$

is Sturmian. The special case $\rho = \alpha$ has additional properties. In this case, we write $\mathbf{s}_\alpha$ for $\mathbf{f}_{\alpha,\alpha}$. The word $\mathbf{s}_\alpha$ is the ***characteristic sequence*** of α. Those words $\mathbf{s}_\alpha$ that are fixed points of morphisms have been characterized by Crisp *et al.* [5].

A morphism h is called ***Sturmian*** if $h(\mathbf{x})$ is Sturmian for every Sturmian word $\mathbf{x}$. The morphism Id_A and the morphism E that exchanges the letters a and b are obviously Sturmian. Let $\tilde{\phi}$ be the morphism defined by

$$\tilde{\phi}(a) = ba \qquad \tilde{\phi}(b) = a$$

It is well-known (see e.g. Séébold [19]) that

Property 2. *The morphisms ϕ and $\tilde{\phi}$ are Sturmian.*

A morphism h is called *weakly Sturmian* if there exists at least one Sturmian word $\mathbf{x} \in A^\omega$ such that $h(\mathbf{x})$ is Sturmian. Obviously every Sturmian morphism is weakly Sturmian. As we shall see, in fact the converse also holds.

3 Results

Notation

Let $m \geq 1$ and $r \geq 1$ be two integers. In the rest of this paper, the following notation will be used:

$$\begin{aligned} w_{m,r} &= b(a^{m+1}b)^{r+1}a^m b(a^{m+1}b)^r a^m b \\ w'_{m,r} &= ab(a^m b)^{r+1}a^{m+1}b(a^m b)^r a^{m+1}b \end{aligned}$$

These words are balanced and primitive. Conversely, every Sturmian word contains as a factor a word $w_{m,r}$ or $w'_{m,r}$ (resp. $E(w_{m,r})$ or $E(w'_{m,r})$) for some $m, r \geq 1$.

The main result of this paper is the following theorem:

Theorem 3. *Let f be a morphism. For every integers m and r with $m, r \geq 1$, the following three conditions are equivalent:*
(*i*) *f is a composition of the morphisms E, ϕ and $\tilde{\phi}$;*
(*ii*) *$f(w_{m,r})$ is a primitive balanced word;*
(*iii*) *$f(w'_{m,r})$ is a primitive balanced word.*

This result shows that in order to test whether a morphism is Sturmian, it suffices to check the image of $w_{m,r}$ for any arbitrary m and r, the shortest being $w_{1,1} = ba^2ba^2baba^2bab$. Thus, we obtain

Corollary 4. *A morphism f is Sturmian iff the word $f(ba^2ba^2baba^2bab)$ is primitive and balanced. In particular, it is decidable whether a morphism is Sturmian.*

Another direct consequence of this result is the following

Theorem 5. *Let f be a morphism. The following conditions are equivalent:*
(*i*) *f is a composition of the morphisms E, ϕ and $\tilde{\phi}$;*
(*ii*) *f is Sturmian;*
(*iii*) *f is weakly Sturmian.*

This result plays a major role in the characterization of morphisms of characteristic sequences associated to irrational numbers.

Proposition 6. *Let f be a morphism, and let α, β be two irrational numbers with $0 < \alpha, \beta < 1$ such that*

$$\mathbf{s}_\alpha = f(\mathbf{s}_\beta).$$

Then f is a product of E and ϕ.

Observe that there is no occurrence of the morphism $\tilde{\phi}$ in the factorization given by this proposition. This is due to the following property of the words s_α:

Property 7. *Let $0 < \alpha < 1$ be an irrational number. Then the word as_α is lexicographically less than all its proper suffixes. Symmetrically, the word bs_α is lexicographically greater than all its proper suffixes.*

From these results, one can obtain rather easily the following characterization of those irrational numbers α whose characteristic sequence s_α is a fixed point of a morphism (which necessarily is Sturmian). This characterization is due to Crisp *et al.* [5]:

Theorem 8. *Let $0 < \alpha < 1$ be an irrational number. The word s_α is a fixed point of a morphism which is not the identity iff the continued fraction development of α has one of the following three forms:*

(*i*) $[0; r_0, \overline{r_1, \ldots r_n}]$, $\quad r_n \geq r_0 \geq 1$;

(*ii*) $[0; 1 + r_0, \overline{r_1, \ldots r_n}]$, $\quad r_n = r_0 \geq 1$;

(*iii*) $[0; 1, r_0, \overline{r_1, \ldots r_n}]$, $\quad r_n > r_0 \geq 1$.

4 Proofs

The most involved part of the paper is the proof of theorem 3. The proof is through three lemmas. We start with a definition. A morphism f is called (m, r)-*balanced* if $f(w_{m,r})$ is balanced or $f(w'_{m,r})$ is balanced. By Theorem 3, these two words are either both balanced or not. The morphism f is *balanced* if it is (m, r)-balanced for some integers $m, r \geq 1$.

Lemma 9. *Let f be a balanced morphism. If $f(a) = a$ and $f(b) \in bA^* \cap A^*b$, then $f(b) = b$.*

Lemma 10. *Let f be a balanced morphism. If $f(a) \in aA^*a$, then $f(b) \in aA^* \cup A^*a$.*

Lemma 11. *Let f and g be two morphism such that $f = \phi \circ g$ or $f = \tilde{\phi} \circ g$, and let $m, r \geq 1$. Then f is (m, r)-balanced iff g is (m, r)-balanced.*

Proof of Theorem 3: Let f be a morphism. Since E, ϕ and $\tilde{\phi}$ are Sturmian (see e.g. Séébold [19]), it is easily seen that (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). By symmetry, it is enough to prove the implication (ii) $\Rightarrow$ (i).

Let f be a morphism such that the word $f(w_{m,r})$ is a primitive balanced word. Since $f(w_{m,r})$ is primitive, $f(a)$ and $f(b)$ are not the empty word ε thus $||f|| \geq 2$ and the result holds for $f = Id_A$ and $f = E$.

Consequently, let us suppose $||f|| \geq 3$. We observe first that $f(a)$ and $f(b)$ start or end with the same letter. Assume indeed that $f(a)$ starts with a and $f(b)$ starts with b (if $f(a)$ starts with b and $f(b)$ starts with a, then consider $E \circ f$). From Lemmas 9 and 10 it follows that if $f(a)$ and $f(b)$ do not end with the same letter, then $f(a)$ ends with b and $f(b)$ ends with a. But in this case,

$f(ab)$ contains the factor bb and $f(ba)$ contains the factor aa, which contradicts the hypothesis $f(w_{m,r})$ balanced. Consequently, let us suppose that $f(a)$ and $f(b)$ both start with the letter a (if it is the letter b then consider the morphism $E \circ f$ and if $f(a)$ and $f(b)$ end with the same letter, ϕ is replaced in what follows by $\tilde{\phi}$). Furthermore, let us suppose that $f(a)$ and $f(b)$ do not contain the factor bb. Then $f(a), f(b) \in \{a,\ ab\}^*$, thus there exist two words x and y such that $f(a) = \phi(x)$, $f(b) = \phi(y)$. Denoting by g the morphism defined by

$$g(a) = x \qquad g(b) = y$$

one obtains $f = \phi \circ g$ (if $f(a)$ or $f(b)$ contains the factor bb then $f = E \circ \tilde{\phi} \circ g$). Now, Lemma 11 implies that $g(w_{m,r})$ is balanced. Furthermore, $g(w_{m,r})$ is primitive (since $f(w_{m,r})$ is so) thus $|g(ab)|_a \neq 0$. Consequently, $||f|| > ||g||$ and the result follows by induction. □

Proof of theorem 5: The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear and we have only to prove (iii) $\Rightarrow$ (i). So let f be a weakly Sturmian morphism and $\mathbf{x}$ a Sturmian word such that $f(\mathbf{x})$ is Sturmian. Furthermore, let us suppose that $\mathbf{x}$ contains the factor aa. In this case, there exist some integers m and r, $m, r \geq 1$, such that $\mathbf{x}$ contains $w_{m,r}$ or $w'_{m,r}$ as a factor. It follows, from theorem 3, that f can be obtained by composition of E, ϕ and $\tilde{\phi}$. If $\mathbf{x}$ contains the factor bb, then the above proof holds for $g = f \circ E$ and, consequently, $f = g \circ E$ has the required property. □

Proof of Lemma 9: Let f be a morphism such that $f(a) = a$ and $f(b) \in bA^* \cap A^*b$, and let m and r be two integers, $m, r \geq 1$ such that $f(w_{m,r})$ or $f(w'_{m,r})$ is a balanced word. Since $f(a) = a$, both $f(w_{m,r})$ and $f(w'_{m,r})$ contain the factors a^m and a^{m+1}, $m \geq 1$, thus $f(b)$ does not contain bb and if $f(b) \neq b$ then $f(b)$ starts (resp. ends) with ba^mb or $ba^{m+1}b$.

For all integers $p \geq 0, p' \geq 1$, define

$$u_{p,p'} = b(a^{m+1}b)^{p+1}(a^mb)^{p'}a^{m+1}, \quad v_{p,p'} = b(a^mb)^{p'}(a^{m+1}b)^p a^m b$$

The word $w_{m,r}$ contains both $u_{r,1}$ and $v_{r,1}$, and $w'_{m,r}$ contains both $u_{0,r}$ and $v_{0,r}$. If $f(b)$ starts with ba^mb then $f(u_{p,p'})$ contains the factor

$$z = a^{m+1}f(b(a^{m+1}b)^p(a^mb)^{p'})a^{m+1}$$

and $f(v_{p,p'})$ contains the factor $z' = f(b(a^mb)^{p'}(a^{m+1}b)^p)a^mba^mb$.

Otherwise, $f(b) = ba^{m+1}bv$ for some word v. But in this case $f(u_{p,p'})$ contains the factor $z = a^{m+1}bvf((a^{m+1}b)^p)a^{m+1}ba^{m+1}$ and, since $p' \geq 1$, $f(v_{p,p'})$ contains the factor $z' = ba^mba^{m+1}bvf((a^{m+1}b)^p)a^mb$.

In both cases, $\delta(z, z') = 2$ which contradicts the hypothesis that f is (m, r)-balanced. Thus $f(b) = b$ and the lemma is proved. □

Proof of Lemma 10 : Let f be a morphism such that $f(a) \in aA^*a$ and let m and r be two integers, $m, r \geq 1$ such that $f(w_{m,r})$ or $f(w'_{m,r})$ is a balanced word. We set, for $k \in \mathbb{N}$,

$$u_k = a^{m+1}b(a^mb)^k a^{m+1}, \quad v_k = ab(a^mb)^k a^m ba$$

$$x_k = b(a^{m+1}b)^{k+1}, \quad y_k = ba^m b(a^{m+1}b)^k a^m b.$$

By construction, $w_{m,r}$ both contains u_0, v_0, and x_r, y_r. Symmetrically, $w'_{m,r}$ both contains u_r, v_r, and x_0, y_0.

Assume, by way of contradiction, that $f(b) \in bA^* \cap A^*b$. If $f(a) = uaav$ then $f(a^{m+1})$ contains the factor $z = aavf(a^{m-1})uaa$ and $f(ba^m b)$ contains the factor $z' = buaavf(a^{m-1})b$. Then $\delta(z, z') = 2$ which contradicts the hypothesis. Thus $f(a)$ does not contain aa and, since $f(a) \neq a$, there exists an integer $n \geq 0$ such that

$$f(a) = (ab)^n aba$$

But, in this case, $f(ab)$ contains bab and $f(aa)$ contains $baab$. Thus $f(b)$ does not contain bb nor a^3, which implies that either $f(b) = b$ or $f(b)$ starts with bab or $baab$.

We shall now prove that $f(b) = (ba)^{n'}b$, for some $n' \geq 0$. This holds if $f(b) = b$. Thus, assume by way of contradiction that $f(b) = uaav$ for some u and v.

Observe first that $f(b)$ starts and ends with $baab$. Indeed, if $f(b)$ starts with bab then $f(ba^{m+1})$ contains the factor $z = aavf(a^m)a$ and $f(ba^m b)$ contains the factor $z' = vf(a^m)bab$. If $f(b)$ ends with bab then $f(a^{m+1}b)$ contains the factor $z = af(a^m)uaa$ and $f(ba^m b)$ contains the factor $z' = babf(a^m)u$. In both cases, $\delta(z, z') = 2$ which contradicts the assumption.

Thus $f(b)$ starts and ends with $baab$. If $m = 1$ then let $f(b) = baabv'$ for some v'. In this case, $f(x_k)$ contains the factor

$$z = a^2bv'f((a^{m+1}b)^k a^{m+1})ba^2$$

and $f(y_k)$ contains the factor

$$z' = bf(a^m)ba^2bv'f((a^{m+1}b)^k a^m)b.$$

Then $\delta(z, z') = 2$ which contradicts the hypothesis.

Otherwise $m > 1$. In this case, both $f(w_{m,r})$ and $f(w'_{m,r})$ contain both of the words $f(a^3)$ and $f(a^2b)$. But $f(a^3)$ contains the factor $baa(ba)^n baab$ and $f(a^2b)$ contains the factor $baa(ba)^{n+1}baab$. Consequently, if $f(b)$ contains as a factor a power of ba, then this power is $(ba)^n$ or $(ba)^{n+1}$. Thus there exist two integers $p, p' \geq 0$, such that $f(b)$ starts with $(baa(ba)^n)^p(baa(ba)^{n+1})^{p'}baab$ (remark that, if $p' = 0$ then $f(b) = (baa(ba)^n)^p baab$).

If $p < m - 1$ then $f(u_k)$ contains the factor

$$z = a(ab)^n aba(ab)^n aba((ab)^n aba)^p f(b(a^m b)^k a^m)a$$

and $f(v_k)$ contains the factor

$$z' = ba(ba)^n f(b(a^m b)^k a^m)(baa(ba)^n)^p baa(ba)^n bab.$$

If $p \geq m - 1$ then $f(b)$ starts with $(baa(ba)^n)^{m-1}baa$ and we denote by v' the word such that $f(b) = bv'$. In this case, $f(x_k)$ contains the factor

$$z = v'f((a^{m+1}b)^k a^m)f(a)(baa(ba)^n)^{m-1}baa$$

and $f(y_k)$ contains the factor

$$z' = bf(a^m(ba^{m+1})^k)bv'f(a^m)b$$

and $\delta(f(a^m), f(a)(baa(ba)^n)^{m-1}) = 0$. Thus, in both cases, $\delta(z,z') = 2$ which contradicts the hypothesis.

Consequently $f(b)$ does not contain aa, thus $f(b) = (ba)^{n'}b$, for some $n' \geq 0$. But in this case, since $m \neq 0$, the word $f(u_k)$ contains the factor

$$z = af((a^m b)^{k+1})(ab)^n abaa$$

and $f(v_k)$ contains the factor

$$z' = babf((a^m b)^{k+1})(ab)^n ab.$$

Again, $\delta(z,z') = 2$ which contradicts the hypothesis, thus $f(b) \in aA^* \cup A^*a$ and the lemma is proved. □

Proof of Lemma 11: The "only if" part is straightforward. For the "if" part, assume $f = \phi \circ g$ (the case $f = \tilde{\phi} \circ g$ could be done exactly in the same way). If $g(w_{m,r})$ is not balanced then there exist two words u and v such that $g(w_{m,r}) = u_1 u u_2 = v_1 v v_2$ with $|u| = |v|$ and $\delta(u,v) = 2$. Furthermore, u can be choosen of minimal length, which implies that there exist $x, y \in A$, $x \neq y$ and $t, t' \in A^*$ such that $u = xtx$, $v = yt'y$ and $\delta(t,t') = 0$. Let us assume $x = a$ and $y = b$ (the other case is exactly the same). Then $g(w_{m,r}) = u_1 ata u_2 = v_1 bt'b v_2$ and $f(w_{m,r}) = \phi(u_1)ab\phi(t)ab\phi(u_2) = \phi(v_1)a\phi(t')a\phi(v_2)$.

If $v_2 \neq \varepsilon$, then $\phi(v_2)$ starts with a and $f(w_{m,r})$ is not balanced, contradiction.

If $v_2 = \varepsilon$, then $bt'b$ is a suffix of $g(w_{m,r})$. Two cases arise:

– If $|bt'b| \leq |g(a^m b(a^{m+1}b)^r a^m b)|$ then $g(a^m b(a^{m+1}b)^r a^m b) = v_1' bt'b$ and $g(w_{m,r}) = g(ba)v_1' bt'bg((a^{m+1}b)^r a^m b)$ which is the same case as $v_2 \neq \varepsilon$.

– If $|bt'b| > |g(a^m b(a^{m+1}b)^r a^m b)|$ then, since $m \geq 1$, one has $|ata| > |g(b(a^{m+1}b)^{r+1})|$. Consequently, there exist three words z, z', z'' with $z \neq \varepsilon$ and $|z'| = |z''|$ such that $ata = z'z$ and $bt'b = zz''$. But $\delta(z', z'') = \delta(u,v) = 2$, and since $z \neq \varepsilon$, one has $|z'| < |u|$ which contradicts the minimality of $|u|$.

If $g(w'_{m,r})$ is not balanced then the same contradiction holds when we compare $|bt'b|$ and $|g(a^m b(a^m b)^r a^{m+1} b)|$, thus the lemma is proved. □

We now turn to the proofs of the number-theoretic applications. Given two infinite words $\mathbf{x} = a_0 a_1 \cdots$ and $\mathbf{y} = b_0 b_1 \cdots$ over the alphabet $A = \{a, b\}$, ordered by $a < b$, we write $\mathbf{x} < \mathbf{y}$ when $\mathbf{x}$ is lexicographically less then $\mathbf{y}$, that is when there exists an integer n such that $a_n < b_n$ and $a_i = b_i$ for $0 \leq i < n$. Property 7 is a consequence of the more general

Lemma 12. *Let $0 \leq \rho, \rho' < 1$ and $0 < \alpha < 1$, with α irrational. Then*

$$\mathbf{f}_{\alpha,\rho} < \mathbf{f}_{\alpha,\rho'} \iff \rho < \rho'.$$

Proof. Since α is irrational, one has $\rho < \rho'$ if and only if there exists an integer n such that $\lfloor \alpha n+\rho' \rfloor = 1+\lfloor \alpha n+\rho \rfloor$ Let m be the smallest integer n satisfying this relation, and set $k = m-1$. Then, setting $\mathbf{f}_{\alpha,\rho} = a_0a_1\cdots$ and $\mathbf{f}_{\alpha,\rho'} = a'_0a'_1\cdots$, one gets $a_j = a'_j$ for $0 \leq j < k$ and $a_k < a'_k$. This proves the lemma. □

Proof of Property 7. Let us prove the first inequality, namely that $a\mathbf{s}_\alpha < \mathbf{x}$ for any proper suffix $\mathbf{x}$ of $a\mathbf{s}_\alpha$. For this, observe that $a\mathbf{s}_\alpha = \mathbf{f}_{\alpha,0}$, and that $\mathbf{x} = \mathbf{f}_{\alpha,n\alpha-\lfloor n\alpha \rfloor}$ for some integer $n > 0$. Since α is irrational, the conclusion follows from the preceding lemma. The other inequality is shown symmetrically. □

Before proceeding to the proof of Proposition 6, recall the following relations which are well-known (see e.g. [3]) :

$$E(\mathbf{s}_\beta) = \mathbf{s}_{1-\beta}, \quad \phi(\mathbf{s}_\beta) = \mathbf{s}_{(1-\beta)/(2-\beta)}$$

Proof of Proposition 6. By induction on the length $||f||$ of f. We may assume that $\mathbf{s}_\beta$ contains the factor aa. Otherwise we replace $\mathbf{s}_\beta$ by $\mathbf{s}_{1-\beta} = E(\mathbf{s}_\beta)$ and f by $f \circ E$. We also can suppose that $\mathbf{s}_\alpha$ contains the factor aa. Otherwise, we replace f by $E \circ f$ and $\mathbf{s}_\alpha$ by $\mathbf{s}_{1-\alpha}$. These normalisations do not increase the length of f. Since $b\mathbf{s}_\beta$ is Sturmian, the word $\mathbf{s}_\beta$ starts with the letter a. Similarly, $\mathbf{s}_\alpha$ starts with the letter a. In particular, $f(a)$ starts with an a, and neither $f(a)$ nor $f(b)$ contain a factor bb.

If the word $f(b)$ also starts with the letter a, then both $f(a)$ and $f(b)$ are products of words in $\{ab, a\}$. Thus $f = \phi \circ g$ for some shorter morphism g, and an appropriate word $\mathbf{s}_\gamma$. To conclude, it suffices to prove that $f(b)$ cannot start with a letter b. Indeed, otherwise $f(a)$ and $f(b)$ finish with the same letter. If this letter is a b, then $\mathbf{s}_\alpha$ contains the factor bb. Thus $f(a)$ and $f(b)$ finish by an a. Now let $r \geq 1$ by the integer such that a^rb is a prefix of $\mathbf{s}_\beta$. Then $a^{r+1}b$ is a factor of $\mathbf{s}_\beta$. The word $af(a^r)b$ is a prefix of $a\mathbf{s}_\alpha$, and $af(a^r)a$ is a factor of $\mathbf{s}_\alpha$. But this shows that $a\mathbf{s}_\alpha$ is lexicographically greater than one of its suffixes. Contradiction. □

We conclude with a proof of Theorem 8. Our proof is shorter than, though not very different from [5]. It will be convenient to introduce the morphism $\gamma = \phi \circ E$. Thus $\gamma(a) = a$, $\gamma(b) = ab$. Clearly, a morphism is a composition of E and ϕ iff it is a composition of E and γ. The morphism γ is used in conjonction with the morphism $\theta_m = \gamma^m \circ E$. We observe that ([3, 22])

$$\mathbf{s}_{\beta/(1+\beta)} = \gamma(\mathbf{s}_\beta), \quad \mathbf{s}_{1/(m+\beta)} = \theta_m(\mathbf{s}_\beta)$$

Proof of Theorem 8. Let

$$\alpha = [0; r_1, r_2, \ldots]$$

be the development into continued fraction of α. Let f be such that $\mathbf{s}_\alpha = f(\mathbf{s}_\alpha)$. Then f is a product of the morphisms γ and E. Clearly $f \neq E$ and f is not a product of γ only. Consequently,

$$f = \gamma^{n_1} E \gamma^{n_2} \cdots E \gamma^{n_k} E \gamma^{n_{k+1}}$$

for some $k \geq 1$ and $n_1 \geq 0$, $n_2, \ldots, n_k \geq 1$, $n_{k+1} \geq 0$. We distinguish two cases.

First case: $n_{k+1} \geq 1$. Then

$$f = \theta_{n_1+1}\theta_{n_2} \cdots \theta_{n_k}\gamma^{n_{k+1}-1}$$

Since s_α is a fixed point,

$$[0; r_1, r_2, \ldots] = [0; 1 + n_1, n_2, \ldots, n_k, n_{k+1} - 1 + r_1, r_2, \ldots]$$

whence $r_1 = 1 + n_1$, $r_2 = n_2, \ldots$, $r_k = n_k$, $r_{k+1} = n_{k+1} + n_1$ and $r_j = r_{j+k}$ for $j \geq 2$. Thus

$$\alpha = [0; r_1, \overline{r_2, \ldots, r_{k+1}}], \qquad r_{k+1} \geq r_1$$

which is case (i) of the theorem.

Second case: $n_{k+1} = 0$. Set $f' = EfE$. Since $s_\alpha = f(s_\alpha)$, one has $f'(Es_\alpha) = Es_\alpha$ and $f(s_\beta) = s_\beta$ where $\beta = 1 - \alpha$. Now

$$f' = E\gamma^{n_1}E\gamma^{n_2} \cdots E\gamma^{n_k} \quad \text{and} \quad n_k \geq 1.$$

This has the same form as above, excepted when $n_1 = 0$. Again, we consider two cases:

First, assume $n_1 = 0$. Then $k \geq 3$ and $f' = \theta_{n_2+1}\theta_{n_3} \cdots \theta_{n_{k-1}}\gamma^{n_k-1}$ whence, using the first case, $\beta = [1 + n_2, \overline{n_3, \ldots, n_{k-1}, n_2 + n_k}]$ and since $n_2 \geq 1$, one gets for $\alpha = 1 - \beta$ the development

$$\alpha = [0; 1, n_2, \overline{n_3, \ldots, n_{k-1}, n_2 + n_k}]$$

This is case (iii) of the theorem.

Finally, assume $n_1 > 0$. Then $f' = \theta_{n_0+1}\theta_{n_1} \cdots \theta_{n_{k-1}}\gamma^{n_k-1}$ with $n_0 = 0$. Applying the first case, we get $\beta = [0; 1, \overline{n_1, \ldots, n_k}]$ and consequently

$$\alpha = [0; 1 + n_1, \overline{n_2, \ldots, n_k, n_1}]$$

which is precisely case (ii) of the statement. □

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