# 1 Solution to Problems of chapter 1

Section 1.1
1.1.1 This monoid is also defined by the relation $ab = 1$. The rules for multiplication follow easily from this definition.
Section 1.2
1.2.1 If $w$ is not primitive, then its period is clearly a proper divisor of its length. Conversely, let $w$ be of length $n$ and period $p$. Then $w = u^m v$ with $|u| = p$ and $|u| < p$. If $p$ is a proper divisor of $n$, then $w = u^m$ with $m > 1$ and thus $w$ is not primitive.
1.2.2 Let $w \notin F(X)$. If $w$ is a letter, $X \subset (A - w)^*$ and thus the coefficients $u_n$ of $f_X(z)$ satisfy $u_n < k^n$. Thus $\rho_X > 1/k$. In the general case, we consider the sets $X_i = \{x \in X \mid |x| \equiv i \pmod{n}\}$ with $n = |w|$. We have for each $i = 0, \ldots, n-1$, $X_i \subset A^i(A^n - w)^*$ and thus $\rho_{X_i} > 1/k$. Since $f_X = \sum_{i=0}^{n} f_{X_i}$, the result follows.
1.2.3 For each $x \in A^*$, the set $xA^\omega$ is open since $xA^\omega = \{y \in A^\omega \mid d(x,y) < 2^{-|x|+1}\}$. Since $XA^\omega$ is the union of the sets $xA^\omega$ for $x \in X$, it is an union of open sets and thus an open set. Conversely, if $U \subset A^\omega$ is an open set, then for each $u \in U$ there is an $n \geq 0$ such that $X$ contains the set $\{v \in A^\omega \mid d(u,v) < 2^{-n}\}$ which is also the set $xA^\omega$ with $x = u_0 \cdots u_n$. Thus $U = XA^\omega$ with $X = \{x \in A^* \mid xA^\omega \subset U\}$.
1.2.4 Suppose first that $u_n$ converges to $x$. Then it is not ultimately constant and $u_n\$^\omega$ also converges to $x$. Conversely, if $u_n\$^\omega$ converges to $x$ and if $(u_n)$ is not ultimately constant, the prefix of length $i$ of $x$ is a prefix of all but a finite number of the $u_n$. Thus the sequence $(u_n)$ also converges to $x$.
1.2.5 Since all intersections are double points, each will be met twice while traversing the curve. Thus each symbol appears exactly twice. The second statement is equivalent to: in a Gauss code, the two occurrences of a symbol are at even distance. To prove this relatively nontrivial assertion, the following path is possible. First consider the planar multigraph $G$ whose vertices are the intersection points and the edges are the arcs of the curve joining two successive intersection points. We have to prove that all cycles in $G$ have even length. The hypotheses on the curve imply that the degree of each vertex is even. Consider the dual graph $G^*$ of $G$. Its faces have all an even number of edges surrouding them. Thus, in $G^*$ all cycles have even length. This implies that $G^*$ is bipartite. Let us color the faces of $G$ in red and black according to this partition of the vertices of $G^*$. The cycle in $G$ corresponding to the travel along the curve has on its right a face of alternating colors. Since it has to come back to a given vertex $a$ by the other arc of the curve than the one it came from, the number of vertices between the two occurrences of $a$ is odd.
Section 1.3
1.3.1 If $i \cdot u = i \cdot v$, then $u^{-1}X = v^{-1}X$. Thus there is a well-defined map $f$ from $Q$ onto $Q(X)$ defined by $f(q) = u^{-1}X$ if $q = i \cdot u$.
1.3.2
1.3.3 If $x \in X$, then $y = f(x)$ is again on the alphabet $A = \{1, 2, 3\}$ by construction. If four consective symbols of $y = i_1 a_1 i_2 a_2 \cdots$ where equal, we would

either have $i_k = a_k = i_{k+1} = a_{k+1}$ or $a_{k-1} = i_k = a_k = i_{k+1}$. In both cases two consecutive $a$'s are equal, a contradiction.

The map $f$ is local and thus both left and right subsequential.

1.3.4 The DeBruijn graph is an Eulerian graph and thus admits Eulerian circuits. The label of such a circuit is a word with the required property (i.e. a DeBruijn sequence).

Section 1.4

1.4.1 The proof is similar to the one for the Thue-Morse morphism (Example 1.5.10). Any block of $x$ is a block of some $f^k(a)$ for some $a \in A$. In $t$, two occurrences of $a$ are separated by at most $2k$ symbols with $k = \max\{|f(b)| \mid b \in A\}$. Thus a second occurrence of $x$ occurs at bounded distance in $t$.

1.4.2 One may remark that the property is a well-known one for partial orders. Indeed, the statement is equivalent to: a recurrent one-sided infinite word over a finite alphabet is uniformly recurrent iff the restriction of the factor order to the set $F$ of its factors is a well-quasi order. The condition is necessary: let $X$ be an infinite subset of $F$ and let $x \in X$. There is an integer $r$ such that $x$ is a factor of any word of length $r$ of $F$. Since $X$ is infinite and the alphabet finite, there is a word $y$ of $X$ of length exceeding $r$. Then $x$ is a factor of $y$. Let us now prove that the condition is sufficient. We argue by contradiction. If the infinite word is not uniformly recurrent, there is a word $w$ which does not appear at successive bounded distances. Thus, there is a sequence $u_n$ of words of increasing length such that $wu_nw \in F$ and $w$ has only two occurrences in $wu_nw$. Then the set formed by the words $wu_nw$ is an infinite subset of incomparable elements of $F$.

1.4.3 Let $\alpha = \inf_{n\geq 1} a-n/n$. For each $\epsilon > 0$, there is a $k$ for which $a_k/k < \alpha + \frac{1}{2}\epsilon$. Then, for $0 \leq j < k$ and $m \geq 1$,

$$\begin{aligned}\frac{a_{mk+j}}{mk+j} &\leq \frac{a_{mk}}{mk+j} + \frac{a_j}{mk+j} \leq \frac{a_{mk}}{mk} + \frac{a_j}{mk} \\ &\leq \frac{ma_k}{m_k} + \frac{ja_1}{m_k} \leq \frac{a_k}{k} + \frac{a_1}{m} < \alpha + \frac{\epsilon}{2} + \frac{a_1}{m}.\end{aligned}$$

Thus $a_n/n < \alpha + \epsilon$ for large enough $n$.

Section 1.5

1.5.1 A word $w$ of length $k^n + n - 1$ has $k^n$ factors of length $n$. If none of them has a factor in $X$, two of them have to be equal to the same $u$ and thus $w = rst$ with $s = uv = v'u$. Then $v^\omega$ is an infinite word without factor in $X$, a contradiction.

1.5.2 The proof is the same as for Higman's theorem (which corresponds to the case $X = A$). The condition is clearly necessary: if $X$ is avoidable, there are words of arbtrary length without factor in $X$ and thus pairwise incomparable. Conversely, let us choose a sequence $(w_n)$ of incomparable words which is minimal in the following sense: $w_1$ is a word of minimal length such that there is an infinite sequence of incomparable words starting with $w_1$. Then, $w_2$ is a word of minimal length such that there is an infinite sequence of incomparable words starting with $w_1, w_2$. And so on. Since $X$ is unavoidable each word $w_n$, except for a finite number of indeices $n$, has a factor in $X$. An infinity of them have the same word $x$ as the first factor in $X$. This subsequence can be denoted $(w'_n)$

with for each $n$, $w'_n = r_n x s_n$. The sequence $r_n s_n$ has again (beacause the $r_n$ are of bounded length) a subsequence made of paiwise incomparable elements which is earlier than $(w_n)$, a contradiction.
1.5.3 The set $Y$ is the set of minimal words (for the factor ordering) such that $S_X = S_Y$. If $X \neq Y$, there is some $y \in Y - X$. Let $w \in A^*yA^*$ be a word of maximal length which does not have a factor in $X$. Let $w = uuv$. If $v \neq \epsilon$, we reason as in the proof of Proposition 1.6.11 to prove that $X$ is reducible by rule (ii). Otherwise, a symmetric argument shows that $X$ is reducible by rule (iii).
1.5.4 Let first $X$ be a subset of a minimal unavoidable set $Y$. For each $x \in X$, the set $Y - x$ is avoidable and thus there is a periodic infinite word $s$ which avoids $Y - x$ and thus such that $F(s) \cap X = \{x\}$. Conversely, let $S(x)$ be the closure of $s_x$ under the shift and let $S = \cup_{x_i n X} S(x)$. Each $S(x)$ is a subshift of finite type and since the union of two subshifts of finite type is itself of finite type, it follows that $S$ is of finite type. Let $I$ be a finite set of words such that $S = S_I$. Then $X + I$ is unavoidable and there is a minimal unavoidable set $Y$ such that $X \subset I \subset X + I$.
1.5.5 To prove that $\{aa, aba, abb, bbb\}$ is, up to symmetries, the only irreducible unavoidable set with four elements on $\{a, b\}$, we may either inspect the six elements of the family $\mathcal{F}_4$ or proceed directly as follows. Let $X$ be an irreducible unavoidable set with four elements on $\{a, b\}$. Let $n, m \geq 1$ be such that $a^n, b^m \in X$. If $n, m \leq 2$, then $X$ has at most three elements. If $n, m \geq 3$, then $X$ has at least five elements since we have to meet the words of period $ab, aab, abb$. We may thus suppose that $aa, bbb \in X$. It is then easy to see that the only possibilities are to choose the two remaining words as $aba, abb$ or $aba, bba$ which are equivalent by reversal.

To prove the second assertion, we use Theorem 1.6.16 and Dickson's lemma. Recall that this result deals the set of $n$-tuples of nonnegative integers ordered by $(r_1, \ldots, r_n) \leq (r'_1, \ldots, r'_n)$ if $r_i \leq r'_i$ for $1 \leq i \leq n$. It says that any set of pairwise incomparable $n$-tuples is finite. We consider a simple family $\mathcal{F} = \{X_1, \ldots, X_n\}$ with $X_i = u_i$ or $X_i = u_i^*$ for $i = 1, \ldots, n$. An $n$-section $X = \{x_1, \ldots, x_n\}$ of $\mathcal{F}$ has the form $x_i = u_i^{r_i}$ for $1 \leq i \leq n$. If $X$ and $X'$ are two such sets which are irreducible unavoidable sets, then the corresponding $n$-tuples $(r_1, \ldots, r_n)$ and $(r'_1, \ldots, r'_n)$ are incomparable. By Dickson's lemma, there can only be a finite number of irreducible unavoidable sets which are an $n$-section of a given simple family. Thus, by Theorem 1.6.16, the total number of irreducible unavoidable sets with $n$ elements is finite.
1.5.6