CHAPTER 13

Independent Systems of Equations

13.0. Introduction

The notion of a dimension, when available, is a powerful mathematical tool in proving finiteness conditions in combinatorics. An example of this is Eilenberg's Equality Theorem, which provides an optimal criterion for the equality of two rational series over a (skew) field. In this example a problem on words, i.e., on free semigroups, is first transformed into a problem on vector spaces, and then it is solved using the dimension property of those algebraic structures. One can raise the natural question: do sets of words possess dimension properties of some kind?

We approach this problem through systems of equations in semigroups. As a starting point we recall the well known *defect theorem*, see Chapter 6, which states that if a set of n words satisfies a nontrivial relation, then these words can be expressed simultaneously as products of at most $n-1$ words. The defect effect can be seen as a weak dimension property of words. In order to analyze it further one can examine what happens when n words satisfy several independent relations, where independence is formalized as follows: a set E of relations on n words is *independent*, if E, viewed as a system of equations, does not contain a proper subset having the same solutions as E.

It is not difficult to see that a set of n words can satisfy two or more equations even in the case where the words cannot be expressed as products of fewer than $n-1$ words. This proposes an interesting problem: how many independent equations a set of n words can satisfy? In other words, how weak is the above dimension property? A partial answer is given in a fundamental result of words revealed in 1985, namely in the *compactness property* of free semigroups (known as Ehrenfeucht's conjecture): each independent set of equations of words is finite. This is the central theme of this chapter.

For finite systems of equations over a free semigroup, we show that there exist independent systems of $\Omega(n^3)$ equations in n variables. Moreover, in comparison to the defect theorem, we construct a set X of words with $\mathrm{Card}(X) = n$ that satisfies $\Omega(n^2)$ independent relations, and still the words of X cannot be expressed as products of less than $n-1$ words. That is, these relations cause the same defect effect as a single nontrivial relation.

Our central problem generalizes, in a natural way, to all semigroups S. In this setting we consider infinite systems of equations, i.e., pairs of words from a free semigroup of variables, and ask whether, for each such system, there exists a finite subsystem of equations that is equivalent to the given one, that is, whether the subsystem has exactly the same set of solutions in S as the original one.

The compactness property does not hold in all semigroups, an example being the bicyclic semigroup, while it does hold in some other than the free semigroups. In general, no characterization of semigroups satisfying the compactness property is known. This, however, changes if we consider varieties of semigroups or monoids. There is a nontrivial characterization in terms of ascending chains of congruences for a variety to satisfy the compactness property.

13.1. Sets and equations

Let A and Ξ be two finite sets, where the elements of Ξ are called *variables*. Let $(u, v) \in \Xi^+ \times \Xi^+$ be an equation, usually written as $u = v$. Its *solution* in the free semigroup A^+ (resp. in a semigroup S) is a morphism $\alpha: \Xi^+ \to A^+$ (resp. $\alpha: \Xi^+ \to S$) that satisfies $\alpha(u) = \alpha(v)$. Solutions of an (infinite) system of equations E are defined in the obvious way. Let $\mathrm{Sol}(E)$ be the set of all solutions of a system E of equations. Two systems E and E' of equations are said to be *equivalent* if they have the same solutions, $\mathrm{Sol}(E) = \mathrm{Sol}(E')$. Further, a system E of equations is *independent*, if it is not equivalent to any of its proper subsystems.

For simplicity, we often write $x = w$ instead of $\alpha(x) = w$, when $x \in \Xi$ is a variable and α a morphism.

The *combinatorial rank* of a finite subset $X \subseteq A^+$ of words is defined by

$$r_c(X) = \min\{\mathrm{Card}(Y) \mid X \subseteq Y^+\}.$$

Clearly, we have $r_c(X) \leq \max\{\mathrm{Card}(X), \mathrm{Card}(A)\}$.

Example 13.1.1. Consider the following three systems of equations:

$$\begin{aligned} E_1 :&\quad xy = zx \\ E_2 :&\quad xy^i = z^i x, \quad i = 1, 2, \ldots, \\ E_3 :&\quad xyz = zyx, \quad xy^2z = zy^2x. \end{aligned}$$

Here E_1 and E_2 are equivalent, since $xy = zx$ implies, for $i \geq 1$,

$$xy^{i+1} = (xy)y^i = (zx)y^i = zxy^i,$$

which gives, by induction, that $xy^{i+1} = z^{i+1}x$. The system E_3 is independent, since $x = a$, $y = b$ and $z = aba$ is a solution of the first equation of E_3 that is not a solution of the second one, and $x = a$, $y = b$ and $z = abba$ is a solution of the second equation that is not a solution of the first one.

The morphisms $\alpha\colon \Xi^+ \to A^+$ are, in a natural way, in a 1-1 correspondence with the finite ordered subsets $X \subseteq A^+$ with $\mathrm{Card}(X) = \mathrm{Card}(\Xi)$. We exploit this by attaching to a finite ordered subset $X = \{w_1, w_2, \ldots, w_n\}$, a set $\Xi_X = \{x_1, x_2, \ldots, x_n\}$ of variables and a morphism $\alpha_X\colon \Xi_X^+ \to X^+$, for which $\alpha_X(x_i) = w_i$ for all i. Such a surjective morphism is a *presentation* of the semigroup X^+. Now we can view the set X of words as a solution of an equation $u = v$ over Ξ_X, if the morphism α_X is its solution. Further, the *set of relations* satisfied by X is defined as the kernel of the morphism α_X,

$$E(X) = \{(u,v) \in \Xi_X^+ \times \Xi_X^+ \mid \alpha_X(u) = \alpha_X(v)\}. \tag{13.1.1}$$

Clearly, $E(X)$ is a congruence of the free semigroup Ξ_X^+, that is, it is an equivalence relation and a subsemigroup of the direct product $\Xi_X^+ \times \Xi_X^+$.

A subsemigroup X^+ of a free semigroup A^+ is cancellative, and so is the semigroup $E(X) \subseteq \Xi_X^+ \times \Xi_X^+$. Therefore if X^+ satisfies the relations (u_1, v_1) and (u_1u_2, v_1v_2) (or (u_2u_1, v_2v_1)), it also satisfies (u_2, v_2). We say that a relation (u, v) is *reduced*, if it belongs to the minimal generating set $E_{\mathrm{red}}(X)$ of the semigroup $E(X)$:

$$E_{\mathrm{red}}(X) = (E(X) \setminus E(X)^2) \setminus \iota_{\Xi_X},$$

where $\iota_{\Xi_X} = \{(x,x) \mid x \in \Xi_X\}$ is the identity relation of Ξ_X. Clearly, if $(u,v) \in E_{\mathrm{red}}(X)$, then the first variables in u and v are different, and so are the last ones. It is obvious that, as systems of equations over Ξ_X, $E(X)$ and $E_{\mathrm{red}}(X)$ are equivalent. However, $E_{\mathrm{red}}(X)$ need not be a minimal equivalent subsystem of $E(X)$.

Example 13.1.2. Let $X = \{a, ab, ba\} \subseteq \{a,b\}^+$. Then it satisfies the relation (x_1x_3, x_2x_1), and, by the previous example, it also satisfies the relations $(x_1x_3^i, x_2^ix_1)$ for all $i \in \mathbb{N}$. It is not difficult to see that the latter are exactly the reduced relations satisfied by X^+, that is, $E_{\mathrm{red}}(X) = \{(x_1x_3^i, x_2^ix_1) \mid i \in \mathbb{N}\}$. Indeed, the validity of this can be concluded from the finite automaton given in Fig. 13.1 that seeks through all double X-factorizations of words in X^+.

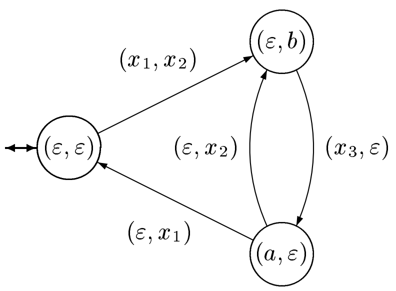

Figure 13.1. A finite automaton for the relations of X

In general, such an automaton can be constructed as follows. The states of the automaton form a subset of the pairs $(u, \varepsilon), (\varepsilon, u)$, where u is a proper suffix of a word in X, and there is a transition

$$(u_1, v_1) \xrightarrow{(x,y)} (u_2, v_2)$$

if $u_1\alpha_X(x)v_2 = v_1\alpha_X(y)u_2$ for $x, y \in \Xi_X \cup \{\varepsilon\}$. The initial and the final state of the automaton is $(\varepsilon, \varepsilon)$. In our example, the automaton has been simplified after the construction.

The proof of the next theorem is left to the reader as an exercise. Example 13.1.2 serves as an illustration.

THEOREM 13.1.3. *The sets $E_{\mathrm{red}}(X)$ and $E(X)$ for a finite set $X \subseteq A^+$ are rational relations.*

Thus the sets $E_{\mathrm{red}}(X)$ and $E(X)$ are rather easy to compute. However, to compute $\mathrm{Sol}(u, v)$ for a given equation $u = v$ maybe very demanding, see Chapter 12.

13.2. The compactness property

In this section we shall prove a compactness result, which states that every system of equations in a free semigroup over a finite set of variables has an equivalent finite subsystem. In other words, each independent system of equations in a free semigroup over a finite set Ξ of variables, is finite. This can be viewed, beside the defect theorem, as another positive dimension property of words.

13.2.1. The proof

In the proof of the compactness result we need Hilbert's basis theorem. For this, let $\mathbb{Z}[X]$ be the ring of polynomials with integer coefficients in a finitely many (commuting) variables X.

THEOREM 13.2.1 (Hilbert's basis theorem). *Let P_i, for $i \geq 1$, be polynomials in $\mathbb{Z}[X]$. There exists a finite subset $P_1, P_2, \ldots, P_t$ of these polynomials such that every P_i can be expressed as a linear combination*

$$P_i = P_1 Q_{i_1} + P_2 Q_{i_2} + \ldots + P_t Q_{i_t},$$

where $Q_{i_j} \in \mathbb{Z}[X]$.

We use Theorem 13.2.1 in the following form. Let $\{P_i = 0 \mid i \geq 1\}$ be a system of polynomial equations, where $P_i \in \mathbb{Z}[X]$. There exists a finite subsystem $\{P_i = 0 \mid i = 1, 2, \ldots, t\}$, every solution of which is a solution of the whole system of equations.

THEOREM 13.2.2. *Every independent system of equations in a free semigroup A^+ over a finite set Ξ of variables is finite.*

Proof. To take advantage of Hilbert's basis theorem, we transform each word equation $u = v$ into a polynomial equation in $\mathbb{Z}[X]$. For this, we use noncommuting matrices of integer polynomials.

Let $\Xi = \{x_1, x_2, \ldots, x_k\}$ be a fixed set of word variables. Since each A^+ can be embedded into B^+, where $B = \{a, b\}$, it is sufficient to solve equations over Ξ in B^+.

Consider the multiplicative semigroup $\mathbb{Z}^{2\times 2}$ of 2×2-matrices over $\mathbb{Z}$. Let $\mathbb{F}$ be the subsemigroup of $\mathbb{Z}^{2\times 2}$ of all matrices of the form

$$M = \begin{pmatrix} 2^m & n \\ 0 & 1 \end{pmatrix}, \qquad \text{where } 0 \le n < 2^m. \tag{13.2.1}$$

Claim 1. *The semigroup $\mathbb{F}$ is free. In fact, the morphism $\mu\colon \{a, b\}^+ \to \mathbb{Z}^{2\times 2}$ defined by*

$$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \textit{and} \qquad \mu(b) = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

is injective, and onto $\mathbb{F}$.

We observe that $M_a = \mu(a)$ and $M_b = \mu(b)$ have the inverses

$$M_a^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad M_b^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 0 & 1 \end{pmatrix}$$

in rational entries. Further, for a matrix M in (13.2.1),

$$M_a^{-1}M = \begin{pmatrix} 2^{m-1} & n/2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_b^{-1}M = \begin{pmatrix} 2^{m-1} & (n-1)/2 \\ 0 & 1 \end{pmatrix}.$$

This yields that $M_a^{-1}M \in \mathbb{F}$ if and only if n is even, and $M_b^{-1}M \in \mathbb{F}$ if and only if n is odd. Consequently, M_a and M_b generate $\mathbb{F}$, and each $M \in \mathbb{F}$ has a unique factorization $M = M_{a_1}M_{a_2}\ldots M_{a_k}$ in terms of the matrices M_a and M_b. This shows that $\mathbb{F}$ is freely generated by M_a and M_b, proving Claim 1.

Next we introduce, for each $x_i \in \Xi$, two commuting *integer variables* y_i and z_i, and denote $X = \{y_i, z_i \mid i = 1, 2, \ldots, k\}$. Further, for each i, let

$$M_i = \begin{pmatrix} y_i & z_i \\ 0 & 1 \end{pmatrix},$$

and let $\mathbb{M}(\Xi)$ be the subsemigroup of the multiplicative semigroup $\mathbb{Z}[X]^{2\times 2}$ generated by the matrices $M_1, M_2, \ldots, M_k$.

Claim 2. *The semigroup $\mathbb{M}(\Xi)$ is free. In fact, the morphism $\varphi\colon \Xi^+ \to \mathbb{M}(\Xi)$, defined by $\varphi(x_i) = M_i$, is an isomorphism.*

For this, consider any $M = M_{i_1}M_{i_2}\ldots M_{i_t} \in \mathbb{M}(\Xi)$. Then

$$M = \begin{pmatrix} y_{i_1}y_{i_2}\cdots y_{i_t} & m_{12} \\ 0 & 1 \end{pmatrix},$$

where the right upper corner entry $m_{12} = z_{i_1} + y_{i_1} z_{i_2} + \ldots + y_{i_1} y_{i_2} \cdots y_{i_{t-1}} z_{i_t}$. From the entry m_{12} we conclude, despite of the fact that the unknowns y_j and z_j commute, that it determines the sequence $i_1, i_2, \ldots, i_t$ uniquely, and Claim 2 follows.

For each morphism $\alpha: \Xi^+ \to B^+$, let $\overline{\alpha} = \mu\alpha\varphi^{-1}: \mathbb{M}(\Xi) \to \mathbb{Z}^{2\times 2}$ so that the following diagram commutes:

$$\begin{tikzcd} \Xi^+ \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & B^+ \arrow[d, "\mu"] \\ \mathbb{M}(\Xi) \arrow[r, "\overline{\alpha}"] & \mathbb{Z}^{2\times 2} \end{tikzcd}$$

Denote for each word $w \in \Xi^+$,

$$\varphi(w) = \begin{pmatrix} P_1(w) & P_2(w) \\ 0 & 1 \end{pmatrix} \in \mathbb{Z}[X]^{2\times 2}.$$

Finally, let $\hat{\alpha}: \mathbb{Z}[X] \to \mathbb{Z}$ be the ring morphism, which is defined by

$$\overline{\alpha}(M_i) = \begin{pmatrix} \hat{\alpha}(y_i) & \hat{\alpha}(z_i) \\ 0 & 1 \end{pmatrix}.$$

It follows that

$$\overline{\alpha}\varphi(w) = \begin{pmatrix} \hat{\alpha}(P_1(w)) & \hat{\alpha}(P_2(w)) \\ 0 & 1 \end{pmatrix}.$$

Now, a morphism $\alpha: \Xi^+ \to B^+$ is a solution of an equation $u = v$ over Ξ if and only if $\overline{\alpha}$ is a solution of the matrix equation $\varphi(u) = \varphi(v)$. We conclude that α is a solution of $u = v$ if and only if the corresponding ring morphism $\hat{\alpha}: \mathbb{Z}[X] \to \mathbb{Z}$ is a solution of the system

$$\begin{cases} P_1(u) = P_1(v) \\ P_2(u) = P_2(v) \end{cases}$$

of integer equations, or equivalently, of the equation $e(u, v) = 0$, where

$$e(u, v) = (P_1(u) - P_1(v))^2 + (P_2(u) - P_2(v))^2.$$

Let then $E = \{(u_i, v_i) \mid i \geq 1\}$ be a system of word equations, and denote by $J = \{e(u_i, v_i) \mid i \geq 1\}$ the corresponding system of polynomial equations. By Hilbert's basis theorem, J has an equivalent finite subsystem, say $J_0 = \{e(u_i, v_i) \mid i = 1, 2, \ldots, t\}$. By the above, if α is a solution of the finite subsystem $E_0 = \{(u_i, v_i) \mid i = 1, 2, \ldots, t\}$, then the corresponding $\hat{\alpha}$ is a solution of J_0, and thus also of J, which gives that α is a solution of E. This shows that E_0 is equivalent to E, proving the theorem. ∎

The above deserves a special comment on the proof method. We have proved there a combinatorial result for semigroups (with one operation) concerning *noncommuting* variables by reducing it to a result for rings (with two operations) concerning *commuting* variables.

13.2.2. Applications

We give two applications of the above compactness theorem. The first of these solves the isomorphism problem for finitely generated subsemigroups of free semigroups. For the proof of this we need an effective restriction of Theorem 13.2.2, see Problem 13.2.3.

LEMMA 13.2.3. *Let $R \subseteq \Xi^+ \times \Xi^+$ be a rational relation (considered as a system of equations). Then an equivalent finite subsystem $R_0 \subseteq R$ can be effectively found.*

Now we state our first application.

THEOREM 13.2.4. *Let $X, Y \subseteq A^+$ be two finite sets of words. It is decidable whether the semigroups X^+ and Y^+ are isomorphic.*

Proof. Since computing the base of X^+ is effective, we may assume that X and Y are the bases of their semigroups X^+ and Y^+. Further, we can suppose that $\mathrm{Card}(X) = n = \mathrm{Card}(Y)$; otherwise X^+ and Y^+ are not isomorphic. Let Ξ be a set of variables with $\mathrm{Card}(\Xi) = n$.

We consider the bijections $\varphi\colon X \to Y$. Note that there are only finitely many of these, since X and Y are finite. We need to decide whether the (unique) extension $\varphi\colon X^+ \to Y^+$ of φ is an isomorphism. For this, consider the representing morphisms, $\alpha_X\colon \Xi^+ \to X^+$ and $\alpha_Y = \varphi\alpha_X\colon \Xi^+ \to Y^+$ for X and Y. Now φ is an isomorphism if and only if it is injective, and this holds if and only if $E(X) = E(Y)$ as sets of relations defined in (13.1.1). By Theorem 13.1.3, $E(X)$ and $E(Y)$ are rational relations. Despite of the fact that the equivalence problem for rational relation is *undecidable*, we can solve the present problem. Considering $E(X)$ and $E(Y)$ as systems of equations, Theorem 13.2.2 states that they have finite equivalent subsystems, and by Lemma 13.2.3, such subsystems, say $E_0(X) \subseteq E(X)$ and $E_0(Y) \subseteq E(Y)$, can be effectively constructed. It is now a simple task to check whether α_X is a solution of $E_0(Y)$. If the answer is positive, then α_X is a solution of $E(Y)$, and $E(X) = \ker(\alpha_X) \subseteq E(Y)$. If the answer is negative, then clearly $E(X) \neq E(Y)$. Similarly, if α_Y is a solution of $E_0(X)$, then $E(Y) \subseteq E(X)$. This proves the theorem. ∎

We emphasize that the above theorem reveals one of the rare cases, where the isomorphism problem for infinite (finitely generated) semigroups is known to be decidable. Indeed, already for multiplicative matrix semigroups with nonnegative entries, it is undecidable.

As a second application of Theorem 13.2.2, we prove a decidability result for monoids of endomorphisms. By an *endomorphism* we mean a morphism $\alpha\colon A^+ \to A^+$ of a free semigroup A^+ into itself.

We first study test sets of languages, which constitute the original language theoretic formulation of the compactness property of Ehrenfeucht's conjecture. Let $L \subseteq A^+$ be a language, i.e., a subset of A^+. We say that a subset $T \subseteq L$ is a *test set* for L, if for all morphisms $\alpha, \beta\colon A^+ \to B^+$:

$$\alpha(u) = \beta(u) \text{ for all } u \in T \iff \alpha(u) = \beta(u) \text{ for all } u \in L.$$

From Theorem 13.2.2 we obtain

THEOREM 13.2.5. *Every set $L \subseteq A^+$ possesses a finite test set.*

Proof. For any alphabet B, let $\bar{B} = \{\bar{a} \mid a \in B\}$ denote its copy, and write for all words $u = a_1 a_2 \ldots a_k \in B^+$, $\bar{u} = \bar{a}_1 \bar{a}_2 \ldots \bar{a}_k \in \bar{B}^+$. Here the function $u \mapsto \overline{u}$ is an isomorphism $B^+ \to \bar{B}^+$.

Let $L \subseteq B^+$, and define $E = \{(u, \overline{u}) \mid u \in L\}$. By Theorem 13.2.2, when E is considered as a system of equations, it has an equivalent finite subsystem E_0. Let $T = \{u \mid (u, \overline{u}) \in E_0\}$. Certainly, T is a finite subset of L. Consider morphisms $\alpha, \beta: B^+ \to A^+$ satisfying $\alpha(u) = \beta(u)$ for all $u \in T$. Let $\Xi = B \cup \overline{B}$ be our set of variables, and define a morphism $\gamma: \Xi^+ \to A^+$ such that $\gamma(x) = \alpha(x)$ and $\gamma(\overline{x}) = \beta(x)$ for all $x \in B$. Now $\gamma(u) = \gamma(\overline{u})$ for all $u \in T$, and hence γ is a solution of the finite system E_0, and therefore γ is a solution of E as well. Consequently, $\alpha(u) = \beta(u)$ for all $u \in L$, which shows that T is a finite test set of L. ∎

Using Makanin's result, see Chapter 12, one can prove (see Problem 13.2.4)

LEMMA 13.2.6. *Let $L_1, L_2 \subseteq A^+$ be finite sets such that $L_1 \subseteq L_2$. It is decidable whether L_1 is a test set of L_2.*

With a help of the above lemma, we now show

THEOREM 13.2.7. *Let H be a finitely generated monoid of endomorphisms of a free semigroup A^+, and let w be a word in A^+. It is decidable, for given endomorphisms α, β of A^*, whether $\alpha\gamma(w) = \beta\gamma(w)$ for all $\gamma \in H$.*

Proof. Let H be generated by $G = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, and denote $L(H, w) = \{\gamma(w) \mid \gamma \in H\}$. We consider the *level sets*

$$L_p(H, w) = \{\gamma(w) \mid \gamma \in G^p\},$$

where G^p denotes the set of all compositions $\gamma_{i_j} \gamma_{i_{j-1}} \ldots \gamma_{i_1}$ of at most p morphisms from G. By Theorem 13.2.5, the set $L(H, w)$ has a finite test set, and, consequently, there exists an index p such that $L_p(H, w)$ is a test set for $L(H, w)$.

It follows that there exists an index q, and hence the least index q, such that $L_q(H, w)$ is a test set for $L_{q+1}(H, w)$ as well. We claim that $L_q(H, w)$ is a test set of $L(H, w)$. This is seen inductively as follows. Assume that for morphisms α and β, $\alpha\gamma(w) = \beta\gamma(w)$ for all $\gamma \in H^{q+i}$. Let $\gamma' \in H^i$, $\gamma_j \in G$ and $\gamma'' \in H^q$, and denote $\kappa = \gamma' \gamma_j \gamma'' \in H^{q+i+1}$. Now, by the assumption on α and β,

$$(\alpha\gamma')\gamma''(w) = \alpha\gamma'\gamma''(w) = \beta\gamma'\gamma''(w) = (\beta\gamma')\gamma''(w),$$

and, by the assumption on H^q, also $(\alpha\gamma')\gamma_j\gamma''(w) = (\beta\gamma')\gamma_j\gamma''(w)$, which shows that $\alpha\kappa(w) = \beta\kappa(w)$ as required.

Finally, by Lemma 13.2.6, the index q can be effectively found. Indeed, it is sufficient to check, for $i = 1, 2, \ldots$, whether $L_i(H, w)$ is a test set of $L_{i+1}(H, w)$.

The existence of q guarantees that this checking will end in a positive answer. The test set $L_q(H, w)$ is finite, and therefore the decidability claim follows. ∎

In the *DT0L problem* we are given a word $w \in A^+$, two monoids H_1 and H_2 of endomorphisms of A^* with equally many generators $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\{\beta_1, \beta_2, \ldots, \beta_n\}$, respectively. We ask whether for all sequences $i_1, i_2, \ldots, i_k$ of indices,

$$\alpha_{i_k} \ldots \alpha_{i_1}(w) = \beta_{i_k} \ldots \beta_{i_1}(w). \tag{13.2.2}$$

This problem reduces to Theorem 13.2.7, since, as is easy to see, (13.2.2) holds for all sequences of indices if and only if $\alpha_i \alpha(w) = \beta_i \alpha(w)$ for all i and $\alpha \in H_1$.

COROLLARY 13.2.8. *The DT0L problem is decidable.*

Our proof for the DT0L problem is short. However, it is based on two deep results, namely Makanin's algorithm and the compactness property. Amazingly this is the only known proof for this problem, although, its special case, the celebrated *D0L problem*, where H_1 and H_2 are both generated by a single morphism, has several different proofs.

Corollary 13.2.8 should be compared to the *DT0L language equivalence problem*, where we are given a word w, and two finitely generated monoids H_1 and H_2 of endomorphisms of a free monoid A^*, and we ask whether for all $\alpha \in H_1$ there exists an $\beta \in H_2$ such that $\alpha(w) = \beta(w)$. This problem is known to be undecidable.

13.3. Independence of finite systems of equations

As we have mentioned both the defect theorem and the compactness property formalize – although from a different perspective – a weak dimension property of words. A link between these two important results is found by considering independent systems of equations. At the same time this allows to analyze how weak these dimension properties are.

From the point of view of the compactness property, it is natural to ask how large a (necessarily finite) independent system of equations in n variables can be. From the point of view of (extensions of) the defect theorem, a natural question is to ask whether two or more 'different' relations force a larger 'defect effect' than a single equation, or how many different relations still allow nonperiodic solutions, i.e., solutions of combinatorial rank at least two. Here we formalize the notion of *different relations* as the independence of a system of equations and the *defect effect* of a system E of equations in n variables as the number $n - t$, where t is the maximal combinatorial rank of a solution of E.

The goal of this section is to search for answers to these questions. In other words, we want to construct as large as possible independent systems of equations in n variables in general, or requiring in addition that they still possess a solution of a certain rank. Recall that the set of equations was defined to be *independent*, if it was not equivalent to any of its proper subsets.

At this point we emphasize that for any solution $X = \alpha(\Xi)$ of a system E of equations, the (combinatorial) rank of X is a property of X, i.e., of *a* solution of E. The independence of E, in turn, is a property of *all* solutions of E. Therefore attempts to fulfill our goal face a problem of relating a particular solution of E to all solutions of it.

Using Makanin's algorithm, see Chapter 12, Theorem 13.1.3 and Problem 13.2.3, it is not difficult to conclude

THEOREM 13.3.1. *For each finite set $X \subseteq A^+$, one can effectively find an independent subset $E_I(X)$ of $E(X)$, which is equivalent to $E(X)$.*

From the point of view of our goal, Theorem 13.3.1 is not really helpful. Indeed, the set $E_I(X)$ need not be unique, and, moreover, to find $E_I(X)$ in practice is very difficult, as is illustrated by the challenging Problem 13.3.1.

We now define formally the central notion of this section. For this, let Ξ be a set of variables and denote $n = \mathrm{Card}(\Xi)$. For $t \leq n$, define

$$D_t(n) = \sup\{\mathrm{Card}(E) \mid E \text{ is an independent system over } \Xi \text{ having a solution of combinatorial rank at least } n - t\}.$$

Now, for all t with $1 \leq t \leq n-1$, $D_1(n) \leq D_t(n) \leq D_{n-2}(n)$, and therefore the two most natural choices for the parameter t are the values $t = 1$ and $t = n-2$. The former corresponds to the case where the defect effect is minimal, that is, equal to 1, while the case $t = n-2$ corresponds to the case where nonperiodic solutions are required to exist. The topic of this section is to search for lower bounds of the value $D_t(n)$.

13.3.1. In free semigroups

We start with an example.

EXAMPLE 13.3.2. Let $\Xi = \{x, y\} \cup \{p_i, q_i, z_i \mid i = 1, 2, \ldots, n\}$ be a set of variables, and let E be the following system of equations over Ξ:

$$E: \quad xp_jz_kq_jy = yp_jz_kq_jx \quad \text{for } j, k = 1, 2, \ldots, n.$$

Then $\mathrm{Card}(E) = n^2$ and $\mathrm{Card}(\Xi) = 3n + 2$. We claim that

(i) E has a solution of combinatorial rank $3n + 1$, and

(ii) E is independent.

The condition (i) is easy to verify. Indeed, choose $x = y$, which makes the equations of E trivial, so that a required solution can be found over the free semigroup having $3n + 1$ generators.

The essential part is to prove (ii). For this, we have to show that, for each pair (j, k), there exists a solution of the system

$$E(j, k) = E \setminus \{xp_jz_kq_jy = yp_jz_kq_jx\},$$

which is not a solution of E. Here is such a solution:

$$\begin{aligned} x &= b^2ab, \\ y &= b, \\ p_t &= \begin{cases} ba & \text{if } t = j\ , \\ bab & \text{otherwise}\ , \end{cases} \\ z_\ell &= \begin{cases} bab^2 & \text{if } \ell = k\ , \\ b & \text{otherwise}\ , \end{cases} \\ q_t &= \begin{cases} ba & \text{if } t = j\ , \\ a & \text{otherwise}\ . \end{cases} \end{aligned} \tag{13.3.1}$$

To find (13.3.1) is not obvious, but to verify that it is a required one, is easy. Indeed, we compute for $t = j$ and $\ell = k$,

$$xp_jz_kq_jy = b^2ab \cdot ba \ldots \neq b \cdot ba \cdot bab^2 \ldots = yp_jz_kq_jx.$$

Therefore (13.3.1) is not a solution of E. For the remaining cases, we compute

$$\begin{aligned} t \neq j, \ell \neq k: &\ b^2ab \cdot bab \cdot b \cdot a \cdot b = (bba)^3 b = b \cdot bab \cdot b \cdot a \cdot b^2ab, \\ t \neq j, \ell = k: &\ b^2ab \cdot bab \cdot bab^2 \cdot a \cdot b = (bba)^4 b = b \cdot bab \cdot bab^2 \cdot a \cdot b^2ab, \\ t = j, \ell \neq k: &\ b^2ab \cdot ba \cdot b \cdot ba \cdot b = (bba)^3 b = b \cdot ba \cdot b \cdot ba \cdot b^2ab, \end{aligned}$$

and, indeed, (13.3.1) is a solution of $E(j,k)$.

The above example yields

THEOREM 13.3.3. (i) *$D_1(n) = \Omega(n^2)$ in A^+ with* $\mathrm{Card}(A) = \infty$.
(ii) *$D_{n-2}(n) = \Omega(n^3)$ in A^+ with* $\mathrm{Card}(A) \geq 2$.

Proof. Part (i) of the claim follows directly from Example 13.3.2. Note that here we have to solve the equations over an infinitely generated free semigroup.

To prove part (ii), we modify Example 13.3.2 slightly by introducing n copies of x and y, say x_i and y_i for $i = 1, 2, \ldots, n$, and by setting

$$E': \qquad x_ip_jz_kq_jy_i = y_ip_jz_kq_jx_i \qquad \text{for } i, j, k = 1, 2, \ldots, n.$$

Accordingly we extend the solutions (13.3.1) by setting

$$\begin{aligned} x_t &= \begin{cases} b^2ab & \text{if } t = i\ , \\ a & \text{otherwise}, \end{cases} \\ y_t &= \begin{cases} b & \text{if } t = i\ , \\ a & \text{otherwise}\ . \end{cases} \end{aligned} \tag{13.3.2}$$

Then we have a system E' of cardinality n^3 over $5n$ variables, which, moreover, by the computations of Example 13.3.2, is independent. It contains nonperiodic solutions, namely, those specified by $x_i = y_i$. Hence, also (ii) is valid, and here indeed A can be binary. ∎

We remark that in the part (i) of the previous theorem we used an infinite generating set A. Indeed, this is unavoidable if the definition of $D_1(n)$ is defined based on the combinatorial rank. However, the definition can be based, for instance, on the *prefix rank*, which, for a subset $X \subseteq A^+$, is defined as the cardinality of the least prefix code Y such that $X \subseteq Y^+$. If $D_1(n)$ were defined using the prefix rank, then in the part (i) we could choose a binary generating set A. This is due to the fact that countably generated free semigroups can be embedded into a 2-generator one using a prefix code as an embedding.

13.3.2. In free monoids

Lower bounds of Section 13.3.1 can be improved if the equations are solved in free monoids instead of free semigroups. This is no surprise, since if the empty word is available, it is essentially easier to find nontrivial equations, and hence also independent systems of equations, having a given set as a solution.

EXAMPLE 13.3.4. Let

$$\Xi = \{y\} \cup \{x_i, p_i, q_i, x'_i, p'_i, q'_i, x''_i, p''_i, q''_i \mid i = 1, 2, \ldots, n\}$$

be a set of variables, and let E be the following system of equations,

$$E: \ yx_j p_k q_\ell x'_j p'_k q'_\ell x''_j p''_k q''_\ell = x_j p_k q_\ell x'_j p'_k q'_\ell x''_j p''_k q''_\ell y \quad \text{for } j, k, \ell = 1, 2, \ldots, n.$$

Then E is over $9n + 1$ variables, and it has n^3 equations. Let us denote by $e(j, k, \ell)$ the equation of E for the triple (j, k, ℓ). As in Example 13.3.2, we show that

(i) E has a solution of combinatorial rank $9n$ in A^*, and
(ii) E is independent.

Part (i) is again clear: fix $y = \varepsilon$, so that all equations of E become trivial over $\Xi \setminus \{y\}$. To prove the part (ii), we fix an equation from E, say $e(j_0, k_0, \ell_0)$, and search for a solution of the system $E \setminus \{e(j_0, k_0, \ell_0)\}$ which is not a solution of the whole E. Such a solution is provided by

$$\begin{array}{rcl} y &=& ababa, \\ x_{j_0} &=& p_{k_0} = q_{\ell_0} = ab, \\ x'_{j_0} &=& p'_{k_0} = q'_{\ell_0} = a, \\ x''_{j_0} &=& p''_{k_0} = q''_{\ell_0} = ba, \\ z &=& \varepsilon \text{ for all others.} \end{array} \tag{13.3.3}$$

Indeed, this is not a solution of $e(j_0, k_0, \ell_0)$, since $ababa \cdot ab \ldots \neq ab \cdot ab \cdot ab \ldots$. On the other hand, for any triple $(j, k, \ell) \neq (j_0, k_0, \ell_0)$, we obtain one of the following identities when substituting (13.3.3) into $e(j, k, \ell)$,

$$ababa = ababa,$$

$$ababa \cdot ab \cdot a \cdot ba = ab \cdot a \cdot ba \cdot ababa,$$

$$ababa \cdot ab \cdot ab \cdot a \cdot a \cdot ba \cdot ba = ab \cdot ab \cdot a \cdot a \cdot ba \cdot ba \cdot ababa.$$

Note that in the above factorizations we have omitted factors that are equal to ε.

By the above, we can formulate

THEOREM 13.3.5. (i) $D_1(n) = \Omega(n^3)$ *in* A^* *with* $\mathrm{Card}(A) = \infty$.
(ii) $D_{n-2}(n) = \Omega(n^4)$ *in* A^* *with* $\mathrm{Card}(A) \geq 2$.

The proof of Theorem 13.3.5 is analogous to that of Theorem 13.3.3. Note also that the remark made after Theorem 13.3.3 applies to the present case.

13.3.3. In free groups

The problems of this section change drastically when free semigroups are replaced by free groups, although both of these satisfy the compactness property, cf. Example 13.5.5. First, instead of considering the values $D_t(n)$ in the free groups, it is more meaningful to consider only the maximal cardinality of independent systems of equations. Second, the independent systems can be unboundedly large in free groups.

THEOREM 13.3.6. *Let n be a positive integer. There exists an independent system E_n of n equations in a free group using six variables.*

Proof. Let $\Xi = \{x, y, z, \bar{x}, \bar{y}, \bar{z}\}$ be a set of variables. Denote $\bar{u} = \bar{x}_m \bar{x}_{m-1} \ldots \bar{x}_1$ for each $u = x_1 x_2 \ldots x_m$, and $\bar{\bar{x}} = x$, where $x_i \in \{x, y, z\}$. Let $[u, v] = \bar{u}\bar{v}uv$ correspond to the commutator word of u and v for $u, v \in \Xi^+$, and define inductively,

$$[v_1, \ldots, v_{k+1}] = [[v_1, \ldots, v_k], v_{k+1}].$$

Let $v_k = \bar{z}^k x^k \bar{y} z^k$ for $k \geq 1$, and let $w_i = [v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]$. For each $n \in \mathbb{N}$, consider the system E_n of equations,

$$E_n : \qquad (w_i)^2 = w_i \quad \text{for } i = 1, 2, \ldots, n.$$

It is now easy to verify that for two generators a, b of a free group,

$$x = a,\ \bar{x} = a^{-1},\ y = a^j,\ \bar{y} = a^{-j},\ z = b,\ \bar{z} = b^{-1}$$

is a solution of $(w_i)^2 = w_i$, if $i \neq j$, but not of $(w_j)^2 = w_j$. This shows that E_n is independent. ∎

REMARK 13.3.7. For groups G one usually prefers *group equations* $w = \varepsilon$, where w is an element of the free group $\Xi^{(*)}$ generated by the variables in Ξ, that is, w is a word over the alphabet $\Xi \cup \Xi^{-1}$, where $\Xi^{-1} = \{x^{-1} \mid x \in \Xi\}$. In this case a solution $\alpha \colon \Xi^{(*)} \to G$ of an equation $w = \varepsilon$ is required to respect the inverses, $\alpha(x^{-1}) = \alpha(x)^{-1}$ for all $x \in \Xi$. It is straightforward to show, see Problem 13.3.5, that a group G satisfies the compactness property for group equations if and only if G satisfies it for the ordinary semigroup equations.

In the proof of the above theorem the equations $(w_i)^2 = w_i$ can now be replaced by the equations $w_i = \varepsilon$, and the variables $\bar{x}, \bar{y}, \bar{z}$ can be replaced by the expressions x^{-1}, y^{-1}, z^{-1} that respect the group inversion. With these modifications Theorem 13.3.6 can be rephrased using only three variables.

13.4. Semigroups without the compactness property

We now consider systems of equations in arbitrary semigroups. A semigroup S is said to satisfy the *compactness property*, if for every finite set Ξ of variables, each system $E \subseteq \Xi^+ \times \Xi^+$ of equations is equivalent in S to a finite subsystem $E' \subseteq E$, that is, if every independent system of equations is finite.

In this section we shall demonstrate, via several examples of semigroups, that the compactness property of the previous section does not hold in general.

EXAMPLE 13.4.1. Let $A = \{a, b\}$, and $\Xi = \{x, y, z\}$. The monoid $\mathrm{Fin}(A^*)$ of all nonempty finite subsets of the free monoid A^* does not satisfy the compactness property. Indeed, the system E of equations

$$xy^i x = xz^i x \qquad \text{for } i \geq 1$$

over three variables does not have an equivalent finite subsystem in $\mathrm{Fin}(A^*)$. To see this, define, for each $n \geq 1$, a morphism $\sigma_n\colon \Xi^+ \to \mathrm{Fin}(A^*)$ as follows:

$$\begin{aligned}
\sigma_n(x) &= \{a^j \mid 0 \leq j \leq 2n+2\},\\
\sigma_n(y) &= \{a^i b a^j \mid 0 \leq i+j < n, \quad \text{or}\\
&\qquad\qquad 0 \leq i \leq 2n+2 \text{ and } n+1 \leq j \leq 2n+2, \quad \text{or}\\
&\qquad\qquad n+1 \leq i \leq 2n+2 \text{ and } 0 \leq j \leq 2n+2\},\\
\sigma_n(z) &= \{a^i b a^j \mid 0 \leq i, j \leq 2n+2\}.
\end{aligned}$$

Now for all $i \geq 1$,

$$\sigma_n(xy^i x) \subseteq \sigma_n(xz^i x) = \{a^{r_1} b a^{r_2} b \ldots b a^{r_i} \mid 0 \leq r_k \leq 4n+4,\ k = 1, 2, \ldots, i\}.$$

We leave it as an exercise to show that $\sigma_n(xy^i x) = \sigma_n(xz^i x)$ for all $i < n-1$. However, $\sigma_n(xy^{n+1}x) \neq \sigma_n(xz^{n+1}z)$, since $(ba^n)^n b \in \sigma_n(xz^{n+1}x)$, but $(ba^n)^n b \notin \sigma_n(xy^{n+1}x)$. These show that E does not have a finite equivalent subsystem in $\mathrm{Fin}(A^*)$.

EXAMPLE 13.4.2. Let $\mathbf{B}$ be the monoid of functions generated by $\alpha, \beta\colon \mathbb{N} \to \mathbb{N}$,

$$\alpha(n) = \max\{0, n-1\}, \qquad \beta(n) = n+1.$$

The product of $\mathbf{B}$ is the ordinary composition of functions. This monoid is called the *bicyclic monoid*, and it has a simple monoid presentation $\langle a, b \mid ab = 1\rangle$.

Let $\iota\colon \mathbb{N} \to \mathbb{N}$ be the identity function. Now $\alpha\beta = \iota$, but $\beta\alpha \neq \iota$. Let $\gamma_i = \beta^i \alpha^i$ for all $i \geq 0$. We have

$$\gamma_i(n) = \begin{cases} i & \text{if } n \leq i,\\ n & \text{if } n > i,\end{cases}$$

and we observe that $\gamma_i \gamma_j = \gamma_{\max\{i,j\}}$. In particular, each γ_i is an idempotent of $\mathbf{B}$, i.e., $\gamma_i^2 = \gamma_i$. Consider the system $E \subseteq \Xi^+ \times \Xi^+$ consisting of the equations

$$x^i y^i z = z \qquad \text{for } i \geq 1$$

in the variables $\Xi = \{x, y, z\}$. For a fixed j, the morphism δ_j defined by $\delta_j(x) = \beta$, $\delta_j(y) = \alpha$ and $\delta_j(z) = \gamma_j$, is a solution of $x^i y^i z = z$ for all $i \leq j$, but δ_j is not a solution of $x^{j+1}y^{j+1}z = z$. We conclude that the system E does not have an equivalent finite subsystem, and therefore the bicyclic monoid does not satisfy the compactness property.

This example extends directly to the bicyclic semigroup (without the identity element).

EXAMPLE 13.4.3. In this example we give a finitely generated semigroup S such that it and its ideal I both satisfy the compactness property but the Rees quotient S/I does not.

Let $S = A^+$ for $A = \{a, b\}$, and define

$$I = A^*\{ab^k ab^j a \mid 1 \leq j \leq k\}A^*.$$

It is plain that I is an ideal of A^+, and that both A^+ and I, as a subsemigroup of A^+, satisfy the compactness property. Let $\Xi = \{x, y, z\}$ be a set of variables, and E the following system of equations,

$$E\colon \quad xy^k xz^j x = xy^k xz^j xx \qquad \text{for } k, j \geq 1.$$

Let $E' \subseteq E$ be any finite subsystem, and set

$$m = \max\{j \mid xy^k xz^j x = xy^k xz^j xx \text{ in } E'\}.$$

Define a morphism $\alpha\colon \Xi^+ \to A^+/I$ by $\alpha(x) = a$, $\alpha(y) = b^m$, $\alpha(z) = b$. Now, by the definition of I, $\alpha(xy^k xz^j x) = \alpha(xy^k xz^j xx)$ in A^+/I for each $xy^k xz^j x = xy^k xz^j xx$ from E', but $\alpha(xyxz^{m+1}x) \neq \alpha(xyxz^{m+1}xx)$. We conclude that E' is not equivalent to E. This proves that A^+/I does not satisfy the compactness property.

We obtain other semigroups that do not satisfy the compactness property after we prove some necessary conditions for this property in the next section, see Examples 13.5.7 and 13.5.11.

13.5. Semigroups with the compactness property

In this section, we search for connections between the compactness property and some classical notions of semigroup theory. We shall give, apart from positive examples, some necessary conditions for a semigroup to guarantee that it satisfies the compactness property. These conditions provide examples of semigroups that defy the compactness property.

Also, it turns out that all the monoids in a *variety* satisfy the compactness property if and only if these monoids satisfy the maximal condition on congruences. Such a characterization does not hold for individual monoids (or semigroups). Indeed, the free semigroups do not satisfy the maximal condition on congruences, but, as we have seen, they do satisfy the compactness property. The bicyclic semigroup, on the other hand, is an example of a semigroup that does satisfy the maximal condition, but does not satisfy the compactness property.

13.5.1. An extension of the proof

As is shown in Theorem 13.5.1, the compactness property is preserved under some natural operations on semigroups. On the other hand, the second case of the theorem shows that the compactness property fails on some other basic operations.

THEOREM 13.5.1. (i) *The class of semigroups that satisfy the compactness property is closed under taking isomorphic images, subsemigroups and finite direct products.*

(ii) *The class of semigroups that satisfy the compactness property is not closed under taking morphic images or infinite direct products.*

Proof. The proof of (i) is fairly easy, and it is left to the reader as an exercise.

That the compactness property is not inherited by the morphic images (or by the quotients) holds simply because the free semigroup $\{a,b\}^+$ satisfies the compactness property, but, as we have seen, its morphic image $\mathbf{B}$ (the bicyclic semigroup) does not.

Consider the semigroup $F = \mathrm{Fin}(A^*)$ of all nonempty finite subsets of A^* from Example 13.4.1. For each $k \geq 0$, define a relation θ_k on F by

$$X\theta_k Y \iff X \cup A^{[k]} = Y \cup A^{[k]},$$

where $A^{[k]} = \{w \in A^* \mid |w| \geq k\}$. It is immediate that θ_k is a congruence on F, that is, for all $X, Y, Z \in F$, $X\theta_k Y$ implies that also $ZX\theta_k ZY$ and $XZ\theta_k YZ$. Furthermore, the quotient F/θ_k is a finite semigroup, and therefore it satisfies the compactness property.

Let $S = \prod_{k=0}^{\infty} F/\theta_k$ be the direct product of these finite semigroups, and define a morphism $\alpha: F \to S$ by its projections,

$$\pi_k \alpha(X) = X\theta_k,$$

where π_k is the projection of S onto F/θ_k, and $X\theta_k \in F/\theta_k$ is the congruence class of X with respect to θ_k.

For any two distinct $X, Y \in F$, we have $X\theta_k \neq Y\theta_k$ for $k = \max\{|X|, |Y|\}$, and therefore the morphism α is an embedding of F into S. We conclude from Example 13.4.1 and the case (i) of the present theorem that $S = \prod_k F/\theta_k$ does not satisfy the compactness property although each of the semigroups F/θ_k does so. ∎

EXAMPLE 13.5.2. By Theorem 13.2.2, finitely generated free semigroups satisfy the compactness property. For a countably generated free semigroup A^+, where $A = \{a_1, a_2, \ldots\}$, the mapping $\alpha: A^+ \to \{a,b\}^+$ defined by $\alpha(a_i) = a^i b$ for all i, is an embedding of A^+ into the free semigroup $\{a,b\}^+$. Therefore the compactness property holds also for countably generated free semigroups.

EXAMPLE 13.5.3. A *trace monoid* M, often referred to as a *free partially commutative monoid*, is a monoid that has a presentation $\langle A \mid ab = ba \ ((a,b) \in R)\rangle$, where R is a symmetric relation on the finite set $A = \{a_1, a_2, \ldots, a_k\}$ of generators. A trace monoid M can be embedded into the k-folded direct product $P^{(k)} = \{a,b\}^* \times \ldots \times \{a,b\}^*$. To see this, let for each $i = 1, 2, \ldots, k$, ν_i be the vector defined by the conditions,

$$\pi_j(\nu_i) = \begin{cases} a & \text{if } i = j, \\ \varepsilon & \text{if } (a_i, a_j) \in R \text{ and } i \neq j, \\ b & \text{otherwise,} \end{cases}$$

where π_j denoted the j-th projection of $P^{(k)}$ into $\{a,b\}^*$. Then it is plain that $\nu_j\nu_i = \nu_i\nu_j$ for $i \neq j$ if and only if $(a_i, a_j) \in R$, which shows that the monoid generated by the vectors ν_i, for $i = 1, 2, \ldots, k$, is isomorphic to M. Now, the claim follows from Theorem 13.5.1.

The above examples are special cases of a general theorem which we now prove as an extension of Theorem 13.2.2.

THEOREM 13.5.4. *Let R be a commutative Noetherian ring R containing an identity element. If a semigroup S can be embedded in the multiplicative matrix semigroup $R^{n\times n}$, then it satisfies the compactness property.*

Proof. We give a detailed outline of the proof. Indeed, polynomials over such a commutative Noetherian ring R satisfy Hilbert's basis theorem, and therefore R can be used instead of $\mathbb{Z}$ in the proof of Theorem 13.2.2. In the case $n = 2$, for each variable x_i in $\Xi = \{x_1, x_2, \ldots, x_k\}$, we introduce a matrix

$$M_i = \begin{pmatrix} x_{i1} & x_{i2} \\ x_{i3} & x_{i4} \end{pmatrix},$$

where $x_{i1}, x_{i2}, x_{i3}, x_{i4}$ are commuting variables. Let the set of these new variables be $X = \{x_{ij} \mid i = 1, 2, \ldots, k, j = 1, 2, 3, 4\}$. The subsemigroup $\mathbb{M}$ of $R[X]^{2\times 2}$ generated by the matrices M_i is a free semigroup (see Problem 13.2.2). If S is a semigroup such that there exists an embedding $\mu: S \to R^{2\times 2}$, then the commuting diagram of the proof of Theorem 13.2.2 takes the form:

$$\begin{tikzcd} \Xi^+ \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & S \arrow[d, "\mu"] \\ \mathbb{M} \arrow[r, "\overline{\alpha}"] & R^{2\times 2} \end{tikzcd}$$

The proof of Theorem 13.2.2 can now be easily modified for Theorem 13.5.4. The general case is treated in a similar way. ∎

As an illustration of the above, we provide

EXAMPLE 13.5.5. It is easy to see that the matrices

$$M_a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a free subgroup of $\mathbb{Z}^{2\times 2}$. Further, as is well known, every finitely generated free group can be embedded into a free group generated by two elements, and therefore Theorem 13.5.4 yields that the finitely generated free groups satisfy the compactness property.

13.5.2. Necessary conditions

The next result is motivated by Example 13.4.2.

An element e of a semigroup S is an *idempotent*, if it satisfies $e^2 = e$. The idempotents of a semigroup S can be partially ordered by defining: $e \leq f$ if and only if $fe = e = ef$. We say that S satisfies the *chain condition on idempotents*, if each subset of idempotents of S contains a maximal and a minimal element, i.e., each chain $\ldots e_{i-1} < e_i < e_{i+1} \ldots$ of idempotents is finite.

THEOREM 13.5.6. *Let S be a finitely generated semigroup satisfying the compactness property. Then it satisfies the chain condition on idempotents.*

Proof. Let S be generated by n elements, and let $\mu\colon \Xi^+ \to S$ be a natural morphism onto S, where $\mathrm{Card}(\Xi) = n$.

Suppose first that $e_1 > e_2 > \ldots$ is an infinite descending chain of idempotents of S. Therefore $e_i e_j = e_{\max\{i,j\}}$ for all $i, j \geq 1$. Now, for each $i \geq 1$, let $w_i \in \Xi^+$ be a word such that $\mu(w_i) = e_i$, and denote $Y = \Xi \cup \{y\}$. Consider the system E of equations $w_i y = y$, with $i \in \mathbb{N}$, over Y. For each $j \geq 1$, let $\alpha_j\colon Y^+ \to S$ be a morphism such that $\alpha_j(x) = \mu(x)$ for $x \in \Xi$, and let $\alpha_j(y) = e_j$. Now $\alpha_j(w_i y) = e_i e_j$ for all i and j. Consequently, α_j is a solution of $w_i y = y$ for all i with $i \leq j$, but α_j is not a solution to $w_{j+1} y = y$. We conclude that E does not have an equivalent finite subsystem in S.

The case of an infinite ascending chain $e_1 < e_2 < \ldots$ of idempotents is treated analogously. Hence our proof is complete. ■

Theorem 13.5.6 yields immediately another example of semigroups that do not satisfy the compactness property.

EXAMPLE 13.5.7. The free inverse semigroups do not satisfy the chain condition on idempotents, and therefore the compactness property fails for these. Indeed, the *free monogenic inverse semigroup*, which is generated by one element as an inverse semigroup, has a semigroup presentation

$$FI_1 = \langle a, b \mid a = aba,\ b = bab,\ a^m b^{m+n} a^n = b^n a^{n+m} b^m,\ n, m \geq 1 \rangle.$$

Here $a^n b^n$ is an idempotent for $n \geq 1$, and $a^n b^n \cdot a^m b^m = a^n b^n = a^m b^m \cdot a^n b^n$, i.e., $a^n b^n \leq a^m b^m$ for all $n \geq m$.

Next we look for a connection between the compactness property and certain types of congruences of semigroups. A congruence θ of a semigroup S is called *nuclear*, if it is induced by an endomorphism, i.e., if $\theta = \ker(\alpha)$ for an endomorphism $\alpha: S \to S$. In other words, a congruence θ of S is nuclear, if the quotient S/θ is isomorphic to a subsemigroup of S.

LEMMA 13.5.8. *Let S be a semigroup with the compactness property, and let Ξ be a finite alphabet. Then each sequence $\alpha_i: \Xi^+ \to S$ of morphisms with $\ker(\alpha_i) \subset \ker(\alpha_{i+1})$, for $i = 1, 2, \ldots$, is finite.*

Proof. Assume that $(\alpha_i)_{i \geq 0}$ is an infinite sequence of morphisms such that $\ker(\alpha_i) \subset \ker(\alpha_{i+1})$. Consider a system $E = \{(u_i, v_i) \mid i = 1, 2, \ldots\}$ of equations, where $(u_i, v_i) \in \ker(\alpha_{i+1}) \setminus \ker(\alpha_i)$ for each i. Clearly, E has no equivalent finite subsystem in S. This proves the lemma. ∎

We say that a semigroup S satisfies the *maximal condition on nuclear congruences*, if each ascending chain $\theta_1 \subset \theta_2 \subset \ldots$ of its nuclear congruences is finite. We obtain our second necessary condition.

THEOREM 13.5.9. *Let S be a semigroup with the compactness property. Then the finitely generated subsemigroups of S satisfy the maximal condition on nuclear congruences.*

Proof. Suppose that S_0 is a finitely generated subsemigroup of S such that, for each $i \geq 1$, $\alpha_i: S_0 \to S_0$ is an endomorphism satisfying $\ker(\alpha_i) \subset \ker(\alpha_{i+1})$ for all $i \geq 1$. Let $\mu: \Xi^+ \to S_0$ be a natural morphism onto S_0. Consequently, $\ker(\alpha_i \mu) \subset \ker(\alpha_{i+1} \mu)$ for all $i \geq 1$, and the claim follows from Lemma 13.5.8. ∎

Theorem 13.5.9 can be used to formulate another necessary condition for the compactness property. A semigroup S is said to be *Hopfian*, if it is not isomorphic to a quotient S/θ for any of its nontrivial congruences θ. Equivalently, S is Hopfian, if every surjective endomorphism $\alpha: S \to S$ is an automorphism.

THEOREM 13.5.10. *Let S be a semigroup with the compactness property. Then every finitely generated subsemigroup of S is Hopfian.*

Proof. Assume that S_0 is a non Hopfian finitely generated subsemigroup of S, and let $\alpha: S_0 \to S_0$ be a noninjective endomorphism onto S_0. Now the nuclear congruences $\theta_i = \ker(\alpha^i)$ for $i \geq 1$ form a properly ascending chain, and hence the claim follows by Theorem 13.5.9. ∎

Note that if a semigroup S satisfies the compactness property, then S itself need not be Hopfian, if it is infinitely generated. Indeed, a countably generated free semigroup A^+ satisfies the compactness property, but it is not Hopfian.

EXAMPLE 13.5.11. One of the simplest non Hopfian semigroups is the so called *Baumslag-Solitar group* which has a group presentation $G = \langle a, b \mid b^2 a = ab^3 \rangle$.

For simplicity, we use group equations introduced in Remark 13.3.7. Let $[g, h] = g^{-1}h^{-1}gh$ be the *commutator* of the elements $g, h \in G$, and denote by ε_G the identity element of G. The morphism $\alpha: G \to G$ defined by $\alpha(a) = a$ and $\alpha(b) = b^2$ is surjective, because $\alpha([a, b^{-1}]) = b$. However, it can be shown that $\alpha(g) = \varepsilon_G$ for $g = [a^{-1}ba, b] \neq \varepsilon_G$, and therefore α is not injective.

Let $\Xi = \{x, y\}$, and denote by $\Xi^{(*)}$ the free group generated by Ξ. Let $u = [x^{-1}yx, y]$. Define a (group) morphism $\beta: \Xi^{(*)} \to \Xi^{(*)}$ by $\beta(x) = x$ and $\beta(y) = [x, y^{-1}]$. We obtain a system E of equations $\{\beta^i(u) = \varepsilon \mid i \geq 1\}$, which has no equivalent finite subsystem. Now if $\gamma: \Xi^{(*)} \to G$ is defined by $\gamma(x) = a$ and $\gamma(y) = b$, then $\alpha^i\gamma(\beta^j(u)) = \varepsilon_G$ for $j < i$, but $\alpha^i\gamma(\beta^i(u)) \neq \varepsilon_G$.

13.5.3. The compactness property for varieties

For convenience, in the rest of this chapter we shall consider monoids and groups rather than semigroups.

Recall that a class $\mathcal{V}$ of monoids (resp. groups) is a *variety*, if it is closed under taking submonoids (resp. subgroups), morphic images, and arbitrary direct products. By Birkhoff's theorem, a variety of monoids becomes defined by a set of *identities* $u \equiv v$, which are equations $(u, v) \in \Xi^*$ such that every morphism $\alpha: \Xi^* \to M$ with $M \in \mathcal{V}$ is a solution of (u, v). Note that here the set Ξ of variables is allowed to be infinite, although the equations are required to be finite. For instance, the identity $x_1x_2 \equiv x_2x_1$ defines the variety of all commutative monoids.

A monoid M satisfies the *maximal condition on congruences*, if each set of congruences of M has a maximal element. The following general result is easy to prove using Zorn's lemma.

Lemma 13.5.12. *The following conditions are equivalent for a monoid M.*

(i) *M satisfies the maximal condition on congruences.*
(ii) *Each ascending chain $\theta_1 \subset \theta_2 \subset \ldots$ of congruences of M is finite.*
(iii) *For each congruence θ of M generated by a subset $E \subseteq \theta$ there exists a finite subset $E' \subseteq E$ such that E' generates θ.*

Using the above lemma, we obtain the following characterization.

Theorem 13.5.13. *A variety $\mathcal{V}$ of monoids satisfies the compactness property if and only if each finitely generated monoid $M \in \mathcal{V}$ satisfies the maximal condition on congruences.*

Proof. We first recall that, for all $n \geq 1$, a variety $\mathcal{V}$ has a monoid V_n generated by an n-element subset B satisfying the following extension property: for any mapping $\gamma_B: B \to M$ with $M \in \mathcal{V}$, there exists a unique morphism $\gamma: V_n \to M$, which is an extension of γ_B, i.e., $\gamma(b) = \gamma_B(b)$ for all $b \in B$. Such a monoid V_n is a *free monoid of* $\mathcal{V}$. The extension property yields that each morphism $\alpha: \Xi^* \to M$, with $M \in \mathcal{V}$ and $\mathrm{Card}(\Xi) = n$, can be factored as $\alpha = \beta\mu$, where $\mu: \Xi^* \to V_n$ is the natural morphism onto V_n and $\beta: V_n \to M$ is a morphism.

Suppose now that each $M \in \mathcal{V}$ satisfies the maximal condition on congruences. Let $E = \{(u_i, v_i) \mid i \geq 1\} \subseteq \Xi^* \times \Xi^*$ be a system of equations, where $\mathrm{Card}(\Xi) = n$. Further, let θ be the congruence on V_n generated by the relation $\mu(E) = \{(\mu(u_i), \mu(v_i)) \mid i \geq 1\}$, i.e., θ is the smallest congruence on V_n containing $\mu(E)$. By assumption and Lemma 13.5.12, θ is generated by a finite subset E'' of $\mu(E)$. Clearly, $E'' = \mu(E')$ for a finite subset E' of E. Now, if $M \in \mathcal{V}$ and $\alpha = \beta\mu: \Xi^* \to M$ is a solution to E', then $E' \subseteq \ker(\alpha)$ and hence $E'' = \mu(E') \subseteq \ker(\beta)$, which implies that $\theta \subseteq \ker(\beta)$. In particular, $\mu(E) \subseteq \ker(\beta)$, and, consequently, $E \subseteq \ker(\alpha)$. Therefore M satisfies the compactness property.

To prove the converse, let $\mathcal{V}$ be a variety of monoids, and assume that $M \in \mathcal{V}$ is a finitely generated monoid that does not satisfy the maximal condition on congruences. Let $\theta_1 \subset \theta_2 \subset \dots$ be an ascending chain of congruences of M, and let $\alpha_i: M \to M_i = M/\theta_i$, for each $i \geq 1$, be a surjective morphism with $\ker(\alpha_i) = \theta_i$. Each quotient M_i is in $\mathcal{V}$, since $\mathcal{V}$ is a variety. Let again $\mu: \Xi^* \to M$ be a natural morphism from the free monoid Ξ^* onto M.

We observe that the congruences $\theta_i' = \ker(\alpha_i\mu)$ of Ξ^*, for $i \geq 1$, form a properly ascending chain. For each $i \geq 2$, choose a pair $(u_i, v_i) \in \theta_i' \setminus \theta_{i-1}'$, and let $E = \{(u_i, v_i) \mid i \geq 2\}$. Consider the direct product $\Pi_{i\geq 1} M_i$. Since $\mathcal{V}$ is a variety, also $\Pi_{i\geq 1} M_i \in \mathcal{V}$. Let $\beta_i: M_i \to \Pi_{i\geq 1} M_i$ be the natural embedding, and define $\gamma_i = \beta_i\alpha_i\mu: \Xi^* \to \Pi_{i\geq 1} M_i$. Now $\gamma_i(u_j) = \gamma_i(v_j)$ for all $j \leq i$, but $\gamma_i(u_{i+1}) \neq \gamma_i(v_{i+1})$, and therefore E does not have an equivalent finite subsystem in $\Pi_{i\geq 1} M_i$. ∎

Theorem 13.5.13 is interesting in the sense that it provides a nontrivial characterization when a variety, i.e., all monoids in that variety, satisfies the compactness property. No similar characterizations are known for individual monoids (or semigroups).

By Redei's Theorem, the finitely generated commutative monoids satisfy the maximal condition on congruences. Hence we have the following corollary of Theorem 13.5.13.

COROLLARY 13.5.14. *Every commutative monoid satisfies the compactness property.*

By Theorem 13.2.2 and Corollary 13.5.14, the compactness property holds in the extremes with respect to commutativity, namely, for the free semigroups as well as for the commutative semigroups. Both results are based on Hilbert's basis theorem. Note also that in neither of these cases the semigroups need be finitely generated.

As a difference between these cases we mention that it is not known whether arbitrary large independent systems of equations with n variables exist in free semigroups, while for commutative semigroups such are easy to find, see Problem 13.3.3.

The proof of Theorem 13.5.13 does not use the property that varieties are closed under taking submonoids. Since (monoid) morphic images and direct products of groups are groups, the proof applies also to groups. Further, in a group G, the congruence class containing the identity element is a normal subgroup of G that determines the congruence θ. Therefore we obtain

THEOREM 13.5.15. *A variety $\mathcal{V}$ of groups satisfies the compactness property if and only if each finitely generated group of $\mathcal{V}$ satisfies the maximal condition on normal subgroups.*

EXAMPLE 13.5.16. For groups Corollary 13.5.14 can be much improved. Let G be a group, and let $[a,b] = a^{-1}b^{-1}ab$ be the commutator of the elements $a, b \in G$. The *metabelian groups* form a variety defined by a single identity $[[x_1,x_2],[x_3,x_4]] \equiv \varepsilon$. Clearly, every abelian group is metabelian. Moreover, by Hall's theorem, every finitely generated metabelian group satisfies the maximal condition on normal subgroups. Therefore the metabelian groups satisfy the compactness property. This result is interesting, since the free semigroups can be embedded into a free metabelian group. This gives the original proof of Albert and Lawrence to Theorem 13.2.2 for free semigroups. However, the proof of Hall's theorem uses a variant of Hilbert's basis theorem.

EXAMPLE 13.5.17. The nilpotent groups satisfy the compactness property, although they do not form a variety. Indeed, the smallest variety that contains all nilpotent groups consists of all groups. However, each nilpotent group belongs to a variety (nilpotent groups of class n, for some n) that does satisfy the maximal condition on normal subgroups.

Problems

Section 13.1

13.1.1 If a system $E \subseteq \Xi^+ \times \Xi^+$ of equations does not have an equivalent finite subsystem (in a semigroup S, in general), then E has an infinite subsystem $E' = \{(u_i, v_i) \mid i = 1, 2, \ldots\}$ ordered in such a way that for each j there exists a solution of the system $u_i = v_i$ for $i = 1, 2, \ldots, j$, which is not a solution of the equation $u_{j+1} = v_{j+1}$.

13.1.2 Show that the system $xy^iz = zy^ix$, $i = 1, 2, 3$, is dependent in Σ^*.

13.1.3 Prove Theorem 13.1.3.

*13.1.4 Classify the relations defined by three generator subsemigroups of a free semigroup. (See Spehner 1976)

Section 13.2

13.2.1 Let S be a semigroup, $\Xi = \{x, y\}$, and let $k \geq 0$ and $m > 0$ be two fixed integers. Denote $I = \{k + jm \mid j \geq 0\}$. Show that the system of equations $x^i = y^i$ $(i \in I)$ in S is equivalent to its finite subsystem $x^i = y^i$ $(i \in \{k + jm \mid j < k\})$.

13.2.2 Let R be any commutative nontrivial ring with an identity 1 ($\neq 0$) element. Let $X = \{x_{ij} \mid 1 \leq j \leq k,\ 1 \leq i \leq 4\}$ be a set of commuting variables for R. Show that the matrix semigroup $\mathbb{M}(\Xi)$ of $R[X]^{2\times 2}$, generated by the matrices

$$M_i = \begin{pmatrix} x_{i1} & x_{i2} \\ x_{i3} & x_{i4} \end{pmatrix},$$

is a free semigroup.

13.2.3 Prove Lemma 13.2.3: each rational relation $R \subseteq \Xi^+ \times \Xi^+$ has (as a system of equations) an equivalent finite subsystem $R_0 \subseteq R$ that can be effectively found.

*13.2.4 Prove Lemma 13.2.6: it is decidable whether L_1 is a test set of L_2 for finite sets $L_1 \subseteq L_2$. For this, one applies Makanin's result in proving that the equivalence of two finite systems of equations is decidable. (See Culik II and Karhumäki 1983)

13.2.5 The famous $2n$-conjecture for D0L systems states that if $\alpha^i(w) = \beta^i(w)$ for all $i < 2n$ (where $\alpha, \beta\colon A^* \to A^*$ are morphisms with $\mathrm{Card}(A) = n$ and $w \in A^*$ is a word), then $\alpha^i(w) = \beta^i(w)$ for all $i \geq 0$. Prove, by modifying the system of equations in Section 13.3.2, that such a conjecture does not hold for HD0L systems, that is, for morphic images of D0L systems. Indeed, show that if a function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies

$$\gamma_1\alpha^i(w) = \gamma_2\beta^i(w) \text{ for all } i < f(n) \Rightarrow \gamma_1\alpha^i(w) = \gamma_2\beta^i(w) \text{ for all } i \geq 0$$

for all words $w \in A^*$ and morphisms $\alpha, \beta, \gamma_1, \gamma_2\colon A^* \to A^*$, where $\mathrm{Card}(A) = n$, then $f(n) = \Omega(n^4)$. (See Plandowski 1995)

*13.2.6 Show that the $2n$-conjecture holds for D0L systems over binary alphabets. (See Karhumäki 1981)

Section 13.3

*13.3.1 Show that any language over a binary alphabet has a test set of cardinality three. This problem on three variables is completely open! In particular, does there exist an independent system of three equations in three variables that has a solution of rank 2 over a free semigroups A^+? (For the binary case, see Ehrenfeucht, Karhumäki, and Rozenberg 1983b)

13.3.2 Show that for all $n, m, p, q \geq 1$,

$$D_p(n) + D_q(m) \leq D_{p+q}(n+m).$$

Use this inequality to prove that for $0 < c < 1$, $D_{\lceil cn \rceil}(n) = \Omega(n^4)$ in free monoids, and $D_{\lceil cn \rceil}(n) = \Omega(n^3)$ in free semigroups.

13.3.3 Show that there are arbitrary large independent systems of equations with n variables in the commutative semigroup $S = \{z \mid \exists n\colon z^n = 1\}$ of the complex roots of unity.

**13.3.4 Improve the lower bounds of Theorems 13.3.3 and 13.3.5. This is an open problem.

13.3.5 Show that a group G satisfies the compactness property for group equations $w = \varepsilon$ with $w \in \Xi^{(*)}$ if and only if G satisfies it for the semigroup equations $u = v$ with $u, v \in \Xi^+$.

13.3.6 Prove Theorem 13.3.1.

Section 13.4

*13.4.1 Two elements a and b of a semigroup S form an *inverse pair*, if $a = aba$ and $b = bab$. In this case, the elements ab and ba are idempotents of S. Show that if S contains an inverse pair a, b such that $ba < ab$ in the partial ordering of the idempotents, then the subsemigroup of S generated by a and b is isomorphic to the bicyclic monoid, and therefore S does not satisfy the compactness property. (See Petrich 1984, p.432.)

13.4.2 (i) Prove that a variety $\mathcal{V}$ of monoids satisfies the compactness property if and only if the (relative) free monoids of this variety satisfy the maximal condition on congruences.

(ii) Show that the bicyclic monoid **B** satisfies the maximal condition on congruences. Show also that **B** is a monoid that does not satisfy the compactness property, but all its proper quotients do.

Section 13.5

13.5.1 Prove the claim of Example 13.5.5.

13.5.2 For a semigroup S that is not a monoid, let S^ε be the monoid obtained from S by adding an identity element ε_S to it. Show that S satisfies the compactness property if and only if the monoid S^ε does so.

13.5.3 Show that a monoid S satisfies the compactness property if and only if each sequence $\alpha_i \colon \Xi^+ \to \Pi^\infty S$ of morphisms with $\ker(\alpha_i) \subset \ker(\alpha_{i+1})$ is finite for all finite Ξ. (See Harju, Karhumäki, and Plandowski 1997b.)

13.5.4 Let us say that a class $\mathcal{S}$ of semigroups satisfies the compactness property *uniformly*, if each system of equations $E \subseteq \Xi^+ \times \Xi^+$ has a finite subsystem $E' \subseteq E$ such that E' is equivalent to E in all semigroups $S \in \mathcal{S}$. Show that if a variety $\mathcal{V}$ of monoids (or groups) satisfies the compactness property, then it satisfies it uniformly. In particular, if a variety $\mathcal{V}$ satisfies the compactness property, and S is a semigroup that is *locally* $\mathcal{V}$ (i.e., each finitely generated subsemigroup of S is in $\mathcal{V}$), then S satisfies the compactness property. (See Harju et al. 1997b.)

*13.5.5 Show that the finite semigroups generated by two elements do not satisfy the compactness property uniformly. (They do satisfy the compactness property, but not uniformly. For this one can use a result, due to Munn, stating that the finitely generated free inverse semigroups are residually finite. See Harju et al. 1997b.)

Notes

For a general source in the theory of semigroups and groups, we refer to Howie 1976 and Magnus et al. 1966, respectively. General references of combinatorics of words are Lothaire 1983 and Choffrut and Karhumäki 1997.

For the statement and the proof of the equality theorem, see Eilenberg 1974, or for a generalization using skew fields. For the defect theorem, see Chapter 6, and the references given there.

The compactness theorem, Theorem 13.2.2, was conjectured by A. Ehrenfeucht in the beginning of 1970's in a language theoretic setting, see Theorem 13.2.5. Its reformulation in systems of equations is due to Culik II and Karhumäki 1983. Theorem 13.2.2 was proved independently by Albert and Lawrence 1985b and Guba 1986. The present proof follows the ideas of the proof by Guba 1986. As mentioned in the notes of Chapter 12, this techniques was originated by Markov in the 1950's. Hilbert's basis theorem is proved in many 'standard' textbooks on algebra. For the proof of the basis theorem formulated for Noetherian rings, see e.g. Kostrikin and Shafarevich 1990 (p.45) and, for a different proof, Cohn 1989 (p.318). Albert and Lawrence 1985b proved the compactness property by embedding the free semigroups into free metabelian groups, for which a variant of Hilbert's basis theorem was proved by Hall 1954.

The applications in Section 13.2.2 are treated in Harju and Karhumäki 1986, Choffrut, Harju, and Karhumäki 1997, Culik II and Karhumäki 1983 and Culik II and Karhumäki 1986. The undecidability of the isomorphism problem for semigroups of nonnegative integer matrices follows, for instance, from the undecidability of freeness for these semigroups, see Klarner, Birget, and Satterfield 1991. The isomorphism problem as well as many other algorithmic problems on semigroups, are treated in a more general setting in Kharlampovich and Sapir 1995. For the decidability problems on iterated morphisms, especially the D0L and DT0L systems, see Rozenberg and Salomaa 1980. The isomorphism problem is open for subsemigroups of free semigroups A^+ generated by a regular set.

The results on the independent systems of equations for free semigroups and monoids in Section 13.3 are due to Karhumäki and Plandowski 1994. Theorem 13.3.6 for free groups was proved by Albert and Lawrence 1985a. Sizes of independent systems of equations in various semigroups have been studied by Karhumäki and Plandowski 1996.

Example 13.4.1 for the semigroup of all finite subsets of a free semigroup is due to Lawrence 1986. For the other examples in Section 13.4, see Harju et al. 1997b and Harju, Karhumäki, and Petrich 1997a.

The compactness property for free groups, see Example 13.5.5, was proven by Guba 1986 and De Luca and Restivo 1986. Since every free semigroup can be embedded into a free group, Theorem 13.2.2 follows from this result. We refer also to Stallings 1986 for a more general approach of the compactness property.

For the theory of varieties needed in Section 13.5, see e.g. Cohn 1981. The results concerning the compactness property in varieties are due to Albert and Lawrence 1985a and Harju et al. 1997b. A short proof of Redei's theorem, needed in Corollary 13.5.14 for commutative semigroups, is given by Freyd 1968. This proof is based on Hilbert's basis theorem.

We have treated the compactness property for groups only cursorily. For more information on this topic, see Baumslag, Myasnikov, and Roman'kov 1997, where the groups satisfying the compactness property are called *equationally noetherian*. In particular, Baumslag et al. 1997 show that if a group G has a subgroup of finite index that satisfies the compactness property, then so does G. Also the authors construct a large class of groups that do not satisfy the compactness property. Indeed, they show that if G is any nonabelian group and H is any infinite group, then their wreath product $G \wr H$ does not satisfy the compactness property.