SYMMETRIZATION OPERATORS ON POLYNOMIAL RINGS

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The Hecke algebra of the symmetric group $\mathfrak{S}(n + 1)$ can be defined as the quotient of the free algebra $C\langle D_1, \ldots, D_n\rangle$ by the Coxeter relations $D_iD_j = D_jD_i$ if $|j - 1| \geq 2$, $D_iD_{i+1}D_i = D_{i+1}D_iD_{i+1}$ together with the Hecke relation $D_iD_i = qD_i + r$.

A more concrete approach realizes this algebra as an algebra of rational symmetrizing operators on the ring of polynomials (i.e., for those who prefer, the equivariant cohomology or Grothendieck ring of the flag manifold).

In this note, we characterize the operators satisfying the above relations plus some extra conditions. They constitute a five (homogeneous) parameter family which admits several interesting degenerate cases such as considered by [1-3]. We also give the expression of any rational symmetrizer in the basis of permutations and in the basis of divided differences. Finally, we study a Leibnitz-type formula generalizing an identity of Bernstein/Gelfand/Gelfand [2].

According to modern practice, all operators act on their <u>left</u>.

Let σ be the transposition exchanging the letters a, b. Let P, $Q \neq 0$ be two rational functions of a, b. To σ, P, Q, associate the rational operator D_σ: $f \to fD_\sigma = fP + f^\sigma Q$ acting on rational functions.

Four known examples (see [4]) are: σ, the transposition of a and b;

∂, the divided difference: $f\partial = (f - f^\sigma)/(a - b)$;

π, the convex symmetrizer: $f\pi = (fa - f^\sigma b)/(a - b)$;

$1 - \pi$, the complement of π: $f(1 - \pi) = (-f + f^\sigma)b/(a - b)$.

We now consider two operators on different pairs of letters. If these two pairs are disjoint, the operators commute. On the contrary, suppose that the two pairs are (a, b) and (b, c); let σ be as above, and let τ be the transposition of b, c. Let also D and D' be the associated operators: $fD = fP(a, b) + f^\sigma Q(a, b)$, $fD' = fP'(b, c) + f^\tau Q'(b, c)$, where P' and $Q' \neq 0$ are rational functions in b, c.

<u>THEOREM 1.</u> Let a, b, c be three letters, D and D' the associated operators. Suppose that D is invertible, that $P \neq 0$, and that the operators satisfy Coexter's relation $DD'D = D'DD'$. Then the necessary and sufficient condition that the operators D and D' preserve the ring of polynomials C[a, b, c] is that there exist scalars α, β, γ, δ, η with $\alpha\delta - \beta\gamma \neq 0$, $\eta \neq 0$, $\eta \neq \alpha\delta - \beta\gamma$ such that $P(a, b) = (\alpha a + \beta)(\gamma b + \delta)/(a - b) = P(a, b)$ and $Q = \eta - P = Q'$.

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In that case, D and D' satisfy the same Hecke relation $DD = D(\alpha\delta - \beta\gamma) + \eta(\eta - \alpha\delta + \beta\gamma)$.

The full discussion, too long to be developed here, contains the noncommutative computations of [5] and shows that the inequalities on the numerical parameters are not needed to ensure Coxeter relations.

Remark. The operator given by the above theorem can also be written $D \doteq \partial(\alpha a + \beta)\cdot(\gamma b + \delta) + \sigma\eta$. It can be obtained from the operator $\pi + \eta'\sigma$ through the homography $\{a, b\} \to \{\rho(a), \rho(b)\}$, with $\rho(x) = (\alpha x + \beta)/(\gamma x + \delta)$. In fact the image of π by the homography ρ is the operator $f \to f\pi^0 = [f\rho(a) - f^{\sigma}\rho(b)]/[\rho(a) - \rho(b)] = f[\partial(\alpha a + \beta)(\gamma b + \delta)/(\alpha\delta - \beta\gamma) + \sigma]$.

It is a puzzling fact that the operator for $\alpha = 1$, $\gamma = 0$ acts on the Grothendieck ring of the flag manifold, whereas the operator for $\alpha = 0 = \gamma$ acts on the cohomology ring.

Let $A = \{a_1, a_2, \ldots, a_{n+1}\}$ be an alphabet. For each permutation $\mu \in \mathfrak{S}(A)$, there exists a divided difference ∂_μ which can be expressed as a product of the elementary divided differences ∂_{σ_i}, where σ_i is the transposition of a_i and a_{i+1} (see [2-4]).

Let E be the algebra of operators ∇: $C[A] \to C[A]$ of the type $\nabla = \sum_{\zeta \in \mathfrak{S}(A)} \zeta R_\zeta$, where the coefficients R_ζ are rational functions of the elements of A. In particular, the divided differences ∂_μ belong to E. Conversely, one can express any permutation $\zeta \in \mathfrak{S}(A)$ in terms of the ∂_μ.

We consider a second alphabet $Z = \{z_1, \ldots, z_{n+1}\}$, the mi-résultante $X = \prod_{i+j \leq n+1} (a_i - z_j)$, and we denote by θ the specialization $z_1 \to a_1, \ldots, z_{n+1} \to a_{n+1}$. It is easy to check the following lemma, denoting by Δ the Vandermonde $\Delta = \Pi_{i<j}(a_i - a_j)$ and by ω the maximal element of $\mathfrak{S}(A)$, i.e., the involution exchanging each a_i with a_{n+2-i}.

LEMMA 2. For any $\mu \in \mathfrak{S}(A)$, $\mu \neq \omega$, the operators $X\mu\theta$ and $X\partial_\mu\theta$ are null; moreover, $X\omega\theta = \Delta\omega$ and $X\partial_\omega\theta = \omega$.

This instantly allows one to decompose any element of E in the basis $\{\zeta\}$ or the basis $\{\partial_\mu\}$:

PROPOSITION 3.

1) E is a Pol(A)-free module of basis $\{\partial_\mu,\ \mu \in \mathfrak{S}(A)\}$.

2) any element ∇ of E can be written

$$\nabla = \sum_\zeta (X\nabla\partial_{\omega\zeta}\theta\omega)\,\partial_{\zeta^{-1}}.$$

3) Let $\nabla = \Sigma\zeta\, R_\zeta$ be an element of E. Then, for each μ, $X\omega\mu^{-1}\nabla\theta = (-1)^{\ell(\omega)}\Delta R_\mu$.

COROLLARY 4. The coefficients of the ∂_ν in the basis $\{\mu\}$ are the same as the coefficients of the μ in the basis $\{\partial_\nu\}$ up to sign and to the factor Δ, i.e.,

$$\partial_\nu\Delta\,(-1)^{l(\omega)} = \Sigma\mu\,(X\omega\mu^{-1}\partial_\nu\theta).$$

This property of self-inversion does not seem to have been noticed (cf. [1, Prop. 4.24], [2, Th. 5.9], [6]). Lemma 2, Proposition 3, and Corollary 4 could be formulated in purely geometrical terms because the mi-résultante can be interpreted geometrically as the class of the diagonal embedding of the flag manifold.

Example: Symmetric group $\mathfrak{S}(a, b, c)$. Let σ_1 be the transposition of (a, b), σ_2 that of (b, c), ∂_1, ∂_2 the corresponding divided differences. Then $1\Delta = (a-b)(a-c)(b-c)$; $\partial_1\Delta = (1-\sigma_1)\cdot(a-c)(b-c)$; $\partial_2\Delta = (1-\sigma_2)(a-b)(a-c)$; $\partial_1\partial_2\Delta = (1-\sigma_1)(a-c) + (\sigma_1 - 1)\sigma_2(a-b)$; $\partial_2\partial_1\Delta = (1-\sigma_2)(a-c) + (\sigma_2 - 1)\sigma_1\cdot(b-c)$; $\partial_1\partial_2\partial_1\Delta = 1 - \sigma_1 - \sigma_2 + \sigma_1\sigma_2 + \sigma_2\sigma_1 - \sigma_1\sigma_2\sigma_1$. On the other hand, $\sigma_1 = (a-b)\partial_1 - 1$; $\sigma_2 = (b-c)\partial_2 - 1$; $\sigma_1\sigma_2 = (a-b)(a-c)\partial_1\partial_2 - (a-b)\partial_1 - (a-c)\partial_2 + 1$; $\sigma_2\sigma_1 = (a-c)(b-c)\partial_2\partial_1 - (a-c)\partial_1 - (b-c)\partial_2 + 1$; $\sigma_1\sigma_2\sigma_1 = \omega = \Delta\partial_\omega - (a-b)(a-c)\partial_1\partial_2 - (a-c)(b-c)\partial_2\partial_1 + (a-c)(\partial_1 + \partial_2) - 1$.

The algebra E gives rise to a noncommutative finite-differential calculus, but only the special cases $A \to \{a, a, \ldots\}$, $A \to \{\varepsilon, 2\varepsilon, 3\varepsilon, \ldots\}$ or $A \to \{a, aq, aq^2, \ldots\}$ have been the object of systematic studies.

As an example, we have for the general case the following Leibnitz-type formula: Let p, k be two positive integers, $\tau_1, \ldots, \tau_p$ be p transpositions, and $\partial_1, \ldots, \partial_p$ the associated divided differences. Given a matrix $\nabla = (\nabla_{ij})_{1\leq i\leq k,\, 1\leq j\leq p}$ of operators $\nabla_{ij} \in E$, and given k functions $f_1, \ldots, f_k$, we denote by $f_1, \ldots, f_k\nabla$ the product $(f_1\nabla_{11}\cdot \ldots \cdot\nabla_{1p})\ldots(f_k\nabla_{k1}\cdot\ldots\cdot\nabla_{kp})$.

Propositions 5. Given k functions $f_1, \ldots, f_k$ and p divided differences $\partial_1, \ldots, \partial_p$, we have that $f_1\cdot\ldots\cdot f_k\partial_1\ldots\partial_p = \Sigma f_1\ldots f_k\nabla$, summed on the k^p matrices ∇ such that for every j: $1 \leq j \leq p$, the j-th column of ∇ is of the type $\tau_j, \ldots, \tau_j, \partial_j, 1, \ldots, 1$, i.e., a sequence of τ_j, exactly one time ∂_j, followed by the identity operator.

When k = p and when $f_1, \ldots, f_k$ are polynomials of degree 1, one obtains the following especially interesting formula due to Bernstein/Gelfand/Gelfand [2, Theorem 3.12], writing ∂^τ for the divided difference: $f \to (f - f^\tau)/(x - y)$, τ being the transposition exchanging the letters x and y.

Proposition 6. Let μ be a permutation of length $\ell(\mu) = p$, and let $f_1, \ldots, f_p$ be polynomials of degree one. Then $f_1\ldots f_p\,\partial_{\mu^{-1}} = \Sigma\,(f_1\partial_{\tau_1})\ldots(f_p\partial_{\tau_p})$, summed on all reduced decompositions of μ as a product of transpositions, i.e., all products $\mu = \tau_1, \ldots, \tau_p$, $\ell(\tau_1, \ldots, \tau_j) = j\,\forall j$: $1 \leq j \leq p$.

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