This file contains corrections and complements to the book.

1 Preliminaries

- p. 28. Insert ‘provided the automaton is complete’ after ‘The matrix $M/k$ is stochastic’.
- p. 30. Replace lines 3–17 by:

  Applying by induction the theorem to $U$ and $W$, we obtain nonnegative eigenvectors $u$ and $w$ for the eigenvalues $\rho_U$ and $\rho_W$ of $U$ and $W$. We prove that $\max(\rho_U, \rho_W)$ is an eigenvalue of $M$ with some nonnegative eigenvector.

  If $\rho_U \geq \rho_W$, then $\rho_U$ is an eigenvalue of $M$ with the corresponding eigenvector $[u]$. If $\rho_U < \rho_W$, then we show that $\rho_W$ is an eigenvalue of $M$ for the eigenvector $[w']$, where

  $$u' = \left(\sum_{n \geq 0} U^n \rho_W^{-n-1}\right)Vw = (\rho_W I - U)^{-1}Vw.$$  

  Since $\rho_U < \rho_W$, the series $\sum_{n \geq 0} U^n \rho_W^{-n}$ converges in view of Proposition 1.9.3, and it converges to a matrix with nonnegative coefficients because each $U^n$ has nonnegative coefficients. If follows that $u'$ has nonnegative coefficients. Moreover

  $$Vw = (\rho_W I - U)u' = \rho_W u' - U u',$$

  showing that $M [w'] = \rho_W [w']$. This shows that $\rho_M \geq \max(\rho_U, \rho_W)$.

  Conversely, if $\lambda$ is an eigenvalue of $M$ with corresponding eigenvector $[w']$, then $\lambda$ is an eigenvalue of $W$ if $v \neq 0$, and is an eigenvalue of $U$ if $v = 0$. This proves that $\rho_M = \max(\rho_U, \rho_W)$.

- p. 31. 12–13 replace by: Recall that the adjacency matrix of a complete deterministic automaton over a $k$-letter alphabet has spectral radius $k$ and...

- p. 37. 2 of proof of Proposition 1.10.10 : remove the last ‘x’.
2 Codes

- p. 74 \( \ell \). 16 : Insert ‘\( pqt^2 F_{D_a}(t) \)' after ‘\( F_a(t) F_{D_a}(t) F_b(t) \)’
- p. 102 \( \ell \). 3 of Exercise 2.4.2 : Replace ‘prefix of \( w \)’ by ‘prefix \( u \) of \( w \)’
- p. 102 In Exercise 2.4.3, replace the second sentence by: Let \( D = D_n \) be the Dyck code on \( A \) (Example 2.2.12). Show that one has
  \[
  f_D(t) = \frac{n}{2n-1} (1 - \sqrt{1 - 4(2n-1)t^2}),
  \]
  \[
  f_{D^*}(t) = \frac{1-n + n\sqrt{1 - 4(2n-1)t^2}}{1-4n^2t^2}.
  \]

3 Prefix codes

- p. 114 Figure 3.8(b) : Replace the label ‘\( a \)’ by ‘\( b \)’ on the last edge of the path of length 3.
- p. 117 \( \ell \). -8 : Replace ‘minimal automata’ by ‘minimal automaton’
- p. 157 \( \ell \). -7 : Replace ‘\( B \)’ by ‘\( B \) of the proof of Lemma 3.8.6’
- p. 173 Exercise 3.8.2 \( \ell \). 1 : Add ‘s’ to ‘length’
- p. 173 Exercise 3.8.2 \( \ell \). 3 : Insert ‘3.8.1 and’ before ‘3.6.4’
- p. 173 add the following exercise, due to Staiger (2007). It shows that for a any infinite prefix code, there is a maximal prefix code on the same alphabet which has the same length distribution.

Exercise 3.8.3 Let \( X \) be an infinite prefix code. Let \( x_1, x_2, \ldots \) be an enumeration of \( X \) by nondecreasing lengths. Set \( \ell_n = |x_n| \). Let \( X_1 \subset X_2 \subset \cdots \) be the strictly increasing sequence of prefix codes defined as follows. Set \( X_1 = \emptyset \). Assume that \( X_n \) is already defined and define \( X_{n+1} \) as follows.

Set \( m = \text{Card}(X_n) \) and \( \ell = \ell_{m+1} \). Let \( \{u_1, \ldots, u_t\} \) be the set of words of length \( \ell \) without any prefix in \( X_n \). For \( 1 \leq i \leq t \), let \( v_i \) be a word such that \( u_iv_i \) has length \( \ell_{m+1} \). Then \( X_{n+1} = X_n \cup \{u_1v_1, \ldots, u_tv_t\} \).

Let \( X' \) be the union of the \( X_n \). Show that:
1. the length distribution of \( X \) and \( X' \) are the same.
2. the set \( X' \) is a maximal prefix code.

4 Automata

- p. 194 Example 4.3.5 : ‘the code \( X = \)’ instead of ‘the code \( C = \)’

5 Deciphering delay

- p. 214 \( \ell \). 15 : Insert ‘with \( a \in A \)’ at the beginning of the line
- p. 221 Add the following exercise which is a result from Simon (1990).

Exercise. A rectangular band is a semigroup of the form \( I \times \Lambda \) for two sets \( I, \Lambda \) with the multiplication

\[
(i, \lambda)(j, \mu) = (i, \mu)
\]

for \( i, j \in I \) and \( \lambda, \mu \in \Lambda \).
Let $f : A^+ \rightarrow S$ be a morphism from $A^+$ onto a rectangular band. Show that for any $s \in S$, the semigroup $f^{-1}(s)$ is of the form $X^+$ where $X$ is a code with verbal deciphering 1.

Solution. Assume that $xyu = x'y'$ with $x, x', y, y' \in X^+$ and $u \in A^*$. Assume that $x = x'v$. Then $y' = vyu$ implies $f(v)Rf(y') = s$ and $x = x'v$ implies $f(v)\mathcal{L}f(x) = s$. Thus $f(v) = s$ which implies $v \in X^*$. This shows that $x = x'v$.

6 Bifix codes

• p. 227 § 11 : ‘Proposition’ instead of ‘Theorem’
• p. 229 § 9 : ‘any parse of $v$’ instead of ‘any parse of $u$’
• p. 230 § 2 : Replace ‘Theorem 3.1.6’ by ‘Proposition 3.1.3’, and insert ‘by Proposition 3.1.6’ before ‘$1 − A$’.
• p. 233 § 5 : ‘for $k = 0, 1$’ instead of ‘for $k = 0, 1, 2$’
• p. 234 § -8 : ‘Corollary’ instead of ‘Proposition’
• p. 245 § 1 : ‘$H = A^* \mathcal{X} A^*$’ instead of ‘$H = A^* \setminus \mathcal{X} A^*$’
• p. 274 § 7 : Insert ‘Exercise 6.1.2 is from Reutenauer (1979)’

7 Circular codes

• p. 297 Add the following exercises for Section 7.1.

Exercise 1.

Let $B_n$ be an alphabet with $n$ elements and let $\tilde{B}_n = \{b \mid b \in B_n\}$. Let $A_n = B_n \cup \tilde{B}_n$. Consider the congruence $\equiv$ of $A_n^*$ generated by all the relations $bb \equiv 1$ for $b \in B_n$. Let $M$ be the corresponding quotient monoid and let $\varphi : A^* \rightarrow M$ be the corresponding morphism. The set $\varphi^{-1}(1)$ is a free submonoid generated by a bifix code $D^*_n$ called the restricted Dyck code. Let $R = A_n^* \setminus A_n^*\{bb \mid b \in B_n\} A_n^*$. Show that $R$ is a set of representatives of the classes modulo $\equiv$.

Identify $M$ and $R$. Show that an element $w \in R$ is right-invertible (resp. left-invertible) if and only if $w \in B_n^*$ (resp. $w \in \tilde{B}_n^*$). Deduce that if $ww, vv \in D^*_n$, then $u, v \in D^*_n$. Conclude that $D^*_n$ is a circular code.

Solution. By induction on the length of $u \in R$. If $w \equiv 1$ for some $v \in R$, we have $u = u'b$ and $v = bv'$ with $b \in B$ and $u'v' \equiv 1$. By induction $u' \in B^*$. Thus $u \in B^*$.

Exercise 2. Let $D_n'$ be the restricted Dyck code as above. Show that one has the following disjoint union.

$$D_n'^* \setminus \{1\} = \bigcup_{b \in B} bD_n'^*bD_n'^*.$$
that the value \( h_1(t) = \frac{1}{2} (1 - \sqrt{1 - 4t^2}) \) is consistent with the value given for \( F_{D_a}(t) = h_1(t/2) \) for \( p = q = 1/2 \) in Example 2.4.10.

Using the binomial formula, as in the derivation of Equation (3.13), show that \( g_n(t) = \sum_{k \geq 0} C_k n^k t^{2k} \), where \( C_k = \frac{1}{k+1} \binom{2k}{k} \) is the \( k \)-th Catalan number (see Table 3.1 p. 129). Thus

\[
g_1(t) = 1 + t^2 + 2t^4 + 5t^6 + 14t^8 + 42t^{10} + 132t^{12} + \ldots, \
g_2(t) = 1 + 2t^2 + 8t^4 + 40t^6 + 224t^8 + 1344t^{10} + 8448t^{12} + \ldots
\]

In particular, \( g_1(t) = \sum_{k \geq 0} C_k t^{2k} \) and \( C_k \) is the number of words of length \( 2k \) in \( D_1^* \). Give a direct bijection between the set of words of length \( 2k \) in \( D_1^n \) and \( B_n^k \), and the Cartesian product of the set of words of length \( 2k \) in \( D_1^* \) with \( B^k_n \).

Solution. The words in \( D_1^* \) may classically viewed as well-parenthesized expressions, with \( n \) different types of parenthesis; each such word, of length \( 2k \), defines a unique word of length \( 2k \) in \( D_1^* \), by matching the opening and closing parenthesis; the sequence of length \( k \) of opening parenthesis, from left to right, defines a word of length \( k \) in \( B^*_n \). This gives the desired bijection. The various Dyck and restricted Dyck codes are described in more detail in (Berstel, 1979).

Exercise. Let \( X \) be a circular code. For \( x \in X \) and \( n \geq 0 \), let \( g_{x,n} \) be the number of words of length \( n \) having an interpretation \((s, y, p)\) with \( x = ps \) and \( p \) nonempty. Show that

\[
g_{x,n} = |x| \text{Card}(X^* \cap A^{n-|x|})
\]

Deduce from this equality a direct proof of Equation (7.14).

Solution. Let \( S = \{ u \in X^* \mid \exists x \in X \text{ such that } u^* = x \} \) and \( u^*(z) = \sum_{n \geq 0} u_n z^n \). Since \( X \) is circular, any word in \( S \) has a unique interpretation \((s, y, p)\) such that \( ps \in X \) and \( p \) nonempty. Thus \( g_{x,n} = |x| u^*_{n-|x|} \) which proves (1). Since \( p_n = \sum_{x \in X} g_{n,x} \), we obtain

\[
p_n = \sum_{x \in X} g_{n,x} = \sum_{x \in X} |x| u^*_{n-|x|}
\]

\[
= \sum_{m=0}^{n} m u_m u^*_{n-m}.
\]

This shows that \( p(z) = z u^*(z) u^*(z) \) whence Formula (7.14).

10 Synchronization

Exercise. Add the following exercise for Section 7.3 (see Stanley, 1997).

Exercise. Let \( X \) be a circular code. For \( x \in X \) and \( n \geq 0 \), let \( g_{x,n} \) be the number of words of length \( n \) having an interpretation \((s, y, p)\) with \( x = ps \) and \( p \) nonempty. Show that

\[
g_{x,n} = |x| \text{Card}(X^* \cap A^{n-|x|})
\]

Deduce from this equality a direct proof of Equation (7.14).

Solution. Let \( S = \{ u \in X^* \mid \exists x \in X \text{ such that } u^* = x \} \) and \( u^*(z) = \sum_{n \geq 0} u_n z^n \). Since \( X \) is circular, any word in \( S \) has a unique interpretation \((s, y, p)\) such that \( ps \in X \) and \( p \) nonempty. Thus \( g_{x,n} = |x| u^*_{n-|x|} \) which proves (1). Since \( p_n = \sum_{x \in X} g_{n,x} \), we obtain

\[
p_n = \sum_{x \in X} g_{n,x} = \sum_{x \in X} |x| u^*_{n-|x|}
\]

\[
= \sum_{m=0}^{n} m u_m u^*_{n-m}.
\]

This shows that \( p(z) = z u^*(z) u^*(z) \) whence Formula (7.14).
and for deterministic automata. The notion of synchronizing pair is an extension of the definition of synchronizing word to codes which are not prefix. It is due to Schützenberger (1979b).

- p. 395 Section 10.6 Notes ℓ. 3: Insert before ‘However’ the sentence ‘This is Theorem 10.2.11.’

11 Groups of codes
- p. 401 Delete ‘It is not known ...thin maximal codes’.
- p. 412 ℓ. 5: Replace ‘Example 3.6.6’ by ‘Example 3.6.3’
- p. 415 Remark 11.4.5, ℓ. 3: Insert a space between ‘X’ and ‘is’
- p. 433 It has been shown by Yun Liu (2012) that the generalization of Proposition 11.1.6 for a code which is not prefix is false. Proposition 11.2.3 already appears as Property 2 in Schützenberger (1964) with the hypothesis that \( G(X) \) is abelian. It has been shown in Liu (2012) that the corresponding statement for a code which is not prefix is false.

13 Densities
- p. 452, ℓ. 1: Replace the first paragraph by:

A real valued function \( \mu \) defined on a Boolean algebra of sets \( \mathcal{F} \) is additive if for any disjoint sets \( E, F \in \mathcal{F} \), one has \( \mu(E \cup F) = \mu(E) + \mu(F) \). It is called countably additive if

\[
\mu\left( \bigcup_{n \geq 0} E_n \right) = \sum_{n \geq 0} \mu(E_n)
\]

for any sequence \( (E_n)_{n \geq 0} \) of pairwise disjoint sets in \( \mathcal{F} \) such that \( \bigcup_{n \geq 0} E_n \in \mathcal{F} \). If \( \mu \) is additive and takes nonnegative values, then it is monotone in the sense that if \( E \subset F \) for \( E, F \in \mathcal{F} \), then \( \mu(E) \leq \mu(F) \) since indeed \( \mu(F) = \mu(E \cup (F \setminus E)) = \mu(E) + \mu(F \setminus E) \geq \mu(E) \).

- p. 452, ℓ. 8: Replace Proposition 13.1.3 by:

Let \( \mu \) be a countably additive function defined on a Boolean algebra \( \mathcal{F} \) of sets. Then

\[
\mu\left( \bigcup_{n \geq 0} E_n \right) \leq \sum_{n \geq 0} \mu(E_n)
\]

for any sequence \( (E_n)_{n \geq 0} \) of sets in \( \mathcal{F} \) such that \( \bigcup_{n \geq 0} E_n \in \mathcal{F} \).

- p. 456: Replace Proposition 13.1.13 by:

The function \( \mu \) satisfies \( \mu(A^\omega) = 1 \) and is countably additive.

- p. 457 ℓ. 8: Add ‘The second inequality holds by Proposition 13.1.3 since, by Lemma 13.1.12, \( \mathcal{F} \) is a Boolean algebra.’

- p. 457 ℓ. 10: Replace the sentence by: A function \( \nu \) defined on a family of sets \( \mathcal{F} \) is called countably subadditive if for any sequence \( (E_n)_{n \geq 0} \) of sets in \( \mathcal{F} \) such that \( \bigcup_{n \geq 0} E_n \in \mathcal{F} \), one has \( \nu(\bigcup_{n \geq 0} E_n) \leq \sum_{n \geq 0} \nu(E_n) \).

- p. 459 ℓ. 15: Replace ‘Then Equation (13.1) holds’ by ‘Then the equation of line 6 holds’

- p. 495 proof of Proposition 14.1.2, ℓ. 1: Replace ‘Let \( X, Y \)’ by ‘Let \( X, Z \)’
14 Polynomials of finite codes

- p. 525, l. -6 : Replace ‘\( \tau m = \tau m + \tau' m \)’ by ‘\( \sigma m = \tau m + \tau' m \)’
- p. 526 Example 14.7.3 : The matrices \( \alpha \) and \( \beta \) should be

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & -1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & -1 & 1 & -2 \\
0 & -1 & 1 & -2
\end{bmatrix}.
\]

- p. 534 Add to the Notes: The argument of the proof of Theorem 14.7.5 is well-known in group representation theory. The map from \( V \) to \( V e \) is called in Green (2007) the Schur functor. Theorem 14.7.5 itself has been generalized in Perrin (2013). The statement holds replacing the submonoid generated by a bifix code by a set \( S \) such that the minimal automata of \( S \) and of its reversal are strongly connected.

Solution of exercises

- p. 543 Solution 3.6.5 : \( \ell \). 1, replace ‘Let \( w \in A^* \) be such that’ by ‘Let \( u \in Z^* \) and \( w \in A^* \) be such that’

\( \ell \). -1, replace ‘We have shown that \( w \in U \)’ by ‘We have shown that \( u \in U \)’.

- p. 544 Solution 3.8.1 : \( \ell \) 3 of p. 544 : Replace ‘\( v_{n+p} = v_n k^p - \sum_{i=1}^p u_{n+i} k^{p-i} \)’ by ‘\( v_n = k^p - \sum_{i=1}^n u_{n-1} k^{i-1} \)’, and insert \( \ell \). 4, before ‘Using’ the sentence ‘It implies that \( v_{n+p} = v_n k^p - \sum_{i=1}^p u_{n+i} k^{p-i} \)’.

- p. 552 Solution 6.1.2 : replace lines 5–9 by

‘Next, if (ii) holds, consider \( x \in H \cap A^* \). Then \( x = h_1^{'i} \cdots h_n^{'i} \) with \( n \geq 0, h_i \in X \) and \( \epsilon_i = \pm 1 \). We may assume that \( n \) is chosen minimal. Assume that \( \epsilon_i = -1 \) for some index \( i \). Since \( X \) is bifix, none of the \( h_i^{-1} \) can cancel completely with \( h_{i-1} \) or with \( h_{i+1} \). Since \( x \in A^* \), there exists an index \( i \) with \( 1 \leq i \leq n \) such that \( \epsilon_i = -1 \) and \( h_i^{'i} \) cancels with its neighbors, that is \( h_i^{'i-1} h_i^{'i+1} \in A^* \). Thus, we have \( \epsilon_{i-1} = 1, \epsilon_{i+1} = 1 \) and \( h_{i-1} = tu, h_i = vu, h_{i+1} = vw \) for \( t, u, v, w \in A^* \). But then \( h_{i-1} h_i^{-1} h_{i+1} = tw \) is in \( X \) by (ii). This contradicts the minimality of \( n \). This shows that \( \epsilon_i = 1 \) for all \( i \) and thus \( x \in X^* \). Thus (iii) holds.’

- p. 568 Solution 9.3.13 : replace the two last paragraphs by:

‘Let \( u = u_1 \cdots u_1 \) and \( v = v_1 \cdots v_2 \). We have \( |u| \leq (s-1)n(n-1)/2 \) and \( |v| \leq (t-1)n(n-1)/2 \). Thus

\[ |uv| \leq (s + t - 2)n(n-1)/2. \]  \hspace{1cm} (2)

Let \( z \in A^* \) be such that \( q_t \xrightarrow{z} p_s \) with \( |z| \leq n - 1 \). Since \( p_s \xrightarrow{u} p_1 \) and \( q_1 \xrightarrow{u} q_t \), we have \( q_1 \xrightarrow{uv} q_t \). This forces \( x_s y_t = 1 \) by unambiguity. Since \( x_s y_t = 1 \), we have

\[ s + t \leq \sum_{q \in Q} (x_q) + \sum_{q \in Q} (y_q) \leq n + 1. \]  \hspace{1cm} (3)
Since the minimal rank of the elements of $M$ is 1, the minimal number of nonzero distinct rows of an element of $M$ is 1. By Exercise 9.3.5., $y_t$ is a column of an element of the monoid $M = \varphi(A^*)$ with minimal number of nonzero distinct rows. Such an element has the form $m = y_t \ell$ where $\ell$ is a row vector. Similarly, $x_s$ is a row of an element of $M$ of the form $n = rx_s$ where $r$ is a column vector. Since the minimal rank of the words in $A$ is 1, we cannot have $\ell r = 0$ which would imply that $0 \in M$. Since $A$ is unambiguous, this forces $\ell r = 1$ and thus $mn = y_t x_s$. This shows that $y_t x_s \in M$.

The word $w = vzuv$ is such that $y_t x_s \leq \varphi(w)$. Since $y_t x_s \in M$, by Exercise 9.3.12, this implies $y_t x_s = \varphi(w)$. Thus $w$ has rank one and by Equations (2) and (3), $|w| \leq (s+t-2)n(n-1)+n-1 \leq (n^2-n+2)(n-1)/2$.

Appendix: Research problems

- p. 593. 8 : Replace the last sentence of the paragraph by ‘It is conjectured that for any finite maximal prefix code $X$ there exist $P, T \subset A^*$ such that
  \[ X - 1 = P(A - 1)T, \]
  where $T$ is the union of $d(X)$ pairwise disjoint maximal prefix sets (see Perrin, Schützenberger (1992)). This is equivalent to say that in Equation (14.7) one has $S = 1$ and the polynomial $Q$ has the form $Q = \sum_{i=1}^{d-1} U_i$, where each $U_i$ is a nonempty prefix-closed set.’
- p. 592 Suppress the first sentence (see the complement to page 433).
- p. 593. -4 : Replace ‘finite set $Y$’ by ‘finite subset $Y$’

References

- p. 596. -10 : Replace ‘Capoceli’ by ‘Capocelli’

Index

- p. 613 Add p. 102 for Dyck code.
- p. 616. 5 : Replace ‘nil-simple semigroup 417’ by ‘nil-simple semigroup 416’

Additional references