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A Remark on Incompletely Specified Automata

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A remark is proved concerning certain groups associated with any finite automaton which satisfies a given incompletely specified auto-m aton.

INTRODUCTION

Let $X*$ (resp. $Y*$) denote the free monoid generated by a fixed input alphabet X (resp. output alphabet Y) and let $\bar{y}$ be an extra symbol not contained in Y. An *incompletely specified automaton* can be characterized formally by a map $\beta : X* \to Y \cup \{\bar{y}\}$ where for each input word $f \in X*$, $\beta f = y$ if the output at the end of f is $y \in Y$, and $\beta f = \bar{y}$ if the output at the end of f is not specified. A map $\beta' : X* \to Y \cup \{\bar{y}\}$ will be said to *satisfy* β iff $\beta f = \beta' f$ for every $f \in X*$ such that $\beta f \neq \bar{y}$. The study of the automata such that their associated map satisfies a given β seems to be a standard topic in automata theory. The Property below is a side remark having its motivation in the point of view taken in (McNaughton, 1960) and in the theory developed in (Krohn and Rhodes, 1963). Further results along the present line have been obtained by L. Verbeek (to appear).

Let β be a fixed map of $X*$ into $Y \cup \{\bar{y}\}$ and, following (Teissier, 1951), let the quotient monoid M_β of $X*$ and the homomorphism $\gamma : X* \to M_\beta$ be defined by the following two conditions:

(i) For any $f, f' \in X*$, if $\beta f \neq \beta f'$ then $\gamma f \neq \gamma f'$.

(ii) If $\bar{\gamma}$ is another homomorphism of $X*$ that fulfils condition i, then M_β is a homomorphic image of $\bar{\gamma} X*$.

If β' is another map of $X*$ into $Y \cup \{\bar{y}\}$, we let the quotient monoid $M_{\beta'}$ and the homomorphism $\gamma' : X* \to M_{\beta'}$ be defined in similar manner.

PROPERTY. *Assume that β' satisfies β and that any submonoid of $M_{\beta'}$ admits minimal quasi-ideals. To each subgroup G of M_β* (Miller and Clifford, 1956) *there correspond a subgroup G' of $M_{\beta'}$ and a normal subgroup H of G that satisfy the following two conditions: G/H is a homo-*

morphic image of G'; *for all* $f, f', f_1, f_2 \in X*$, *the relations* $\gamma f, \gamma f' \in H$ *and* $\beta f_1 f f_2 \in Y$ *imply that* $\beta f_1 f' f_2 = \bar{y}$ *or* $= \beta f_1 f f_2$.

The last condition above has been studied in (Elgot and Rutledge, 1962). Following these authors, we shall say that it expresses the statement that H is "*β-compatible.*"

VERIFICATION OF THE PROPERTY

We keep the notation and hypothesis already introduced and we let ρ denote the map $\gamma'\gamma^{-1}$ of $\mathfrak{P}(M_\beta)$ into $\mathfrak{P}(M_{\beta'})$ which sends each $A \subset M_\beta$ onto $\rho A = \{\gamma' f : f \in X*; \gamma f \in A\} \subset M_{\beta'}$. Thus, for any $A, B \subset M_\beta$ we have

$$\begin{aligned} \rho A \cdot \rho B &= \{\gamma' f f' : f, f' \in X*; \gamma f \in A; \gamma f' \in B\} \\ &\subset \{\gamma' f'' : f'' \in X*; \gamma f'' \in AB\} = \rho(AB). \end{aligned} \tag{1}$$

If G is a subgroup of M_β, we have $GG = G$ and (1) gives $\rho G \cdot \rho G \subset \rho(GG) = \rho G$ showing that the union of ρG with the neutral element of $M_{\beta'}$ is a submonoid of $M_{\beta'}$. By hypothesis this submonoid contains at least one subgroup G' which is a minimal quasi-ideal, i.e. which satisfies

$$G' = (G' \cdot \rho G) \cap (\rho G \cdot G') = G' \cdot \rho G \cdot G'. \tag{2}$$

G' is the desired group and, letting $\bar{\rho} A = G' \cap \rho A$ for any $A \subset G$, we show first that $\bar{\rho}\{g\} \neq \emptyset$ for any $g \in G$. Indeed, there is at least one element of G, say g_1, such that $\rho\{g_1\}$ contains at least one element of G', say g_1'. Let g_2 be the inverse of g_1 in G. We have $g_1 g_2 g g_2 g_1 = g$. Using (1), this shows that $\rho\{g\}$ contains the set $A' = g_1' \cdot \rho\{g_2 g g_2\} \cdot g_1'$, which, because of $g_1' \in G'$ and (2), is a subset of G'. Since $\rho\{g_2 g g_2\} \neq \emptyset$ this proves $\bar{\rho}\{g\} \neq \emptyset$ and the equivalent statement $G \subset \gamma\gamma'^{-1} G' (= \{\gamma f : f \in X*; \gamma' f \in G'\})$.

Let $H = \gamma\gamma'^{-1}\{g_0'\} \cap G$ and $H' = \bar{\rho}\{g_0\}$ where $g_0' = g_0'^2 \in G'$ and $g_0 = g_0^2 \in G$. It is well-known that H is a normal subgroup of G, H' a normal subgroup of G' and that the quotient groups G/H and G'/H' are isomorphic. We recall the proof for the sake of completeness. Indeed, from $g_0 = g_0^2$, $G' \cdot G' = G'$ and (1) we deduce that $H' \cdot H' \subset H'$. Since the union of G' with the neutral element of $M_{\beta'}$ admits minimal quasi-ideals, this shows that H' is a subgroup of G', hence that $g_0' \in H'$. Take an arbitrary element $g_1 \in G$. From $g_1 g_0 = g_0 g_1 = g_1$ and (1) we deduce that both $\bar{\rho}\{g_1\} \cdot H'$ and $H' \cdot \bar{\rho}\{g_1\}$ are contained in $\bar{\rho}\{g_1\}$ and, since $g_0' \in H'$ implies that each of these sets contains $\bar{\rho}\{g_1\}$, we can conclude that $\bar{\rho}\{g_1\} = \bar{\rho}\{g_1\} \cdot H' = H' \cdot \bar{\rho}\{g_1\}$. Now, if g_2 is the inverse of g_1 in G, applying (1) to

$g_1g_2 = g_0$ shows that $\rho\{g_1\} \cdot \rho\{g_2\}$ is contained in H'. It follows that if $h, h' \in \bar{\rho}\{g_1\}$ and $h'' \in \bar{\rho}\{g_2\}$, we have hh'', $h'h'' \in H'$ showing that $\bar{\rho}\{g_1\}$ is contained in a single right coset of H'. Using symmetry and $\bar{\rho}\{g_1\} \cdot H' = H' \cdot \bar{\rho}\{g_1\}$, it follows that H' is a normal subgroup of G'. Further, for any two elements $g_1, g_3 \in G$, relation (1) gives $\bar{\rho}\{g_1\} \cdot \bar{\rho}\{g_3\} \subset \bar{\rho}\{g_1g_3\}$ showing that the restriction of $\bar{\rho}$ to the one-element subsets of G can be considered as a homomorphism of G onto G'/H' and our partial result is proved since, by definition, $H = \gamma\gamma'^{-1}\{g_0'\} \cap G$ is the kernel of this homomorphism.

To conclude the verification it only remains to show that H is β-compatible. However, H is a subset of $\gamma\gamma'^{-1}\{g_0'\}$ and we have only to check that for any $f, f', f_1, f_2 \in X*$ the relations $\gamma f, \gamma f' \in \gamma\gamma'^{-1}\{g_0'\}$ and $\beta f_1ff_2 \in Y$ imply $\beta f_1f'f_2 \in \{\beta f_1ff_2\} \cup \{\bar{y}\}$. The first relation gives $\gamma' f = \gamma' f'$, hence $\beta' f_1ff_2 = \beta' f_1f'f_2$ according to the definition of $M_{\beta'}$ and γ'. Thus $\beta f_1ff_2 = \beta' f_1f'f_2 \in Y$ since $\beta f_1ff_2 \in Y$ and since we have postulated that β' satisfies β. For the same reason we must have $\beta f_1f'f_2 = \{\beta' f_1f'f_2\} \cup \{\bar{y}\}$ and the verification of our property is completed.

EXAMPLES

1. The property is vacuous if it imposes no restriction upon the subgroups of the monoid $M_{\beta'}$. Assuming that M_β is finite we show that a necessary condition for this is the existence of a natural number p such that for all $f, f', f'' \in X*$ the set $\{\beta ff'^n f'' : n \geqq p\}$ contains at most one letter from Y. Indeed, since M_β is assumed to be finite, there corresponds to each $m \in M_\beta$ a natural number p_m such that $\{m^n : n \geqq p_m\}$ is a cyclic subgroup of M_β. Taking $p = \max\{p_m : m \in M_\beta\}$, it follows that for each $f \in X*$ the set $\{\gamma f^n : n \geqq p\}$ is a cyclic group and in order that the property be vacuous each of these groups must be β-compatible, which is precisely the condition given above.

2. Let x_1 and x_2 (resp. y_1 and y_2) be two distinct elements of X (resp. of Y) and let n be a fixed integer at least equal to 5. Further, let $\beta^{-1}y_1$ be the submonoid of $X*$ generated by the set

$$\{x_1^n, x_1^{n-1}x_2x_1, x_1^{n-2}x_2, x_2x_1^{n-1}\} \cup \{x_1^i x_2 x_1^{n-i} : 0 < i < n - 2\}$$

and $\beta^{-1}y_2 = (\beta^{-1}y_1) \cdot x_1$. Computing the syntactic monoid M_β shows that it contains a subgroup G isomorphic to the symmetric group on n objects and that the subgroup of G corresponding to the alternating subgroup is not β-compatible. Thus, by our property, any β' with $M_{\beta'}$ finite which satisfies β must contain a subgroup G' which is isomorphic with G (since for $n \geqq 5$ the symmetric group admits no proper non-trivial normal subgroup except the alternating group). This implies, for instance, that none of the sets $\beta'^{-1}y_i$ $(i = 1, 2)$ can be described within the "L_π-language" of (McNaughton, 1960) since, as it is known, this last requirement would imply that all the subgroups of $M_{\beta'}$ are abelian (for a formulation in the so called "algebraic terminology" of the relevant part of McNaughton's theory see (Petrone et Schützenberger, 1963)).

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