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# An Extension Problem in the Theory of Incomplete Block Designs

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SUMMARY

By generalization of concepts of projective geometry, two combinatorial methods have been studied which may allow the extension of a block design into another one. As an application a new infinite family of block designs has been given.

Let $A$ be the incidence matrix of an incomplete block design with parameters $(\lambda, k, r, v, b)$. By definition $A$ is a $v \times b$ matrix the elements of which are 0 or 1 and satisfy—

$$(1) \text{ for any } j: \sum_{i=1}^{b} a_j^i = r;$$

$$(2) \text{ for any } i: \sum_{j=1}^{v} a_j^i = k;$$

$$(3) \text{ for any } j \text{ and } j' \ (j \neq j') \sum_{i=1}^{b} a_j^i a_{j'}^i = \lambda.$$

It is easily proved that these hypothesis imply—

$$(4) \ vr = bk$$

and

$$(5) \ \lambda(v-1) = r(k-1).$$

If a submatrix $A'$ of $A$ represents another block design with parameters $(\lambda', k', r', v', b')$ we shall say that $A$ is an *extension* of $A'$, and we shall partition $A$ into the four following submatrices:

$$A' = | a_j^i | \text{ for } 1 \leqslant i \leqslant b' \text{ and } 1 \leqslant j \leqslant v',$$
$$B = | a_j^i | \text{ for } b'+1 \leqslant i \leqslant b \text{ and } 1 \leqslant j \leqslant v',$$
$$C = | a_j^i | \text{ for } 1 \leqslant i \leqslant b' \text{ and } v'+1 \leqslant j \leqslant v,$$
$$D = | a_j^i | \text{ for } b'+1 \leqslant i \leqslant b \text{ and } v'+1 \leqslant j \leqslant v.$$

Two types of extension are already known when $A$ is a square matrix and $B$ and $D$ are degenerated into a single column: the *block intersection* ($B$ has only unit elements and $D$ zero elements) and the *block section* ($B$ has only zero elements and $D$ unit elements) (see R. C. Bose, 1939). In this paper we shall try to generalize two methods used in the finite projective geometries.

## I. Algebraic Extensions

We shall say that $A$ is an *algebraic extension* of $A'$ if $B$ and $C$ may be partitioned, respectively, into $x$ and $y$ unit submatrices and $z$ and $t$ lines with zero elements only.

Obviously, unless $A = A'$, $x \neq 0$. We assume further that $y \neq 0$. For $1 \leqslant j \leqslant v'$ and $1 \leqslant i \leqslant b'$ we have

$$(6) \ a_j^{b'+x(j-1)+x'} = \begin{cases} 1 \text{ if } 1 \leqslant x' \leqslant x \\ 0 \text{ otherwise} \end{cases}$$

$$(7) \ a^i_{v'+v(i-1)+v'} = \begin{cases} 1 \text{ if } 1 \leqslant y' \leqslant y \\ 0 \text{ otherwise} \end{cases}$$

Thus

$$(8) \ \lambda = \lambda'; \ b = b' + xv' + z; \ v = v' + yb' + t; \ r = r' + x; \ k = k' + y.$$

Let us consider the submatrices $E_j^i$ of $D$ defined by

$$E_j^i = | a_{v'+y(i-1)+y'}^{b'+x(j-1)+x'} | \text{ with } 1 \leqslant x' \leqslant x;\ 1 \leqslant y' \leqslant y;\ 1 \leqslant i \leqslant b';\ 1 \leqslant j \leqslant v'.$$

From (3) and (8) it follows that each row of $E_j^i$ contains exactly $\lambda' - a_j^i$ unit elements. Hence, if $v' + 1 \leqslant j \leqslant v' + yb'$,

$$(9)\ r - \sum_{i=1}^{b'+xv'} a_j^i = r - 1 - k'(\lambda' - 1) - \lambda'(v' - k') = x + r' + k' - \lambda'v' - 1 = \begin{cases} > 0 \text{ if } z \geqslant 1 \\ = 0 \text{ if } z = 0. \end{cases}$$

If $t \geqslant 1$, we may write for $j \geqslant v' + yb' + 1$

$$(10)\ r = r' + x \geqslant \sum_{i=b'+1}^{b'+xv'} a_j^i = \sum_{j'=1}^{v'} \left( \sum_{i=b'+1}^{b'+xv'} a_j^i a_{j'}^i \right) = \lambda' v'.$$

Apart from these last inequalities little may be said on the general algebraic extensions within our merely arithmetic approach. We shall confine ourselves to a more restrictive case: $A$ will be called a *quadratic extension* of $A'$ if $z = t = 0$. We prove:

*If* B *has no columns with only zero elements, then* C *has no rows with only zero elements.*

By (9), $z = 0$ implies

$$(11)\ x = \lambda'v' - r' - k' + 1$$

but this value is not compatible with (10) so that $t = 0$. From (4), (5) and (11) it follows that

$$(12)\ y = \frac{(k' - 1)x}{(b' - v')\lambda' + k' - 1}.$$

Hence, *all quadratic extensions of a given matrix* A′ *have the same parameters and are square matrices when* A′ *itself is a square matrix.*

Conversely, *if the square matrix* A *may be represented as a quadratic extension of both* $A'_1$ *and* $A'_2$*, these two matrices are square matrices and have the same parameters.*

The first part of the statement (which holds for *any* algebraic extension), follows from (7) and the equation

$$\sum_{j=1}^{v} a_j^i a_j^{i'} = \sum_{j=1}^{v'} a_j^i a_j^{i'} = \lambda \text{ for } 1 \leqslant i < i' \leqslant b'.$$

The second part follows from (11) since by (5), when $r_1 = k_1$ and $r_2 = k_2$, $r = x_1 + r_1 = x_2 + r_2$ is equivalent to $(r_1 - r_2)(r_1 + r_2 - 2) = 0$.

### *Applications*

When $A'$ is the incidence matrix of a plane projective geometry with co-ordinates in a Galois field $GF(p^n)$ (then: $\lambda' = 1$; $k' = r' = p^n + 1$; $v' = b' = p^{2n} + p^n + 1$), it may be proved by enumeration methods that there is an algebraic extension of $A'$ corresponding to the extension of the $GF(p^n)$ into $GF(p^{mn})$ (with $m = 2$ if the extension is a "quadratic" one).

A few other applications are given in Figs. 1, 2, 3 and 4.

### *Remark*

We assumed that $y \neq 0$; the very simple example of the incidence matrix of points with lines in a finite $d$-dimensional projective geometry (with $A$ corresponding to $d - 1$ dimensions) shows that when this condition is not fulfilled, $z = 0$ does not imply necessarily $t = 0$.

## II. DIMENSION EXTENSIONS

Throughout this section it will be assumed that $\lambda' \neq 0$, and that $A'$ is not a matrix with unit elements everywhere. Let us suppose that $B$ may be partitioned into $(x - 1)$ matrices identical with $A'$, $z$ columns with only zero elements and a column with unit elements only: for $1 \leqslant j \leqslant v'$

$$(13)\ a_j^{b'+i'} = \begin{cases} a_j^i \text{ if } i' = x' + (x-1)(i-1) \text{ (with } 1 \leqslant i \leqslant b' \text{ and } 1 \leqslant x' \leqslant x - 1) \\ 1 \text{ if } i = b - b' \\ 0 \text{ otherwise} \end{cases}$$

That implies—

$$(14)\ r = xr' + 1;\ \lambda = x\lambda' + 1;\ b = b'x + z + 1;\ k \geqslant v'.$$

We now prove that *if* B *has no columns with only zero elements,* A *is a square matrix.*

From (4), (5) and (14) a straightforward computation shows that $k \geqslant v'$ is equivalent to $xr'(v' - k')^2(xr' - v' + 1) \leqslant 0$ so that $x \leqslant (v' - 1)/r'$. Then by (14), again, $r \leqslant v'$; but in any design (see R. A. Fisher, 1940) $k \leqslant r$ if $v \neq k$. Thus $r = k = v'$; $b = v = 1 + b'(v' - 1)/r'$; $\lambda = 1 + \lambda'(v' - 1)/r'$.

At the same time, these equations show that the $b^{\text{th}}$ column of $A$ has in $D$ zero elements only.

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FIG. 1.—Quadratic extension of $A'$: $\lambda' = k' = r' = v' = b' = 3$ into $A$: $\lambda = 3$; $k = r = 7$; $v = b = 15$. ($A$ is the incidence matrix of a space projective geometry with co-ordinates in $GF(2)$; the same construction holds for any $A'(\lambda' = k' = r' = v' = b' = p^n + 1)$.)

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FIG. 2.—Quadratic extension of $A'$: $\lambda' = 4$; $k' = 3$; $r' = 4$; $v' = 3$; $b' = 4$ into $A$: $\lambda = 4$; $k = 5$; $r = 10$; $v = 11$; $b = 22$ (this solution is not isomorphic to twice the ($\lambda = 2$; $k = r = 5$; $v = b = 11$) design).

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FIG. 3.—Kümmer's configuration as quadratic extension of: $A'$: $\lambda' = 2$; $k' = r' = 3$; $v' = b' = 4$ into $A$: $\lambda = 2$; $k = r = 6$; $v = b = 16$.

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FIG. 4.—Quadratic extension of $A'$: $\lambda' = 2$; $k' = 3$; $r' = 5$; $v' = 6$; $b' = 10$ into $A$: $\lambda' = 2$; $k = 4$; $r = 10$; $v = 16$; $b = 40$.

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FIG. 5.—Dimension extension of $A'$: $\lambda' = 2$; $k' = r' = 5$; $v' = b' = 11$; into $A$: $\lambda = 5$; $k = r = 11$; $v = b = 23$. (Notice that the group of automorphisms of this solution is not transitive in the rows nor in the columns, so that this solution is not isomorphic with the cyclic one as given by R. C. Bose, 1939.)

$B$ having no columns with zero elements only, let us assume further that a row of $C$ has unit elements only: then $r' \leqslant \lambda$, that is to say $r'(k' - 1) \geqslant r'(r' - 1)$. Using once more the condition $r' \geqslant k'$, one obtains—

(15) $\lambda = r' = k' = x\lambda' + 1;\ r = k = v' = b' = x^2\lambda' + x + 1;\ b = v = x^3\lambda + x^2 + x + 1.$

Instead of this last condition on $C$ and of $z = 0$, we may start conversely from a condition on a single row of $A$. We prove that, *if* (13) *holds,* (15) *follows from the existence of a row of* A *having in* C *unit elements only and in* D *zero elements only.*

This new condition implies $\lambda = r'$ and $r = b'$. From (14) one deduces $r' = x\lambda' + 1$ and $b' = xr' + 1 = x^2\lambda' + x + 1$, so that $k' \left( = \dfrac{r'(r' - \lambda')}{r'^2 - b'\lambda'} \right) = x\lambda' + 1 = r'$. Thus $A'$ must be a square matrix. As $r = v'$, $k > v'$ would imply $k > r$ so that $A$, too, must be a square matrix and direct computation shows then that $z = 0$. At the same time it has been proved that $C$ has a single row with only unit elements. Further, we see that the transposed matrix of $C$ (apart from its last row) must represent a design with parameters: $\lambda_1 = (x - 1)\lambda'$; $k_1 = k'$; $r_1 = (x - 1)r'$. These conditions are obviously fulfilled if $C$ may be partitioned into $(x - 1)$ matrices identical with $A'$ and a row with unit elements only, that is to say, for $1 \leqslant i \leqslant v'$; $1 \leqslant j \leqslant v'$, if

$$(16)\quad a^i_{v'+j'} = \begin{cases} a_j^i \text{ if } j' = x' + (x - 1)(j - 1) \text{ with } 1 \leqslant x' \leqslant x - 1 \\ 1 \text{ if } j' = v - v' \end{cases}$$

Now we can define a *dimension extension* of $A'$ as an extension satisfying (13), $z = 0$ and (16). Let us consider the submatrices $E_j^i$ of $D$ defined for $i$ and $j$ smaller than $v' + 1$ by the following relation:

$$E_j^i = | a_{j^*+x''}^{i^*+x'} | \text{ with : } x' \text{ and } x'' \text{ smaller than } x \text{ and}$$

$$i^* = v' + (i-1)(x-1); \; j^* = v' + (j-1)(x-1).$$

We prove: $E_j^i$ *has zero elements only if* $a_j^i = 1$ *and it is a permutation matrix if* $a_j^i = 0$. That follows from the equation,

$$\sum_{i=v'+1}^{v} a^i{}_j a^i{}_{j^*+x'} = \lambda - \sum_{i=1}^{v'} a^i{}_j a^i{}_{j^*+x'} = 0,$$

the corresponding equality for the rows $j^* + x'$ and $j^* + x''$ $(1 \leqslant x' < x'' \leqslant x - 1)$, and the transposed equation of this last one for the columns $i^* + x'$ and $i^* + x''$.

From these results on $E_j^i$ it follows that (3) between the $j^{\text{th}}$ and $(j'^* + x')$th rows is satisfied if $1 \leqslant j \neq j'^* \leqslant v'$ and $1 \leqslant x' \leqslant x - 1$, for one has

$$\sum_{i=1}^{v} a^i{}_j a^i{}_{j'^*+x'} = \sum_{i=1}^{v'} a^i{}_j a^i{}_{j'^*+x'} + \sum_{i=v'+1}^{v} a^i{}_j a^i{}_{j'^*+x'} = \lambda' + (r' - \lambda') = \lambda.$$

## Applications

(i) Let $A$ be the incidence matrix of points with $(d-1)$ dimensional hyperplanes in a finite projective geometry of $d$ dimensions with co-ordinates in a $GF(p^n)$. The consideration of any block intersection of $A$ shows that $A$ is the dimension extension (with $x = p^n$) of the corresponding matrix $A'$ for a number of dimensions $d' = d - 1$.

(ii) When $x = 2$, the matrices $E_j^i$ are degenerated into $1 \times 1$ matrices, so that $D$ (apart from its last row and column) is the incidence matrix of the complement design of $A'$. In order to prove that such an $A$ is a balanced design we need only to prove that (3) holds between any two $j^{\text{th}}$ and $j'^{\text{th}}$ rows for $v' + 1 \leqslant j < j' \leqslant v - 1$. As the parameter $\overline{\lambda'}$ of the complementary design of $A'$ is given by $\overline{\lambda'} = v' - 2r' + \lambda'$, one has

$$\sum_{i=1}^{v} a_j^i a_{j'}^i = \sum_{i=1}^{v'} a_j^i a_{j'}^i + \sum_{i=v'+1}^{v} a_j^i a_{j'}^i = \lambda' + (v' - 2r' + \lambda') = 2\lambda' + 1 = \lambda.$$

Obviously, the construction which led from $A'$ to $A$ may be applied again to $A$. Thus an infinite family of designs may be obtained each time that a design with $\lambda' = 2^i\mu + 2^{i-1} - 1$, $r' = k' = 2^{i+1}\mu + 2^i - 1$, is known. For $\mu = 0$, one obtains the matrices corresponding to the finite projective geometries with co-ordinates in $GF(2)$. For $\mu = 1$, the two first designs of the family are given in Fig. 5.

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