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A NON-EXISTENCE THEOREM FOR AN INFINITE FAMILY OF SYMMETRICAL BLOCK DESIGNS

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An incomplete block design consists of a set of b subsets or 'blocks', each block containing k different 'varieties' chosen out of v varieties in all, in such a way that each of these varieties occurs the same number r of times or 'replications' in the whole design, and each pair of varieties occurs together in a fixed number λ of blocks. The design is called 'symmetrical', if $v = b$ and $r = k$: then v, r and λ are related by

$$v = 1+\frac{r(r-1)}{\lambda}. \quad (1)$$

We show in this paper that, if v is even, then such a symmetrical design can exist only if $r-\lambda$ is a perfect square.

The design may be schematized by a square $v\times v$ matrix $A = \| a_{ij} \|$ in which columns correspond to blocks and rows to varieties, and $a_{ij} = \alpha$ if the ith variety is a member of the jth block and $a_{ij} = \beta$ if not. Without loss of generality we may suppose that $\alpha = 1$ and $\beta = 0$.

Further, A may be interpreted in a Euclidean space E_v as a set of v row vectors V_i, the coordinates of which are the a_{ij}'s; and the V_i's satisfy the following conditions:

I. The V_i's have the same length $l = \sqrt{r}$, for

$$l^2 = r\alpha^2+(v-r)\beta^2 = r. \quad (2)$$

II. The angle $\phi_{ii'}$ between any two V_i's, has the same value $\cos^{-1}\lambda/r$, for

$$V_i V_{i'} = l^2\cos\phi_{ii'} = \lambda\alpha^2+2(r-\lambda)\alpha\beta+(V-2r+\lambda)\beta^2 = \lambda. \quad (3)$$

From these relations, det $|A|$, the measure of the multiple vector A may be computed by reducing A to a triangular matrix B by rotations in E_v. Thus B is defined by:

If $$i\leqslant j+1, \quad \text{then} \quad b_{ij} = 0, \quad (4)$$

for all i: $$\sum_{j=1}^{n}(b_{ij})^2 = l^2 = r, \quad (5)$$

for all i and i': $$\sum_{j=1}^{n} b_{ij}b_{i'j} = l^2\cos\phi = \lambda. \quad (6)$$

From (4) and (6) it follows immediately that:

For all $i\leqslant i'$ and $j\leqslant i-1$: $$b_{ij} = b_{ij} = c_j, \quad (7)$$

and from (5) and (6) $$b_{ii}^2 = d_i^2 = r-\sum_{j=1}^{i-1}c_j^2, \quad (8)$$

$$d_i c_i = \lambda-\sum_{j=1}^{i-1}c_j^2 = \lambda-r+d_i^2. \quad (9)$$

Hence $$d_{i+1}^2 = d_i^2-c_i^2 = d_i^2-(d_i^2+\lambda-r)^2 d_i^{-2}. \quad (10)$$

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Hence, writing $P_\mu = \prod_{i=1}^{\mu} (d_i)^2$, we get the linear recurrence equation

$$\left.\begin{aligned} \frac{P_{\mu+1}}{P_\mu} &= \frac{P_\mu}{P_{\mu-1}} - \frac{(P_\mu - (r-\lambda)P_{\mu-1})^2}{P_\mu P_{\mu-1}}, \\ P_{\mu+1} &= 2(r-\lambda)P_\mu - (r-\lambda)^2 P_{\mu-1}. \end{aligned}\right\} \tag{11}$$

For the initial conditions $p_1 = r$, $P_2 = r^2 - \lambda^2$, the solution of (11) is

$$P_\mu = (r-\lambda)^{\mu-1}(r+(\mu-1)\lambda), \tag{12}$$

and, finally, we get, by putting $\mu = v = 1 + \frac{r(r-1)}{\lambda}$,

$$\det|A| = \det|B| = P_v^{\frac{1}{2}} = r(r-\lambda)^{\frac{1}{2}(r-1)}. \tag{13}$$

Hence we have the following theorem: if v is even, a symmetrical incomplete block design may exist only if $r-\lambda$ is the square of an integer, for det $|A|$ may be computed directly by rational operations.

It is obvious that the theorem applies only if λ admits 2 as a divisor and that, for any such λ it works for an infinite number of values of r. Obviously too, the theorem may be used indifferently for a design or its complement, for

$$r - \lambda = (V-r) - (v - 2r + \lambda). \tag{14}$$

We have listed below all the cases of application of the theorem for $v \leqslant 100$ or $r \leqslant 20$ ($r \leqslant v/2$, and $r \neq v$ or $v-1$):

	v	r	λ		v	r	λ	v	r	λ
(1)	16	6	2		*34	12	4	70	24	8
(2)	*22	7	2	(3)	40	13	4	*76	25	8
	*46	10	2		96	20	4	66	26	10
	56	11	2		36	15	6	*88	30	10
	*92	14	2		*52	18	6	*94	31	10
	*106	15	2		*58	19	6	64	28	12
	*172	19	2		78	22	6	*86	35	14
								100	45	20

* Impossible by application of the theorem.

(1) Kümmer's configuration.

(2) Already discarded by Q. M. Hussain. (Cf. R. A. Fisher & F. Yates, *Statistical Tables*..., 3rd ed., 1948, pp. 17 and 65.)

(3) Finite geometry modulo 3.