A Historical Perspective on Schützenberger-Pinsker Inequalities

GSI'23, Saint-Malo, August 30th, 2023

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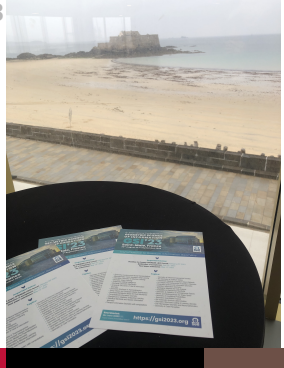

How far is one distribution to another?

Distances $\Delta(p, q)$

- Lévy-Prokhorov
- Fortet-Mourier
- Kantorovich-Rubinstein
  a.k.a. Wasserstein
- Radon
- “Hellinger” (Jeffreys)
- L^1, L^p
- (Ky Fan: between rv’s)
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Divergences $D(p\|q)$

- Rényi
- Bhattacharyya
- “Jensen-Shannon” (Lin)
- “Jeffreys” (sym. Kullback-Leibler)
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- Total variation $\Delta(p, q)$ (“statistical”)

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- Kullback-Leibler $D(p\|q)$ (“relative entropy”)

Definitions

$$\boxed{\Delta(p,q) \triangleq \frac{1}{2}\int |p-q|\,\mathrm{d}\mu} \qquad \boxed{D(p\|q) \triangleq \int p\log\frac{p}{q}\,\mathrm{d}\mu}$$

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- vanishes iff $p = q$;

$\Delta(p,q) = 1 \iff p \wedge q = 0$ μ-a.e. (non-overlapping supports) $\implies D(p\|q) = +\infty$

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- Binary case $P = (p, 1-p)$, $Q = (q, 1-q)$:

$$\boxed{\delta(p,q) = |p-q|} \qquad \boxed{d(p\|q) = p\log\frac{p}{q} + (1-p)\log\frac{1-p}{1-q}}$$

Alternate definitions

Supremum taken all *partitions* into a countable number of $A_i \in \Omega$:

$$\Delta(p,q) \triangleq \frac{1}{2} \underbrace{\sup \sum_i |p(A_i) - q(A_i)|}_{\text{Jordan's total variation of } p-q} \qquad D(p\|q) \triangleq \sup \sum_i p(A_i) \log \frac{p(A_i)}{q(A_i)}$$

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- $\Delta = \frac{1}{2} \int |p-q| \, \mathrm{d}\mu$ already for binary partition $\{p < q\}$, $\{p \geq q\}$.
  $D = \int p \log \frac{p}{q} \, \mathrm{d}\mu$ by *Gel'fand-Yaglom–Perez theorem* (1959)

Nice Properties

- Total variation distance: binary reduction property for partition A, A^c:

$$\boxed{\Delta(p,q) = \sup_A |p(A) - q(A)|}$$

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- Kullback-Leibler divergence: tensorization property for product of probability measures:

$$D\Big(\bigotimes_i p_i \Big\| \bigotimes_i q_i\Big) = \sum_i D(p_i \| q_i)$$

Summary of Properties

$\Delta(p, q)$

- metric ✓
- binary reduction (bounded) ✓
- does not tensorize ✗

$D(p\|q)$

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Which topology is finer?

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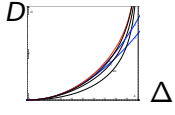
Pinsker's inequality: $\boxed{D \geq \varphi(\Delta)}$ where $\varphi(0) = 0$, φ increasing (convex)



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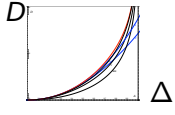

By binary reduction, it is enough to prove it in the binary case: $\boxed{d \geq \varphi(\delta)}$

Why is PInsker's Inequality Useful?

Example: How to distinguish fair ($p = ½$) from unfair ($q \neq ½$) coin with n tosses

To be sure with probability $1 - \epsilon$, we need

$$\Delta(\otimes_{i=1}^{n} P_i, \otimes_{i=1}^{n} Q_i) \geq |(1 - \epsilon) - \epsilon| = 1 - 2\epsilon$$

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$$D(\otimes_{i=1}^{n} P_i \| \otimes_{i=1}^{n} Q_i) = n \cdot d(p\|q) = n \cdot \log \frac{1}{\sqrt{1-4\delta^2}}$$

where $\delta = |p - q|$.

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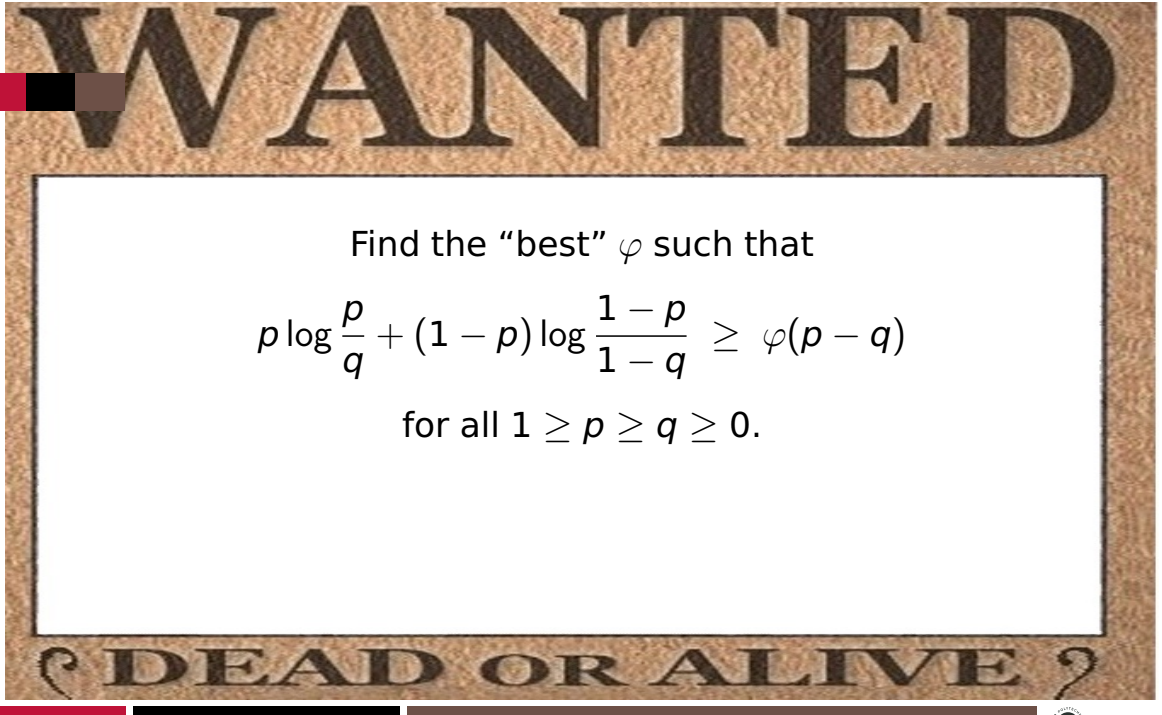
where $\delta = |p - q|$. Then by Pinsker inequality, $\boxed{D \geq \varphi(\Delta)}$, there should be at least

$$n \geq \frac{\varphi(1-2\delta)}{\log \frac{1}{\sqrt{1-4\epsilon^2}}} \text{ tosses.}$$


WANTED
Find the “best” φ such that
$$p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q} \geq \varphi(p-q)$$
for all $1 \geq p \geq q \geq 0$.
DEAD OR ALIVE ?

The classical “Pinsker” Inequality

$$\boxed{D \geq c \cdot \Delta^2}$$

where[1] the optimal (maximum) constant c is $\boxed{c = 2}$.

[1] Assuming natural logarithms—otherwise divide D by $\log e$.

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Pinsker’s book, 1960: Информация и информационная устойчивость случайных величин и процессов

При малых значениях $I(\xi, \eta)$ оказывается полезным следующее неравенство

$$\mathcal{J}(\xi, \eta) \leqslant I(\xi, \eta) + \Gamma\sqrt{I(\xi, \eta)}, \qquad (2.3.3)$$

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- Some math (Verdú, 2014) gives $D \geq c \cdot \Delta^2$ with the suboptimal $c = \frac{1}{102}$.

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- Volkonskii and Rozanov, 1959: $\boxed{D \geq 2\Delta - \log(1 + 2\Delta)}$ first explicit occurence, before the publication of Pinsker’s book!

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- Finally, Csiszár, 1967 proved $c = 2$; can be simplified as a 1-line proof:

$$d(p\|q) = \underbrace{d(p\|p)}_{=0} + \int_p^q \frac{\partial d(p\|r)}{\partial r}\,\mathrm{d}r = \int_p^q \frac{r-p}{r(1-r)}\,\mathrm{d}r \geq 4\int_p^q (r-p)\,\mathrm{d}r = 2(p-q)^2 \quad \square$$

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- grand grand father Paul: renowned chemist, founded the ESPCI, satirized in *Les Palmes de Mr. Schutz*
- during WWII, appointed intern at the psychiatric hospital in Blois; active in resistance activities (worked for the Intelligence Service); and published his first mathematical paper

SÉANCE DU 31 MAI 1943. 71

THÉORIE DES GROUPES. — *Sur la théorie des structures de Dedekind.* Note de M. **Marcel-Paul Schutzenberger**, présentée par M. Gaston Julia

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- 1948 defends his doctoral thesis entitled *Contribution à l'étude du sexe à la naissance* (Contribution to the study of sex at birth)—awarded by the French Academy of Medicine

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- 1952 & 1953, WHO sent him to Asia to combat infectious diseases of tropical countries. Met 2nd wife Hariati Soerosoegondo in Java.

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- 1952, came to information theory from biostatistics:

> APPLICATIONS BIOMÉTRIQUES DE LA THÉORIE DE L'INFORMATION
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CONTRIBUTION

AUX

APPLICATIONS STATISTIQUES

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- invited to MIT by Claude Shannon, he spent 1956-57 there
- at the crossroads of algebra and theoretical computer science: variable-length codes, monoids, automata theory, Kleene-Schützenberger theorem. His seminal paper *Une théorie algébrique du codage* (1958) lays foundations of automata theory and its relationship with rational languages and semi-groups.

Marcel-Paul (Marco) Schützenberger's 1953 Thesis

Dans le cas dichotomique, on a l'inégalité suivante qui semble nouvelle. Ecrivons :

$$D = p(\theta_0) - p(\theta_i) = q(\theta_i) - q(\theta_0)$$

$$\boxed{W \geq 2D^2 + \tfrac{4}{3}D^4.}$$

Posons en effet $2\,p(\theta_0) = 1-x$ et $2\,p(\theta_1) = 1-y$ après avoir choisi p de telle sorte que x soit positif.

On peut développer W en série de puissance de x et de y :

$$2W = (1-x)\,\mathrm{Log}(1+x)/(1-y) + (1+x)\,\mathrm{Log}\,(1+x)(1+y).$$

On trouve :

$$W = \sum_{i=1}^{\infty} (4i^2 - 2i)-1\ (x^{2i} - 2ixy^{2i-1} + (2i-1)y^{2i})$$

Tous les termes sont positifs car le polynome $t^{2i} - 2it + 2i - 1$ a un unique extremum pour $t = 1$ et prend en ce point la valeur 0 .

Bien plus :

$$x^{2i} - 2ixy^{2i-1} + (2i-1)y^{2i} = 4D^2\,(x^{2i-2} + 2x^{2i-3}y + 3x^{2i-4}y^2 + \ldots$$

$$\ldots + (2i-1)y^{2i-2}\ )$$

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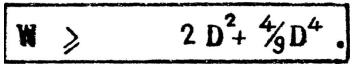

$$D \geq 2\Delta^2 + \frac{4}{9}\Delta^4$$

- optimal constants of Pinsker's inequality, first and second order
- in the binary case
- 7 years before Pinsker's book
- 14 years before Csiszár's first order term
- 17 years before Kullback's correction of the second-order term !

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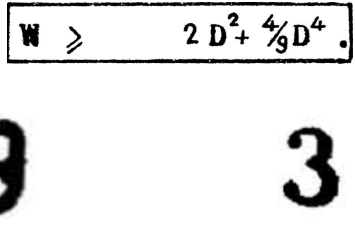
From 4/3 to 4/9



(a) Denominator in the fraction 4/9, zoomed in.

(b) Digits from the same manuscript.

Fig. 2: Schützenberger's correction from "3" to "9": the correction clearly follows the shape of a "3" in the original manuscript.

Schützenberger's identity

Schützenberger's derivation is correct and gives the 1st and 2nd order *optimal* constants based on his identity:

$$d = \sum_{k\geq 1} \frac{x^{2k} - 2kxy^{2k-1} + (2k-1)y^{2k}}{2k(2k-1)} = 2\delta^2 \sum_{k\geq 1} \frac{x^{2k-2} + 2x^{2k-3}y + \cdots + (2k-1)y^{2k-2}}{k(2k-1)}$$

where $d = d(p,q)$, $x = 1-2p$ and $y = 1-2q$.

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- 2003, Fedotov, Harremoës, Topsøe used Schützenberger's identity to derive the *4th-order* optimal constant $7072/42525$

$$\boxed{D \geq 2\Delta^2 + \frac{4}{9}\Delta^4 + \frac{32}{135}\Delta^6 + \frac{7072}{42525}\Delta^8 + \cdots}$$

More Recent Improvements on Schützenberger-Pinsker Inequality

So far the inequalities become vacuous as soon as D gets large, e.g., $D \geq 2\Delta^2$ is vacuous as soon as $D > 2$: Improved bounds should reflect the fact that $\Delta = 1$ (non overlapping supports) imply $D = +\infty$

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- Gilardoni, 2008: best known explicit Pinsker inequality of this kind so far:

$$\boxed{D \geq \log \frac{1}{1-\Delta} - (1-\Delta)\log(1+\Delta)}$$

(simple proof in the paper)

Optimal Schützenberger-Pinsker Inequality

Derived by Fedotov, Harremoës, and Topsøe (2009), only in implicit form, using Legendre–Fenchel transformation as a curve parametrized by hyperbolic trigonometric functions. — Simplified expression in the paper:

Theorem

The optimal Pinsker inequality $D \geq \varphi^*(\Delta)$ is given in parametric form as

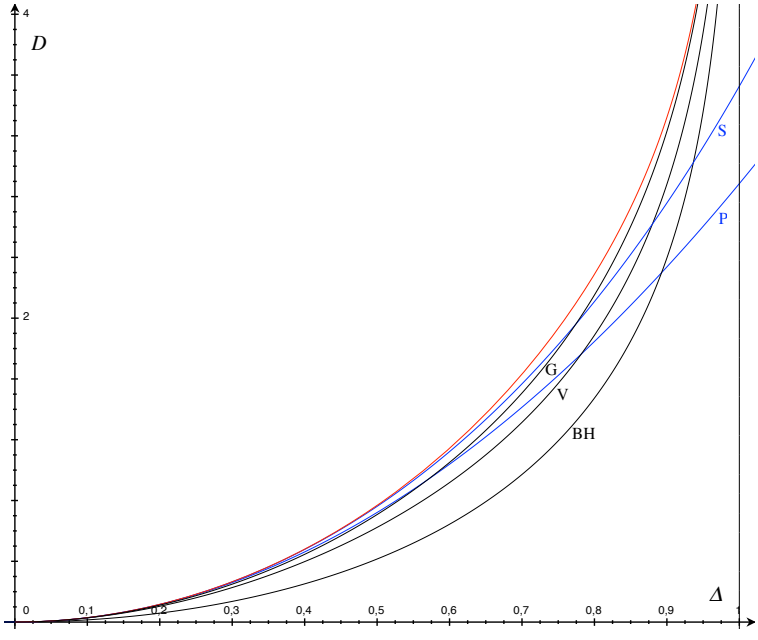
$$\begin{cases} \Delta &= \lambda(1-q)q \\ D &= \log(1-\lambda q) + \lambda q(1+\lambda(1-q))\log e \end{cases} \tag{1}$$

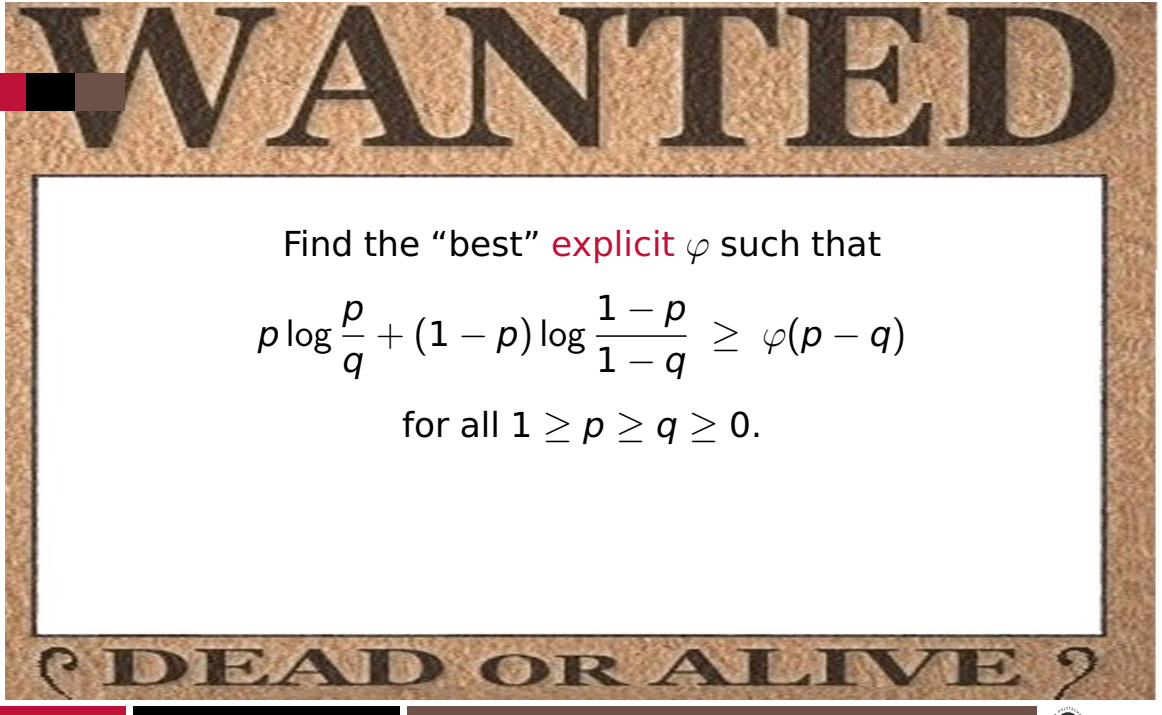
where $\lambda \geq 0$ is the parameter and $q = q(\lambda) \triangleq \frac{1}{\lambda} - \frac{1}{e^{\lambda}-1} \in [0, \frac{1}{2}]$..

Proof.

(simple proof in the paper using Lagrangian) □

Summary




WANTED
Find the “best” explicit φ such that
$$p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q} \geq \varphi(p-q)$$
for all $1 \geq p \geq q \geq 0$.
DEAD OR ALIVE ?

A Historical Perspective on Schützenberger-Pinsker Inequalities

Thank you!

Olivier Rioul

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