From objects to methods

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Abstract. *The Pythagorean School was active in the VI-IV century B.C. in Crotone. One can safely say that there is no documented information on Pythagoras while the existence of a School is well documented. Our article refers only to the results commonly attributed to the Pythagorean School and to the techniques actually used today by computer scientists, in particular theoretical informaticians.*

1 Introduction

This paper is organized as follow:

1. **Introduction**,
2. **Results of Pythagorean School**,
3. **Incommensurability**,
4. **Arithmetics of Pythagorean**,
5. **Fibonacci sequence as a Pythagorean sequence**.
6. **Results and methods of theoretical computer science which can be considered developments of Pythagorean ideas**
7. **Arithmetics of computer**
8. **The algorithm of Knuth, Morris and Pratt**.

The section *Arithmetics of Pythagorean* concerning the so-called arithmogeometry corresponds to section *Arithmetics of computer* while section *Fibonacci sequence as a Pythagorean sequence* correspond to section *Algorithm of Knuth, Morris and Pratt.*

Theoretical computer science appears to be in great debt with the mathematics of Pythagorean. This was an often repeated thought of Marcel-Paul Schützenberger.

2 Results from the School

For the results concerning the so-called arithmo-geometry see the section *Arithmetics of Pythagorean.*

Pythagorean Theorem (*given any triangle ABC then*

$$AB^2 + BC^2 = AC^2$$

if and only if

the angle ABC has a width of 90 degrees,

i.e, ABC is a right-angled triangle) is to be attributed to the School.

The definition of incommensurable quantities is certainly to be attributed to the School even if it is found explicitly formulated only in the Elements of Euclid: *commensurable are called the quantities that are measured from the same measure, and incommensurable those of which there can be no common measure.* Starting from an argument probably used by Pythagoreans, in [13] one conclude that Fibonacci numbers can most likely be attributed to the School, see also Section 5.

Probably the Pythagorean Theorem was used by the School in order to prove the incommensurability of the side and the diagonal of the square.

The definition of the *golden ratio* (*given a segment AB then its internal point C divides it in medium and extreme reason if* $AB : AC = AC : CB$ *and in this case* $\frac{AB}{AC}$ *is the golden ratio*) belongs to the School even if its formalization appears only in the *Elements* of Euclid.

One can say the same for the following proposition: *in a regular pentagon, the diagonals are cut in extreme and average reason and the longest segment of this subdivision has equal length to the side of the pentagon* and the following formulas related to the side, indicated with b, and the diagonal indicated with a, of a regular pentagon

$$b : a = a : (b + a) \qquad (1)$$

and

$$b(b + a) = a^2 \qquad (2).$$

Finally the following fundamental theorem belongs to the School.

Theorem 1 *Side and diagonal of the regular pentagon are incommensurable.*

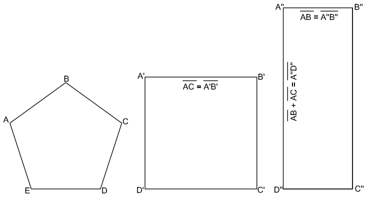

Figure 1

In this figure Theorem 1, a foundamental results of Pythagoreans, is illustrated: the square built on the diagonal a regular pentagon has the same area as the rectangle with dimensions the side and the sum of the side and of the diagonal of the same regular pentagon (see Formula (2) above).

3 Incommensurability.

Historical documents on the discovery of incommensurable segments (or on the discovery of irrationaly) are very rare. Those contemporary with discoveries are almost totally absent. In this section we limit ourselves to making some quotations from the works of only two authoritative scholars, a French and an Italian.

First we recall the words of Paul-Henri Michel (see his book "De Pythagore a Euclide", [11]): *A la notion d'irrationnel correspondent, en grec, plusieurs espressions* **arreton**, *...,* **asummetron**, *...,* **alogon**. *Ces termes ne sont ni rigoureusement synonymes ni parfaitment fixés. Ils répondent à des nuances de pensée et certain d'entre eux ... subissent une évolution sémantique que l'hystorien des irrationnelles ne négligera pas d'obserber. Ceux que nous détachons sont le plus digne d'intérêt, les autres n'apparaissant guère que comme des doublets ou périphrases.*

Arreton - *ineffable, indicible - insiste sur l'impossibilité d'exprimer par un mot ou par un nombre énoncable une quantité donnée.*

Asummetron - *incommensurable, sans commune mesure - suggérant l'idée d'espace et imposant celle d'une comparaison entre deux grandeurs traduit à la fois l'aspect géométrique et le caractère relatif de l'irrationalité.*

Alogon - *dont le premier emploi comme terme mathématique se trouve chez Platone — semble d'abord plus difficile à definir. Ses significations sont multiples comme celles du mot* **logos** *auquel le rattache l'‘étymologie. (Quant Héraclite, par exemple, parle du* **logos** *il a présent à l'esprit l'ensemble des notions que les Pythagoriciens avaient construit autour des idées de proportion, d'harmonie et de rythme.) Mais dès que* *la langue mathématique se précise «* **logos** *» s'y incorpore avec un sens restreint et parfaitement clair, celui que nous donnons encore au mot «* **raison** *».*

In [6] an authoritative Italian author Franciosi expresses himself with the following statements:

It does not seem to me that anyone has advanced the hypothesis of a discovery of the irrational outside the Greek world. [1]

It is known in particular that the discovery of irrationality was a fundamental result of Greek pre-Euclidean mathematics.

The ancient authors repeatedly attribute the discovery of irrationality to the Pythagoreans. That these Pythagoreans are to be identified with Pythagoras himself and his colleagues or with their immediate successors are almost equally attested by the sources.

Franciosi also reports that today scholars almost unanimously recognize the paternity of this fundamental discovery to the Pythagoreans (even if they are divided on the exact moment in which it happened) and he adds: *In my opinion the Greeks came across irrationality facing the problem of doubling the square.*

In short, with small changes, Franciosi says: *One had to ask if taking two square numbers it was possible, by adding them, obtain a third. Sometimes the thing succeeded (e.g. $3^2 + 4^2 = 5^2$), most often not. In particular, it must certainly be considered the case where the two starting numbers were the same. Attempts on small square numbers (4, 9, etc.) naturally proved to be immediately fruitful. But what happened with the big numbers, difficult to achieve practically and to handle? Could not there be any double square number of another? It does not seem to me risky to think that the Pythagoreans at least intuitively concluded that there was none: in ascertaining this impossibility it was the seed of the discovery of the irrational. That the Pythagoreans have demonstrated this also scientifically is theoretically possible. Pythagorean doctrine of the even and odd numbers, allows them that there can not exist a square number that is the double of another one square number. An argument to support the hypothesis that this has actually happened is the fact, unequivocally attested by the sources, that the doctrine of the even and odd numbers furnishes the instrument for the rigorous demonstration of the existence of irrationality in the next phase of the problem, when Pythagorean discovered impossibility to express with whole side and diagonal of the*

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[1] In particular, Franciosi says : *The same O. Neugebauer, the great scholar of Egyptian and Babylonian mathematical texts, notoriously inclined to ascribe to those peoples the achievement of certain results especially in algebra, certainly attributes to the Greeks the discovery of irrational magnitudes.* [12]

square.

Franciosi remembers the paper of von Fritz [14] in which the priority of the discovery of incommensurability is given to the pentagon. The work of Franciosi is full-bodied and complex. We tried to synthesize it but it deserves to be read.

In order to conclude this section we recall that the discovery of incommensurability would not have caused neither dismay nor scandal: the sources that would attest this are all of neopitagoric or even later age, while nothing is said in this sense by Plato and Aristotle.

4 The arithmetics of Pythagorean

Here we limit ourselves to very little information on the so-called arithmo-geometry. The representation of whole numbers was of great importance for the Pythagoreans, see the following Figure 2 where the numbers 2, 4, 6, 8, 10 are represented from top to bottom on the column (a) while on the column (b) the numbers 1, 3, 5, 7, 9 are represented from top to bottom. This representation is a slight variant of the one suggested in [9]. In the book of Michel the numbers are represented in a slightly different way [11] but the basic idea is always the same: an even number is represented with a set of pairs of points while an odd number is a set of pairs completed by an unpaired point.

These and other appropriate representations of the numbers greatly facilitate the demonstrations of theorems that often just require only a look. For example, the proofs of the following Theorems 2 and 3 are evident.

Theorem 2 *Let* a, b *be positive integers. Then* $a + b$ *is even if and only if* a *and* b *have the same parity.*

Theorem 3 *Let* a, b *be positive integers. Then* $a \cdot b$ *is odd if and only if* a *and* b *are both odd.*

Theorem 4 *Let* n *be a positive integer. Then* $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Put $T_n = \frac{n(n+1)}{2}$ and call it n-th *triangular number.* We have that the sum of two consecutive triangular numbers is a square.

Theorem 5 *Let* n *be a positive integer. Then* $T_n + T_{n+1} = (n + 1)^2$

Also a square is the sum of the first n odd numbers.

Theorem 6 *Let n be a positive integer. Then* $1 + 3 + \cdots + 2n - 1 = n^2$

Theorems 2 and 3 (even if very elementary!) have had a crucial importance for the proof of incommensuability. Indeed, in principle, to prove the incommensurability of side and diagonal of a regular pentagon we must consider infinite cases (more precisely ∞^2 cases). But these infinite cases, by focusing on parity, are reduced to just 4 possible cases. Let b the side and a the diagonal of a regular pentagon. The 4 cases are the following: i) b odd and a odd, ii) b odd and a even, iii) b even and a odd and iv) b even and a even! These 4 cases can be easily treated with the help of Theorems 2 and 3 (and Formula (2) above). Even if today we do not pay attention to this step, it is actually a huge leap forward. Indeed, Theorems 2 and 3 have enormous importance: with them not only we easily reach a fundamental conclusion but it should also be borne in mind that they are at the origin of binary arithmetics actually used today by computer scientists! In fact, Pythagorean introduced and studied a new arithmetic structure with 2 elements “odd” and “even”.

We would like it to be taught in European high school that today’s binary arithmetic has roots in Pythagorean mathematics.

The proof of Theorem 5 consists in a suitable juxtaposition of the representations of two consecutive triangular numbers, see Figure 3. Theorem 6 requires only a clever representation of odd numbers, see Figure 4.

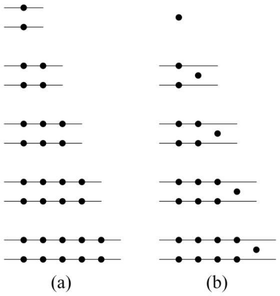

Figure 2

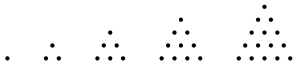

Figure 3

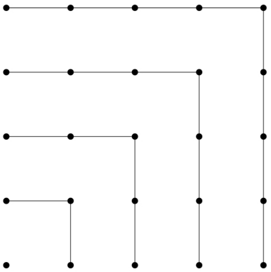

Figure 4

5 Fibonacci sequence as a Pythagorean sequence

The archaic geometry of the first Pythagorean School has considered the following statement to be true: given segments s and t there is a segment u contained exactly n times in s and m times in t, for some suitable integers n and m (this was a direct consequence of their philosophical-mathematical thought).

With Theorem 1 the *Pythagorean* system has been put in crisis. Precisely, a detailed analysis of the "mathematical argument" of Theorem 1 is made in [13]. This leads to the conclusion that the *Pythagorean identity* $b(b+a)-a^2=0$, concerning the side b and the diagonal a of a regular pentagon, and the *Cassini identity* $F_iF_{i+2}-F_{i+1}^2=(-1)^i$, concerning three consecutive Fibonacci numbers, were "almost simultaneously" discovered by the *Pythagorean School.*

So, Fibonacci numbers are most likely of Pythagorean origin.

Moreover, the fact that the just above mentioned *Pythagorean identity* $b^2+ab-a^2=0$ has no integer solutions is enough to conclude that the very beginning of study of *binary quadratic forms* must be attributed to the Pythagorean School.

6 Results and methods of theoretical computer science which can be considered developments of Pythagorean ideas

Non-exhaustive list (on the other hand, such an exhaustive list can not exist!)

Representation of real numbers (decimal, binary, still binary but Sturmian)

Fibonacci number, Fibonacci word, Sturmian words

Result of Knuth Morris and Pratt, results of Crochemore et al. Here there is an elementary exposure to the problem of searching for a given word in a text, question whose analysis involve Fibonacci words, underlining that in practice today all those who use computers derive enormous benefit from the existence of these algorithms.

Number theory that derives from Pythagorean studies on triangular, rectangular numbers, etc. With particular emphasis on prime numbers and their importance in cryptography.

Notion of order, Lyndon's words.

Algorithm theory (The Euclidean algorithm is probably Pythagorean).

In short, all discrete mathematics, including code theory.

7 Arithmetics of computer

Computers' memory stores information as binary sequences. Symbols are stored with a fixed binary word and natural numbers are usually represented by their binary expansion. This is a mere application of the fact that any natural number is the sum of distinct powers of 2.

In more specific applications like text compression it can be more useful to use Fibonacci expansion of natural numbers. Indeed, Fibonacci numbers satisfy the same property as above. The decomposition is made unique if no consecutive Fibonacci numbers are used or equivalently if no consecutive `1`'s are allowed in the expansion. This is known as the Zeckendorf representation. For example, $\texttt{101001}_{\mathrm{Fib}}$ is the expansion of 17 because it is the sum $11+5+1$ of non-consecutive Fibonacci numbers.

Using Fibonacci coding for text compression has several advantages. It produces a better compression than dense codes, it is more robust to errors than Huffman codes and it allows more flexibility for developing modular operations. Additionally, since every `1` is followed by `0` (after some tuning), the latter bit can be used to store extra information in a stream of data.

8 The algorithm of Knuth, Morris, and Pratt

Knuth-Morris-Pratt string matching (KMP) [10] searches a text y for occurrences of a pattern x. The algorithm treats the text sequentially and outputs an occurrence of x as soon as it is found. At a given position on the text, it tries to match the pattern with the aligned factor of the text. When a pattern occurrence is discovered it increases the position by the smallest period of the pattern. If a mismatch is found, that is if the text contains a factor uc where ub is a prefix of x for two distinct letters b and c, a specific period is used to increase the considered position: it is the smallest period of u that is not a period of ub if it exists, else it is the length pf ub. The picture shows what happens in that case where $uc = aabaabaac$ and $ub = aabaabaab$. Periods of $aabaabaa$ are 3, 6, 7 and 8. Since 3 and 6 are also periods of ub, the next position on the text is at distance 7. If ever $c = a$ we get the beginning of a potential match.

$$\begin{array}{lccccccccccccccccccc}
y: & \cdot & \cdot & a & a & b & a & a & b & a & a & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x: & & & a & a & b & a & a & b & a & a & b & a & & & & & & & \\
 & & & & & & & & & & a & a & b & a & a & b & a & a & b & a
\end{array}$$

Before the search, the algorithm processes the pattern to get the appropriate periods associated with each of its prefixes.

All is done in linear time: processing the pattern requires less than $2|x| - 2$ letter comparisons and searching the text needs less than $2|y|$ comparisons. However the algorithm does not work is real time because going from a letter of the text to the next one may require a certain number of comparisons, called the delay. On the example below where letter c is different from a and from b, 4 comparisons are done until the position on the text moves right after the position of c.

$$\begin{array}{lccccccccccccccccc}
y: & \cdot & \cdot & a & b & a & a & b & a & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x: & & & a & b & a & a & b & a & b & a & & & & & & & \\
 & & & & & & a & b & a & a & b & a & b & a & & & & \\
 & & & & & & & & a & b & a & a & b & a & b & a & & \\
 & & & & & & & & & a & b & a & a & b & a & b & a & \\
 & & & & & & & & & & a & b & a & a & b & a & b & a
\end{array}$$

This is a worst case delay obtained with the Fibonacci word $abaababa$.

Theorem 7 *The maximal delay of KMP algorithm searching for a pattern of length m is no more than $\log_\Phi(m+1)$, where Φ is the golden ratio.*

The proof of Theorem 7 draws on Theorem 8 known as the Periodicity lemma (see [1]). The result is rather natural because processing the pattern is similar to Euclid's algorithm applied to words.

Theorem 8 (Fine and Wilf [4]) *Let x be a word with periods p and q. If $p + q - \gcd(p, q) \leq |x|$ then $\gcd(p, q)$ is also a period of x.*

The maximal delay of sequential string-searching can be reduces to the optimal quantity $\min\{k, 1 + \log_2 m\}$ where k is the alphabet size. This is realised by Simon-Hancart algorithm (see [8]).

Another consequence of Theorem 8 that implies again Fibonacci words is the following statement. A primitive-rooted square is a word of the form uu where u is primitive (not of the form v^k with k an integer larger than 1)).

Theorem 9 *A word x has less than $\log_\Phi |x|$ prefixes that are primitively-rooted squares.*

Theorem 9 is a corollary of next result and square prefixes of Fibonacci words meet the bound.

Theorem 10 (Three-square-prefix lemma [2]) *Let u, v and w be three words that satisfy: u^2 is a proper prefix of v^2, v^2 is a proper prefix of w^2, and u is primitive. The $|u| + |v| \leq |w|$.*

For example, $(aabaabaaab)^2$ has three other square prefixes: a^2, $(aab)^2$ and $(aabaaba)^2$. Note that $|aab|+|aabaaba| = |aabaabaaab|$, which shows the inequality in Theorem 10 is optimal.

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