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321*t*. M. P. Schützenberger: *A generalization of the Fréchet-Cramér inequality to the case of Bayes estimation.*

Let $f(x)$ be the a priori density function of x; $g(y|x)$ the conditional density function of y. For fixed x, the set of n independent y-variates is represented by z. The density function of z is $f'(z)$ and $g'(x|z)$ is the a posteriori density function of x, for given z. The a posteriori variance of the Bayes estimate is $v_z^2 = \int (x-\bar{x})^2 g'(x|z)dx$ and $v^2 = E_z v_z^2 = \int v_z^2 f'(z)dz$ is its average over z. $F = \int (\partial f(x)/\partial x)^2 (f(x))^{-1}dx$; $G = E_x G_x$ with $G_x = \int ((\partial/\partial x)g(y|x))^2 (g(y|x))^{-1}dy$; $G' = E_z G_z'$ with $G_z' = \int ((\partial/\partial x)y'(x|z))^2 (g(x|z))^{-1}dx$. The usual assumptions on f and g, which insure that F, G_x, G_z' are finite are made. Since $O = F' = \int ((\partial/\partial x)f'(z))^2 (f'(z))^{-1}dz$, it is easily seen that $F + nG = G'$ (Third London Symposium on Information Theory, 1955, p. 18). Furthermore, it is a classical result that $v_z^2 G_z' \geqq 1$. Thus $v^2 = E_z v_z^2 \geqq (E_z 1/v_z^2)^{-1} \geqq (E_z G_z')^{-1} = (F+nG)^{-1}$, which is the desired inequality that tends to the usual form when n goes to infinity. It reduces to an equality if and only if $v^2 = v_z^2 = (G_z')^{-1}$ for all z, that is, if and only if $g'(x|z)$ is gaussian with variance independent of z. If, furthermore, $y - x = t$ has a distribution $h(t)$ independent of x, this implies that $f(x)$ and $h(t)$ are also gaussian. (This work was supported in part by the Army (Signal Corps), the Air Force (Office of Scientific Research, Air Research and Development Command), and the Navy (Office of Naval Research).) (Received November 5, 1956.)